FORTH

# Modelling approaches for photonic metamaterials

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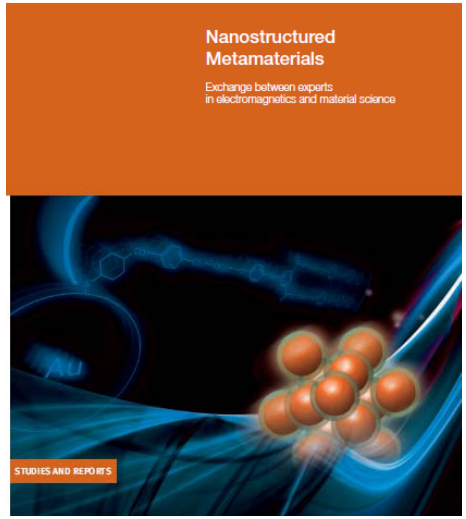
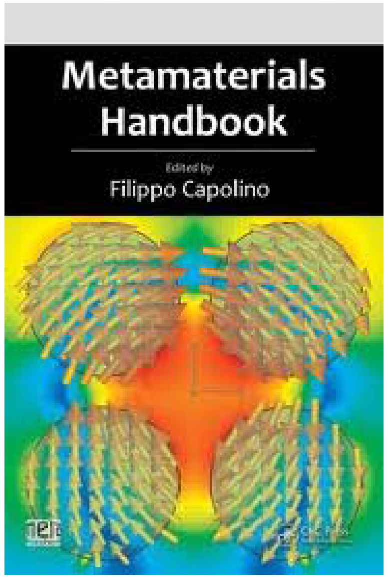
# Main resources

**EU-Brochure on *Nanostructured Metamaterials***
**http://ec.europa.eu/research/industrial_technologies/pdf/metamaterials-brochure_en.pdf**




***Metamaterials handbook* (2 volumes), edited by Filippo Capolino, CRC Press, Taylor & Frances**

**http://www.metamorphose-vi.org**

ECONAM

**EU project on *Electromagnetic Characterization of Nanostructured Metamaterials***

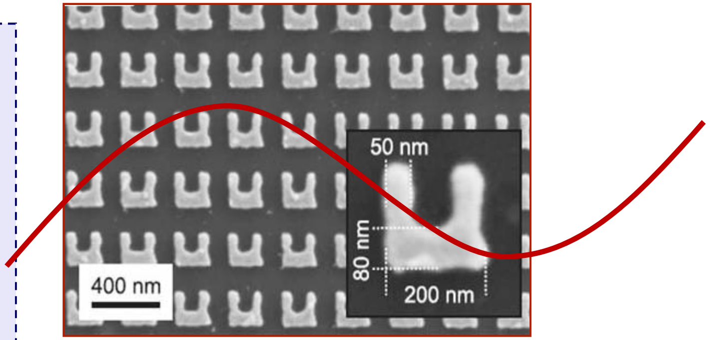
**http://econam.metamorphose-vi.org**

# Metamaterials

**Artificial, structured (in sub-wavelength scale) materials**

**with electromagnetic (EM) properties not-encountered in natural materials**

**EM properties derive from shape and distribution of constituent units (usually metallic & dielectric, of subwavelength scale)**




**EM properties**

$\varepsilon$

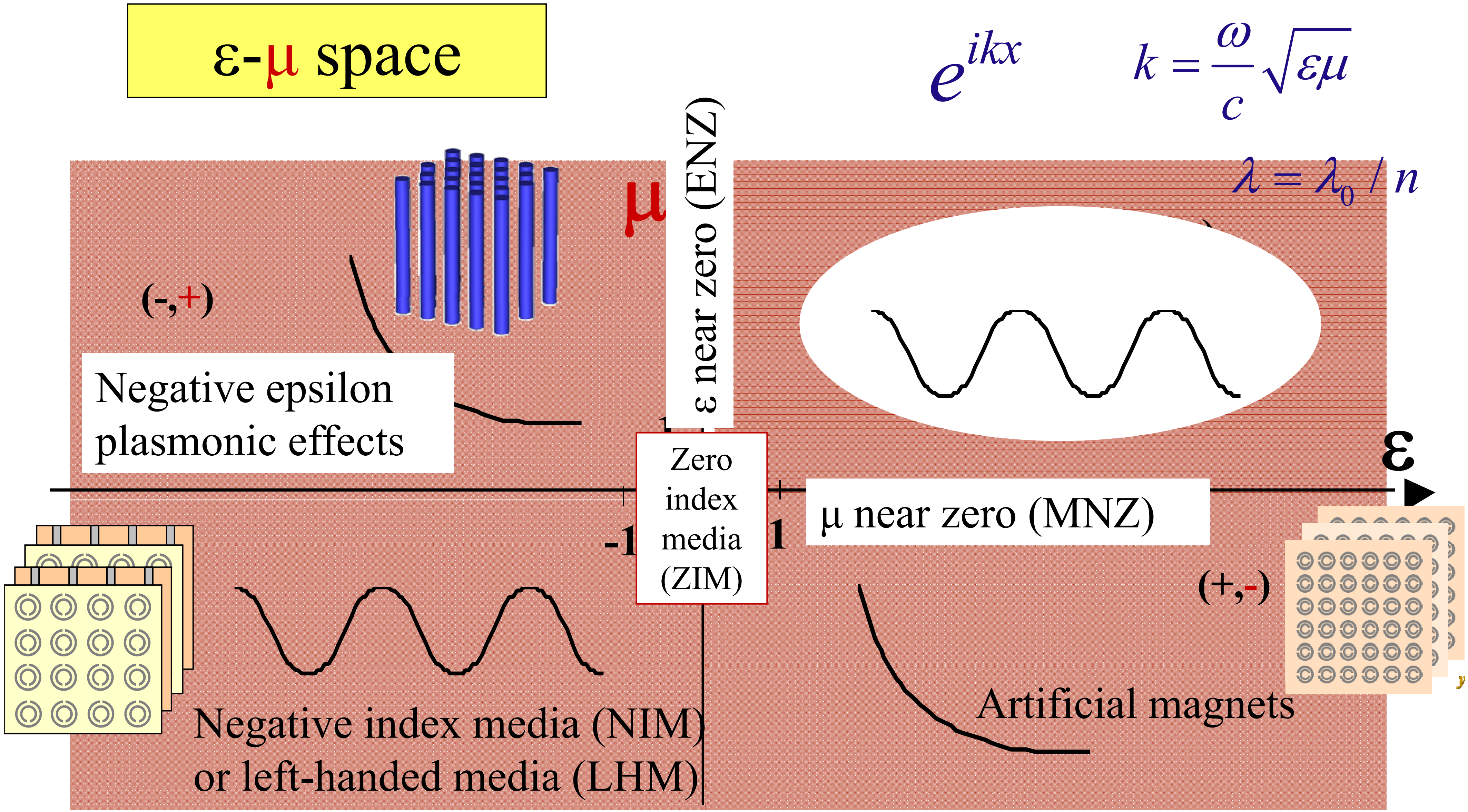
**Electrical permittivity**

$\mu$

**Magnetic permeability**

**Possibility to engineer electromagnetic properties**

# Interesting metamaterial regimes

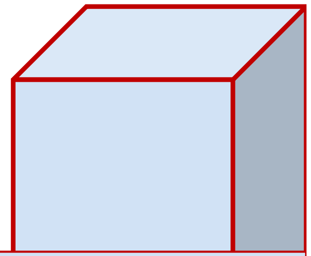


# Modelling approaches

- **Rigorous approaches (detailed numerical techniques, e.g. FDTD, BEM, TMM,..)**

  **Input:** Maxwell equations + detailed geometrical structure + material parameters of components

  **Output:** transmission/reflection, dispersion relation, electromagnetic fields, …



- **RLC circuit modeling**
- **Mixing rules**

- **Homogenization (homogeneous effe approaches:** Search for the equivalent homogeneous medium (ε, μ, κ, …) with the same response as our metamaterial

  - **Direct approaches (first principle approaches):** From microscopic quantities to macroscopic through averaging
  - **Inverse (heuristic) approaches:** From reflection/transmission to material parameters through inversion

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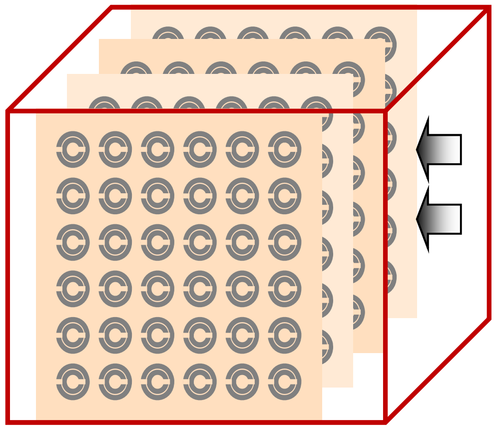
# Modelling essentials

- **Maxwell's equations - determine the propagation of EM waves**
- **Constitutive relations in the constituent media (metals + dielectrics) – represent the electromagnetic response of each material**

$$\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) \qquad \mathbf{B}(\mathbf{r}) = \mu_0\mathbf{H}(\mathbf{r})$$

**→Structure geometry**



- **Boundary conditions**
- **Excitations (if any)**

# Maxwell's equations in matter

Bound and free charges are not separated (ε ,σ, P, J describe both bound and free charges)

$$\nabla \cdot \mathbf{D} = \rho_{ext}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_{ext}$$

Ampere's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{J} = \sigma \mathbf{E} = \frac{\partial \mathbf{P}}{\partial t}$$

$$\varepsilon = \varepsilon_0 + i\frac{\sigma}{\omega}$$

# Rigorous modelling techniques

**Finite Difference Time Domain (FDTD) method and Finite Integration Technique (FIT)**

**Finite Element Method (FEM)**

**Transfer Matrix Method (TMM)**

**Boundary Element Method (BEM) or Method of Moments (MoM)**

**Discrete Dipole Approximation**

***Fourier Modal Method or Rigorous Coupled Wave Analysis***

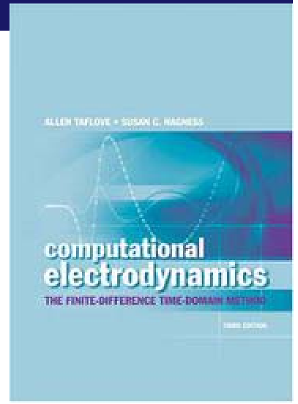
***Multiple Scattering Method***

**Scattering analysis**

**Eigenmode analysis**

**Many free and commercial software packages – see Wikipedia**

# Finite Difference Time Domain (FDTD) Method



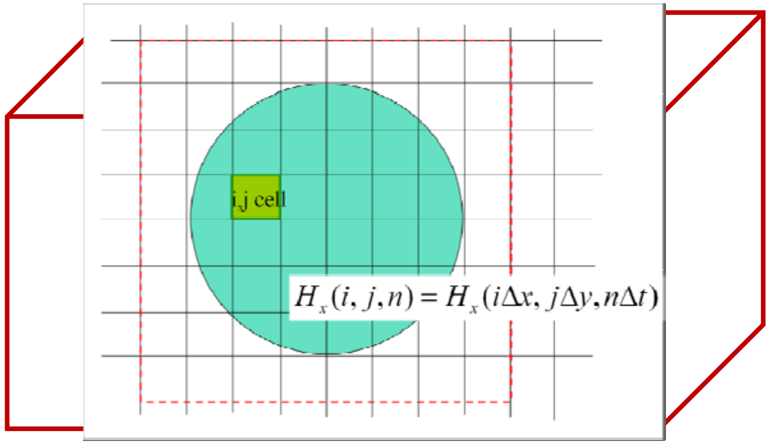

**Treats:** mainly finite systems along propagation direction

**Calculates:** transmission, reflection, fields in time and frequency domain

**Approach:** Discretization of time-dependent Maxwell's equations in both space and time (1D-3D)

$$\mathbf{H}_{i,j}^{n} = \mathbf{H}(i\Delta x, j\Delta y, n\Delta t)$$

$$\mathbf{E}_{i,j}^{n} = \mathbf{E}[(i+1/2)\Delta x, (j+1/2)\Delta y, (n+1/2)\Delta t]$$




Yee's scheme

Maxwell's equations → algebraic difference equations

$$\mathbf{H}(t+\Delta t/2) = \text{Function}(\mathbf{E}(t), \mathbf{H}(t-\Delta t/2))$$

$$\mathbf{E}(t+\Delta t) = \text{Function}(\mathbf{E}(t), \mathbf{H}(t+\Delta t/2))$$

→ $\mathbf{E}(t), \mathbf{H}(t)$

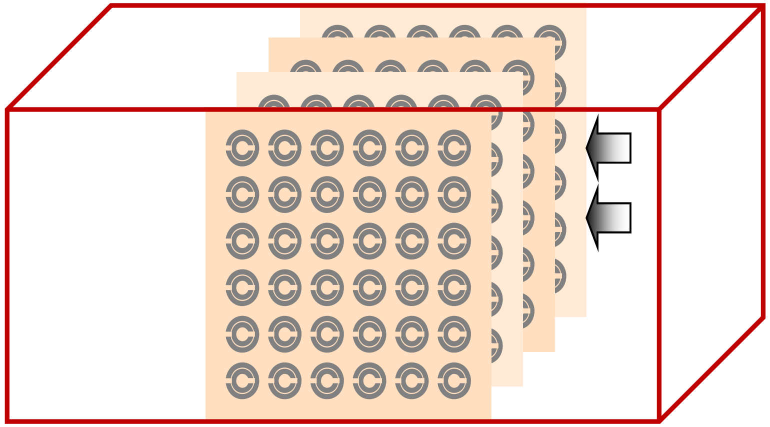
→ $\mathbf{E}(\omega) = \text{FFT}(\mathbf{E}(t))$

$\mathbf{H}(\omega) = \text{FFT}(\mathbf{H}(t))$

# Finite Difference Time Domain (FDTD) Method (2)

## Problem for dispersive materials

$$\mathbf{D}(t) \neq \varepsilon(t)\mathbf{E}(t)$$



Constitutive relations should be discretized taking into account explicit dispersion model

↓

**Additional equations are required**

**Advantages of FDTD**

- With one computation →multifrequency study
- Treats both random and periodic media
- Treats almost arbitrary geometries
- Not heavy computational memory requirements

**Limitations**

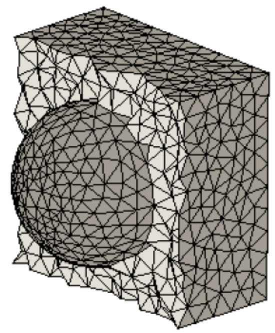
- Restrictions in system size
- Restrictions in materials contrast

# Finite element method (1D-3D)

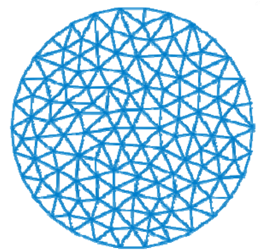
From Yinun Liu, Univ. of Cincinnati

**Treats** mainly finite systems but also infinite

**Calculates** transmission, reflection, fields in time and frequency domain, dispersion relation



**Approach:** Reformulates harmonic Maxwell's equations + boundary conditions → weak form



**Procedure:**

- Divides structure into pieces (elements with nodes)
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities (fields) at the nodes→ matrix inversion

**Commercial packages:**
FEMLAB
CST

**Advantages**

- Treats arbitrary geometries and materials

**Disadvantages**

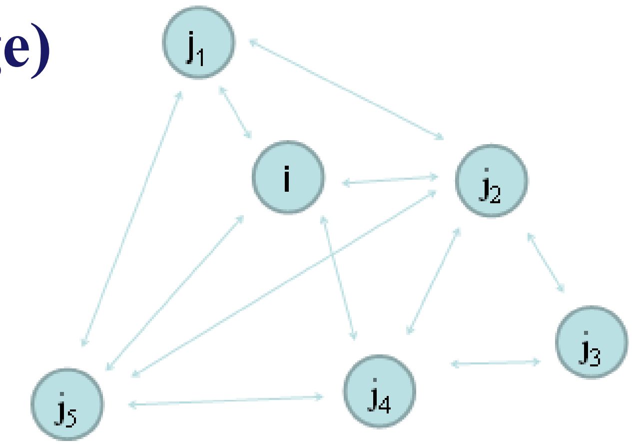
- Not easy implementation

W. B. J. Zimmerman, *Process Modelling and Simulation with Finite Element Methods (World Scientific, Singapore,* 2004).

J. Jin, *The Finite Element Method in Electromagnetics (John Wiley & Sons, Inc., New York, 2002).*

# Multiple scattering method

**Treats:** Both infinite and finite (not very large) systems; both periodic and random systems




**Calculates:** Band structure (ω vs k), equifrequency contours, transmission/reflection, fields

**Main idea:** Incident wave at each scatterer = external field + scattered wave from all the other scatterers

Yannopapas, Stefanou, Moroz, Chan, Sheng, …

**Procedure:** Waves are expanded in vector spherical harmonics; final equation is a non-linear algebraic system (for eigenmodes) or a linear algebraic system (for finite slabs)

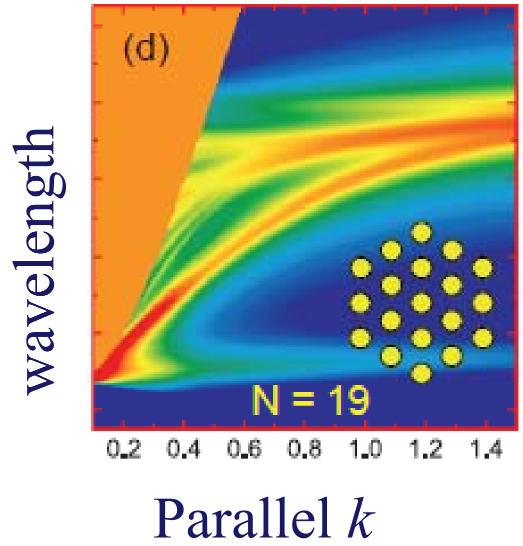
**Variation:**
Layered Multiple Scattering Method

**Disadvantages**
- Heavy algebra
- Treatment of simple-shaped scatterers only

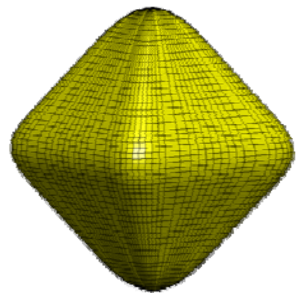
# Boundary Element Method (BEM) or Method of Moments (MoM)

**Treats:** Clusters of arbitrarily-shaped particles (good for particles with small surface to volume ratio)

**Calculates:** Scattering cross-sections, fields, density of states




**Procedure:** Discretizes only the boundaries, expresses the fields vs potentials, potentials vs Green's functions, requires continuity of potentials and parallel fields



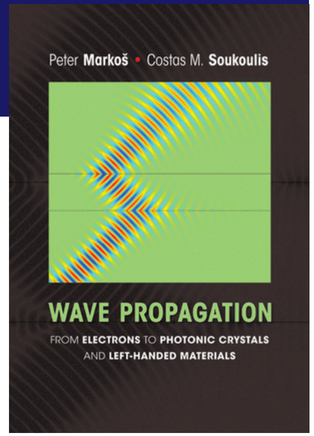
**Disadvantage:** Produces dense matrices (→ suitable for small systems)

Courtesy of V. Myrosnichenko

De Abajo, Craeye, Cappolino, …

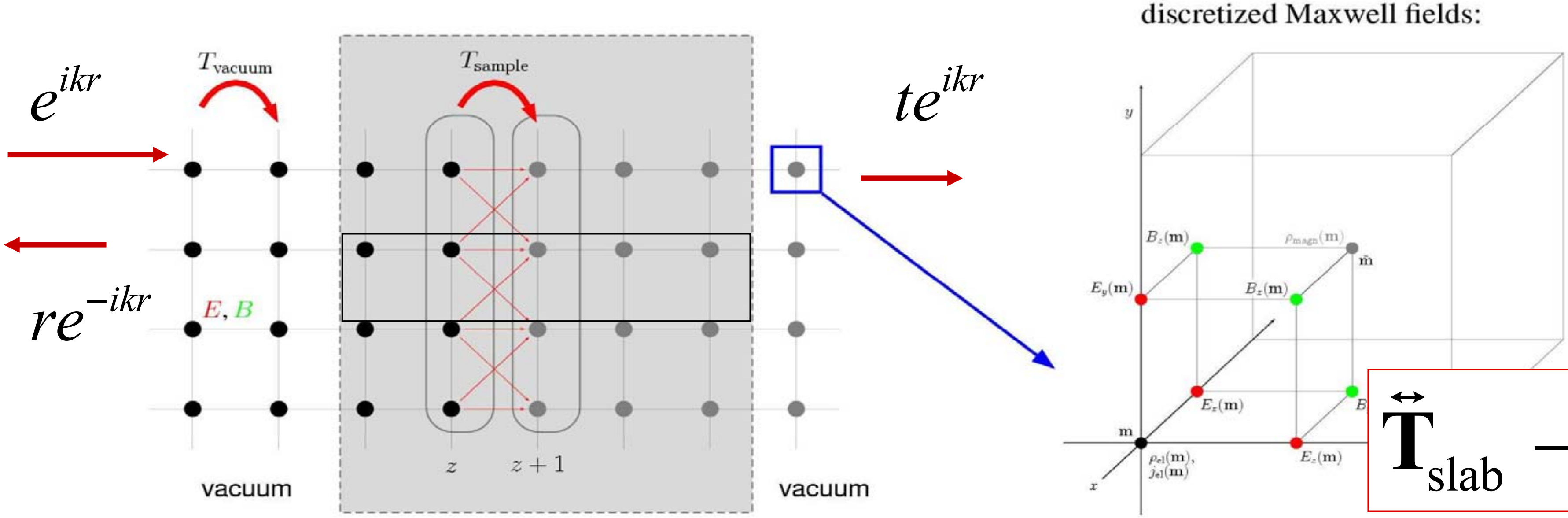
**F. J. García de Abajo and A. Howie, *Phys. Rev. B*, 65, p. 115418, 2002**

# Transfer Matrix Method (1D-3D)




**Treats:** Finite slabs along propagation direction; both periodic and random systems

**Calculates:** transmission/reflection, fields (in frequency domain)




$$\overleftrightarrow{\mathbf{T}}_{slab} \rightarrow r,t$$

**Procedure:** Divides the space in layers. Calculates transfer matrix for each layer

$$\overleftrightarrow{\mathbf{T}}_{slab} = \prod_k \overleftrightarrow{\mathbf{T}}_k$$

$$\begin{pmatrix} E^{k+1} \\ H^{k+1} \end{pmatrix} = \overleftrightarrow{\mathbf{T}}_k \begin{pmatrix} E^k \\ H^k \end{pmatrix}$$

Pendry, Bell, Transfer matrix techniques for EM waves, *Photonic Band Gap Materials*, 1996

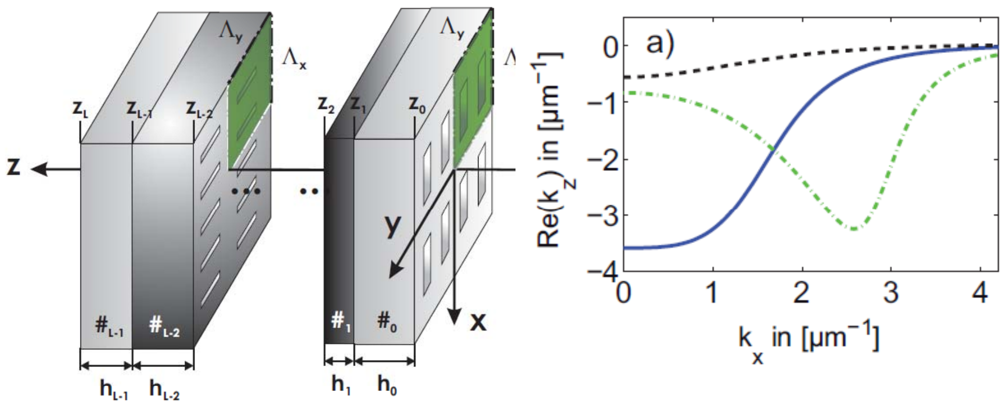
# Fourier Modal Method or Rigorous Coupled Wave Analysis (2D-3D)

**Treats:** Both infinite and finite (in 1D) slabs

**Calculates:** Band structure (ω vs k), equifrequency contours, transmission/reflection, fields in frequency domain (sum of Bloch modes)

**Main idea:** All EM quantities are expanded in Fourier series → Eigenmode problem calculating ω for each k

**Procedure applied to planar metamaterials:** Brakes system into layers, calculates the eigenmodes of a single layer (2D problem) for a given ω, $k_{//}$




From Thomas Paul thesis, Univ. of Jena

P. Lalanne, F. Lederer, G. Shvets, …

L. Li, “New formulation of the Fourier modal method for crossed surface-relief gratings,” J. Opt. Soc. Am. A 14, 2758 (1997).

# Modelling approaches

- Rigorous approaches (detailed numerical techniques, e.g. FDTD, BEM, TMM,..)

  Input: Maxwell equations + detailed geometrical structure + material parameters of components

  Output: transmission/reflection, dispersion relation, electromagnetic fields, …

RLC circuit modeling
Mixing rules

- Homogenization (homogeneous effective medium) approaches: Search for the equivalent homogeneous medium (ε, μ, κ, …) with the same response as our metamaterial

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# Why homogenization?

- Offers **simple** and **physical picture** of the metamaterial response (connects the response to few, well known material parameters)

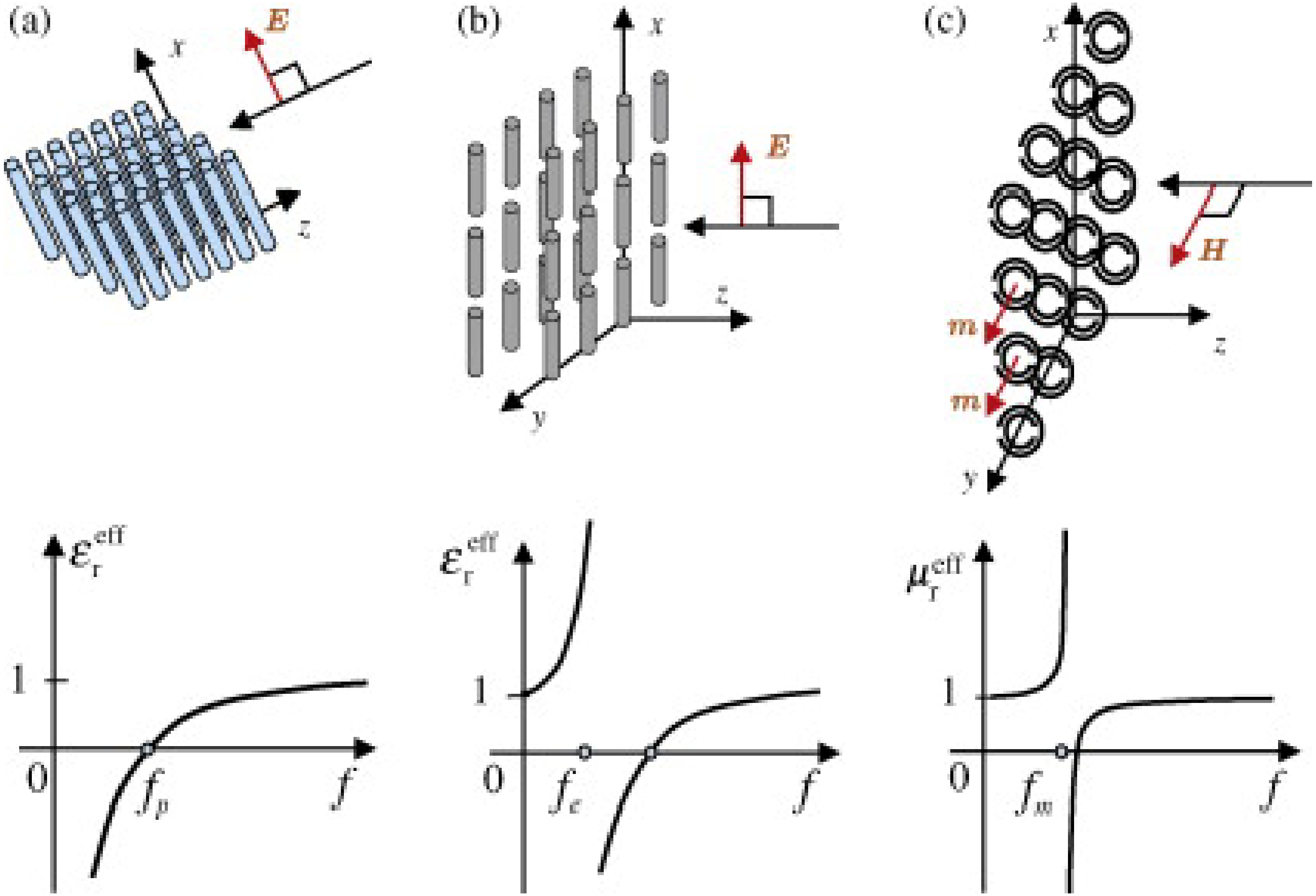
- **Predicts the** metamaterial **response under different conditions** (excitation, environment, total size)

- Offers **path to** metamaterial **optimization** and **design rules**

- Predicts **phenomena** connected with the metamaterial

- Reveals **potential applications/uses** of the metamaterial

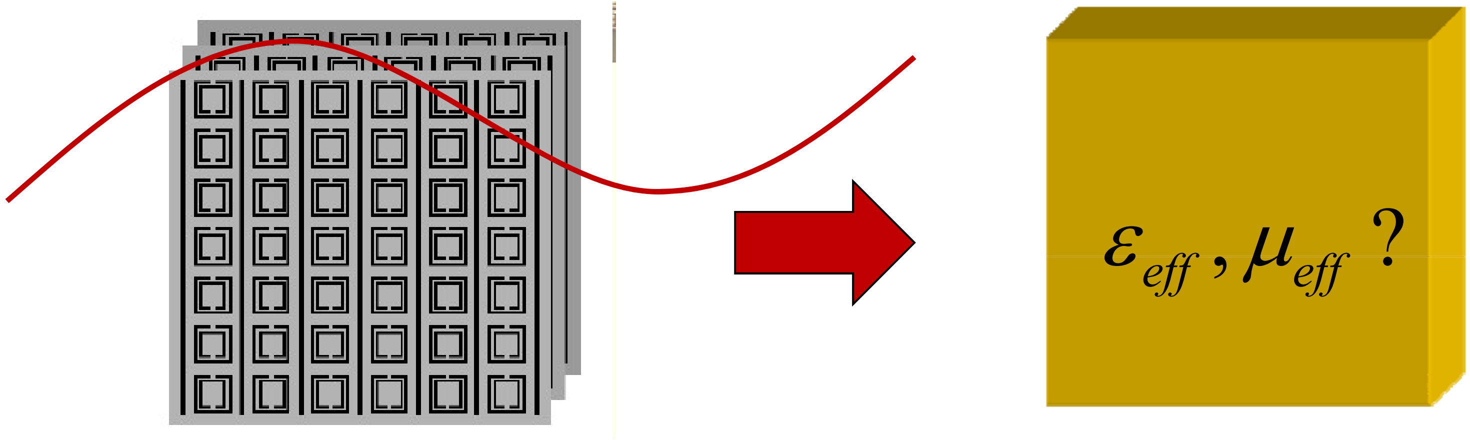
# Typical forms of metamaterial parameters




**Artificial electric metamaterials** **Artificial magnetic metamaterials**

Donzelli et al, Metamaterials 3 (2009)

# Questions on homogenization

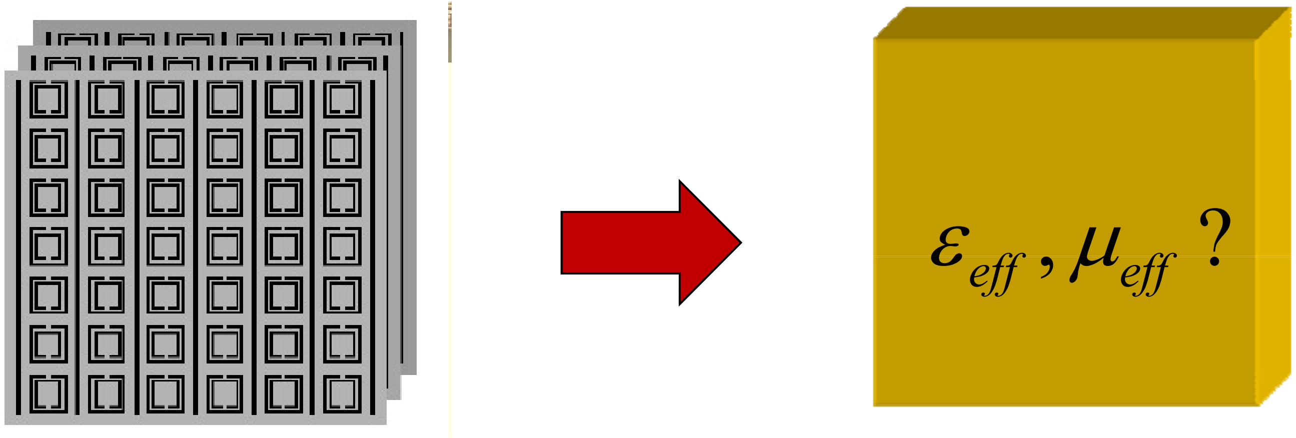



- How many effective parameters are needed to characterize the metamaterial? → metamaterials classification

- Under what conditions a metamaterial can be homogenizable?

- Under what conditions effective medium parameters can be considered characteristic metamaterial parameters?

# Questions on homogenization

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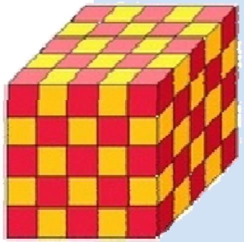
# Metamaterial classification

**For linear, *homogenizable*, metamaterials (in frequency domain)**

## Isotropic

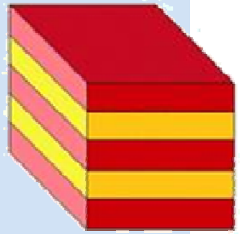


$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

**2-parameters**

## Anisotropic

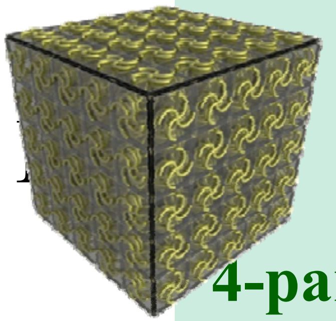


$$\mathbf{D} = \overleftrightarrow{\varepsilon} \mathbf{E}$$

$$\mathbf{B} = \overleftrightarrow{\mu} \mathbf{H}$$

**18-parameters**

## Bi-isotropic

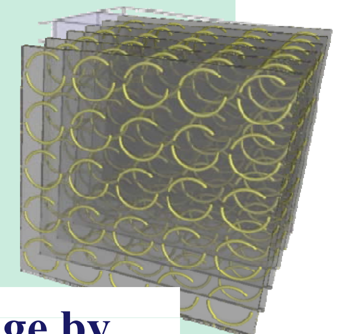


$$\mathbf{D} = \varepsilon \mathbf{E} + \xi \mathbf{H}$$

$$\mathbf{B} = \zeta \mathbf{E} + \mu \mathbf{H}$$

**4-parameters**

## Bi-anisotropic



$$\mathbf{D} = \overleftrightarrow{\varepsilon} \mathbf{E} + \overleftrightarrow{\xi} \mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta} \mathbf{E} + \overleftrightarrow{\mu} \mathbf{H}$$

**For reciprocal media (system properties do not change by exchanging source and receiver position)** $\overleftrightarrow{\varepsilon} = \overleftrightarrow{\varepsilon}^T, \overleftrightarrow{\mu} = \overleftrightarrow{\mu}^T, \overleftrightarrow{\xi} = -\overleftrightarrow{\zeta}^T$

**Greek: isos=equal, tropos=way, i.e. isotropic=behave in equal way for all directions**

# Subclasses of bi-isotropic media: Chiral

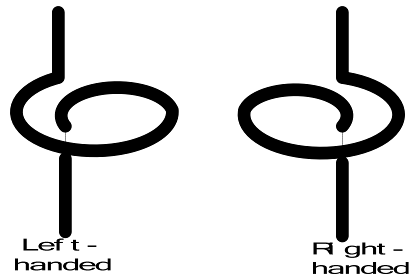
**Chiral media** (no identical with their mirror images)

$$\mathbf{D} = \varepsilon \mathbf{E} + i\kappa \mathbf{H}$$

$$\mathbf{B} = -i\kappa \mathbf{E} + \mu \mathbf{H}$$

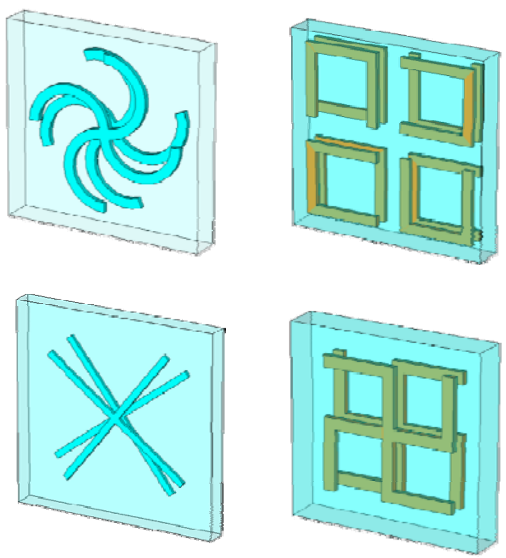
e.g. DNA

$\kappa$=Chirality parameter




Greek: chira=hand

$$n_{\pm} = \sqrt{\varepsilon\mu} \pm \kappa$$



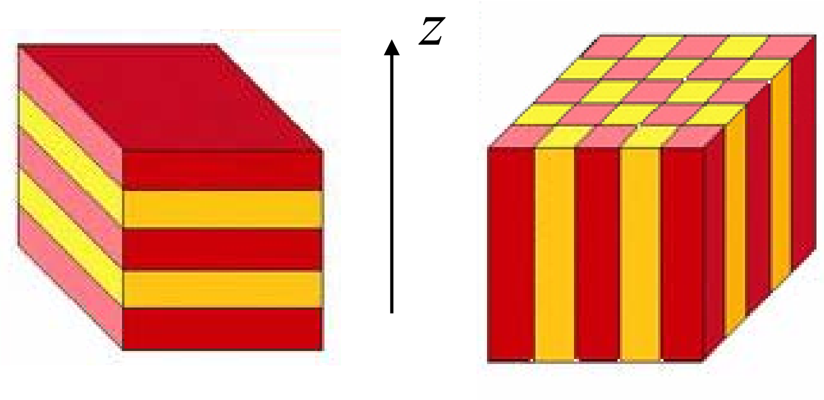
- Different index for left- and right-handed circularly polarized waves (circular birefringence)
- Negative index possibility
- Optical activity (polarization rotation of a linearly polarized wave)
- Circular dichroism (different absorption for left and right circularly polarized light)

Zheludev, Pendry, Tretyakov, Soukoulis, Wegener, Giessen, Shalaev, …

# Subclasses of anisotropic media: Uniaxial

## Uniaxial media

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$




**Birefringence:** **Different index for polarization along *z* and perpendicularly to *z***

**Light polarized along the optic axis is called the extraordinary ray, and light polarized perpendicular to it is called the ordinary ray**

**Important example: hyperbolic dispersion relation metamaterials → Hyperlensing, large DOS**

**Narimanov, Engheta, Shalaev, Smolyaninov, Zhang, …**

# Effective parameters dispersion

**Temporal dispersion**

$$\mathbf{D}(\mathbf{r},t)=\int_0^t \varepsilon_{eff}(\mathbf{r},t-t')\mathbf{E}(\mathbf{r},t')dt'$$

**Response is not instantaneous** ➡ $\varepsilon_{eff}=\varepsilon_{eff}(\omega)$

**Spatial dispersion?**

**Due to the not very small size/wavelength ratio**

$$\mathbf{D}(\mathbf{r},t)=\int_0^r \varepsilon_{eff}(\mathbf{r},\mathbf{r}',t)\mathbf{E}(\mathbf{r}',t)d\mathbf{r}'$$

**Response is not local in space** ➡ $\varepsilon_{eff}=\varepsilon_{eff}(\mathbf{k})$

**Boundary conditions are not fulfilled**

**Strong spatial dispersion is detrimental for homogenization**

# Artificial magnetism, chirality and weak spatial dispersion

Silveirinha, Phys. Rev. B 75, 115104 (2007)

Tretyakov, EU-brochure

## Equivalent descriptions for weak spatial dispersion

$$\mathbf{D} = \varepsilon(\mathbf{k})\mathbf{E} \qquad \mathbf{B} = \mu_0 \mathbf{H}$$

$$\mathbf{D} = \varepsilon\mathbf{E} + \xi\mathbf{H} \qquad \mathbf{B} = \zeta\mathbf{E} + \mu\mathbf{H}$$

Taylor expansion

$$\mathbf{D} = \varepsilon\mathbf{E} + a\mathbf{k}\times\mathbf{E} + \beta\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + \gamma\mathbf{k}\times(\mathbf{k}\times\mathbf{E})$$

$$\mathbf{D} = \varepsilon\mathbf{E} + a\nabla\times\mathbf{E} + \beta\nabla\nabla\cdot\mathbf{E} + \gamma\nabla\times\nabla\times\mathbf{E}$$

$\sim \omega\mathbf{H}$ Chirality term

$\mathbf{B} \sim \nabla\times\mathbf{E}/\omega$ Artificial magnetism

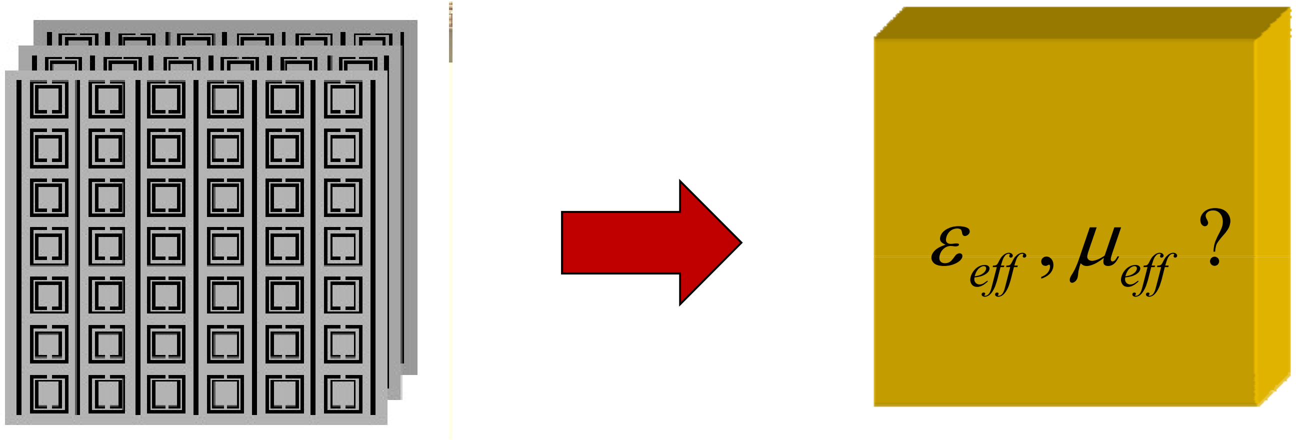
**Artfical magnetism and metamaterial chirality are results of weak spatial dispersion**

Maxwell eqs invariant for

$$\mathbf{D}' = \mathbf{D} + \nabla\times\mathbf{Q}$$

$$\mathbf{H}' = \mathbf{H} - i\omega\mathbf{Q}$$

# Questions on homogenization




- How many effective parameters are needed to characterize the metamaterial? → metamaterials classification

- **Under what conditions a metamaterial can be homogenizable?**

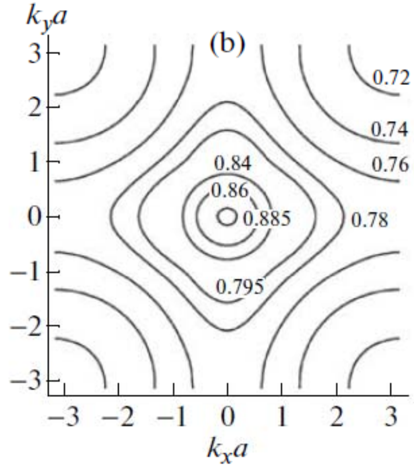
- Under what conditions effective medium parameters can be considered characteristic metamaterial parameters?

# Conditions for homogenization

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} + \overleftrightarrow{\xi}\mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta}\mathbf{E} + \overleftrightarrow{\mu}\mathbf{H}$$

**The metamaterial equifrequency surfaces should be either ellipsoids or hyperboloids**




**The condition is fulfilled in the limit**

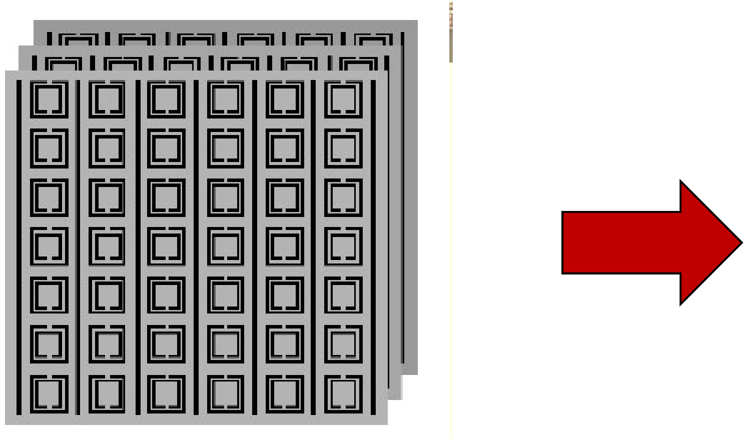
$$q_{eff} = \frac{\omega}{c} n_{eff} < \frac{\pi}{a} \rightarrow \quad \lambda_{eff} = \frac{\lambda_0}{n_{eff}} > 2a$$

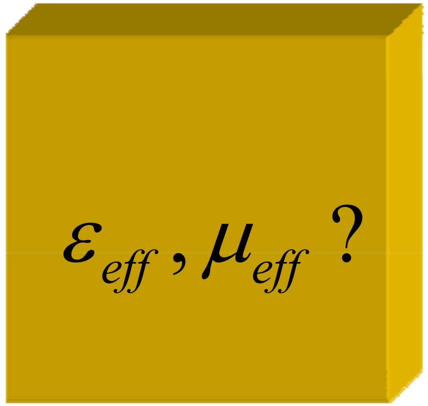
**For resonant structures this may require** $q_0 a << 1 \Rightarrow \lambda_0 >> a$

**For most of today's metamaterials** $\lambda_0 < 10a$

**Validity of effective parameters is questionable or limited**

# Questions on homogenization






- How many effective parameters are needed to characterize the metamaterial? → metamaterials classification
- Under what conditions a metamaterial can be homogenizable?
- **Under what conditions effective medium parameters can be considered characteristic metamaterial parameters?**

# Effective parameters restrictions

## Parameters should obey

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H}$$

**Causality:** **(cause precedes the effect) Polarization P(t) depends on electric field E(t'<t)**

$$\frac{\partial(\omega\varepsilon)}{\partial\omega} \geq 1 \qquad \frac{\partial(\omega\mu)}{\partial\omega} \geq 1$$

**For low-loss *ε*, *μ*, *n* should be increasing functions of frequency**

**Passivity:** **Energy does not grow**

$$\mathrm{Im}(\varepsilon) > 0,\ \mathrm{Im}(\mu) > 0$$

**or**

$$\mathrm{Re}(z) = \mathrm{Re}(\sqrt{\mu/\varepsilon}) > 0$$

$$\mathrm{Im}(n) = \mathrm{Im}(\sqrt{\mu\varepsilon}) > 0$$

# Effective parameters requirements

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} + \overleftrightarrow{\xi}\mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta}\mathbf{E} + \overleftrightarrow{\mu}\mathbf{H}$$

**To be considered as characteristic bulk continuous medium parameters effective parameters should be**

- **Independent of system thickness**
- **Independent of direction of propagation**

+

- **Fulfilling the causality and passivity requirements**

**If no, applicability of parameters is restricted (e.g. to the specific excitation conditions)**

# Homogenization approaches

**Direct approaches: From microscopic quantities (or fields) to macroscopic through "averaging" → material parameters**

$\lambda >> 1000a$

atom  → Bulk metal

$$<\mathbf{D}> = \varepsilon_{eff} <\mathbf{E}>$$

$$<\mathbf{B}> = \mu_{eff} <\mathbf{H}>$$

$$\mathbf{p} = a\mathbf{E}_{loc} \rightarrow \mathbf{P} = (\varepsilon - \varepsilon_0) <\mathbf{E}> \qquad \mathbf{P} = \sum_i \mathbf{p}_i / V$$

$\lambda \approx 10a$

nanosphere  → Nanospheres system

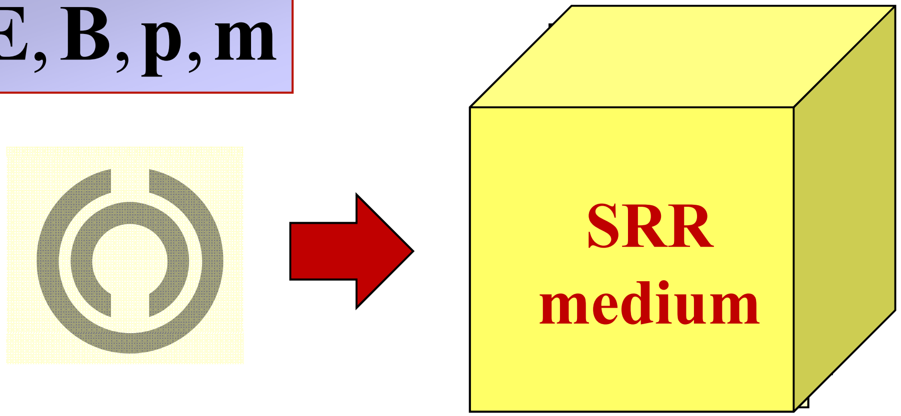
$$\mathbf{P} = (\varepsilon_{eff} - \varepsilon_0) <\mathbf{E}>$$

**Heuristic (inverse) approaches: From reflection/transmission (propagated wave features) to material parameters through inversion → wave parameters**

# From microscopic to macroscopic quantities

**Microscopic**

$\mathbf{E}, \mathbf{B}, \mathbf{p}, \mathbf{m}$




SRR Medium

**Macroscopic**

$\mathbf{D}, \mathbf{P}, <\mathbf{E}>, \mathbf{H}, \mathbf{M}, <\mathbf{B}>$

$$\mathbf{D} = \varepsilon_0 <\mathbf{E}> + \mathbf{P} = \varepsilon_{eff} <\mathbf{E}>$$

$$\mathbf{H} = <\mathbf{B}> / \mu_0 - \mathbf{M} = <\mathbf{B}> / \mu_{eff}$$

**Approach:** Consideration of the metamaterial as collection of electric and/or magnetic dipoles

$$\mathbf{p}.vs.\mathbf{E}_{loc} \rightarrow \mathbf{P}.vs. <\mathbf{E}>$$

$$\mathbf{m}.vs.\mathbf{B}_{loc} \rightarrow \mathbf{M}.vs. <\mathbf{B}>$$

**Problem: Calculate P vs <E> and M vs <B>**

**p.*vs*.$\mathbf{E}_{loc}$ or m.*vs*.$\mathbf{B}_{loc}$ by simple EM formulas**

**RLC circuit formulas**

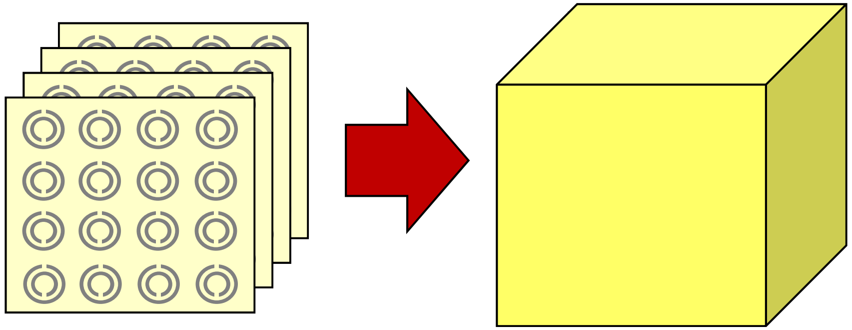
C. Simovski, J. Optics 13, 013001 (2011)

# Averaging Maxwell equations

Smith & Pendry, J. Opt. Soc. Am. B 23, 391 (2006)

**Before average**

$\mathbf{E}(\mathbf{r}), \mathbf{D}, \mathbf{H}, \mathbf{B}, \varepsilon(\mathbf{r})$



**After average**

$<\mathbf{E}>, <\mathbf{D}>, <\mathbf{H}>, <\mathbf{B}>,$

$\varepsilon_{eff}, \mu_{eff}$

$$\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{B}(\mathbf{r}) / \mu_0$$

**Problem**

**Averaging as to fulfill dispersion relation of homogeneous medium**

$$<\mathbf{D}> = \varepsilon_{eff} <\mathbf{E}>$$

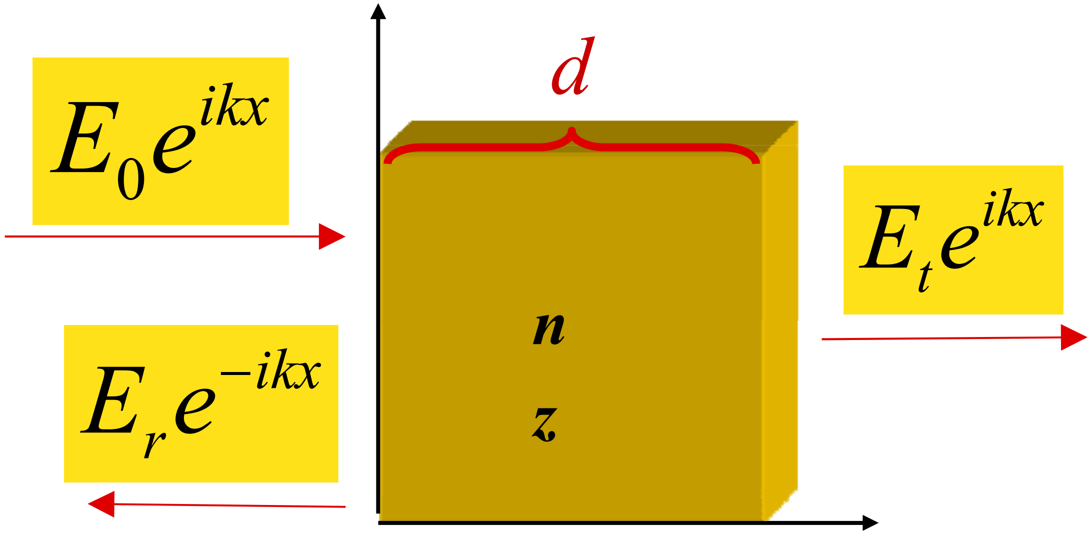
$$<\mathbf{H}> = <\mathbf{B}> / \mu_{eff}$$

**Other approaches: Silveirinha, Alu, Shvets, …**

# Homogenization through S-parameter inversion (1)

## *T* and *R* of a homogeneous slab

Nicholson-Ross-Weir (NRW) method




$$r = E_r / E_0$$

$$t = E_t / E_0$$

*r*: Reflection amplitude

*t*: Transmission amplitude

$$t = \frac{\exp(-ikd)}{\cos(nkd) - \frac{1}{2}\left(z + \frac{1}{z}\right)\sin(nkd)}$$

*k:* free space wave number

$$n = \sqrt{\mu\varepsilon}$$

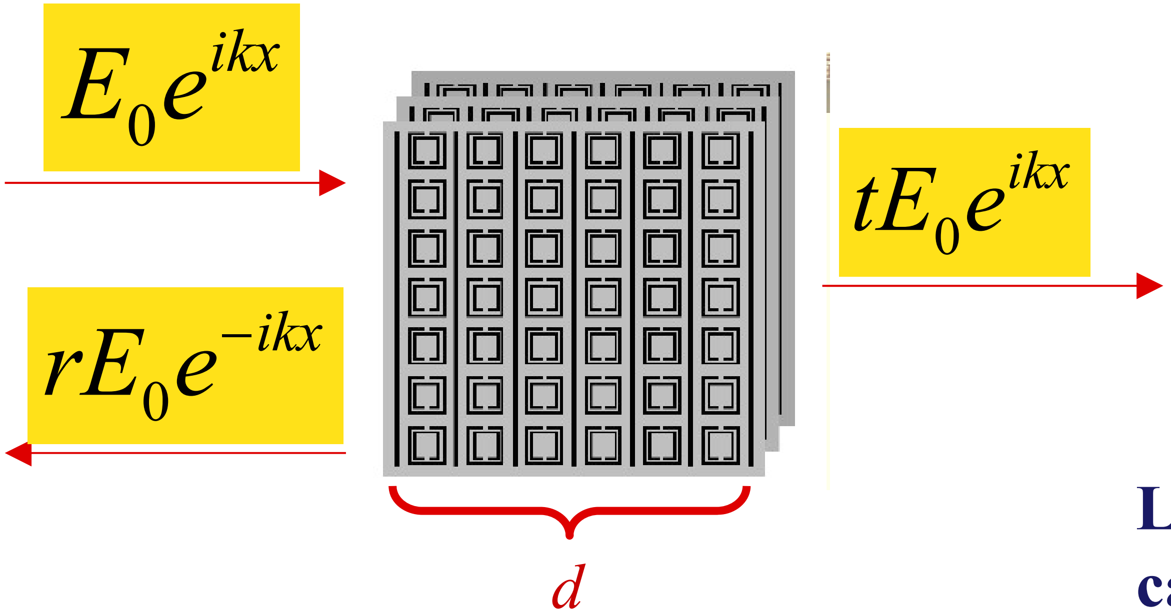
$$r = -t\exp(+ikd)i(z - 1/z)\sin(nkd)/2$$

$$z = \sqrt{\frac{\mu}{\varepsilon}}$$

# Homogenization through S-parameter inversion (2)

$$n = \frac{1}{kd}\cos^{-1}\left(\frac{1}{2t}\left[1-\left(r^2-t^2\right)\right]\right)+\frac{2\pi m}{kd}$$

$$z = \pm\sqrt{\frac{(1+r)^2-t^2}{(1-r)^2-t^2}}$$




$$\varepsilon = n/z$$

$$\mu = nz$$

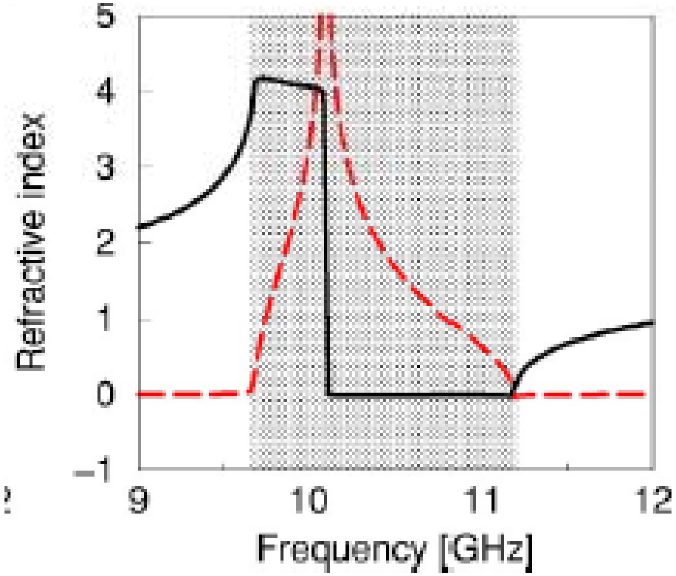
**Lifting ambiguities using causality arguments**

- Re($z$)>0
- Im($n$)>0
- $n$ continuous function of $\omega$
- Apply for small $d$

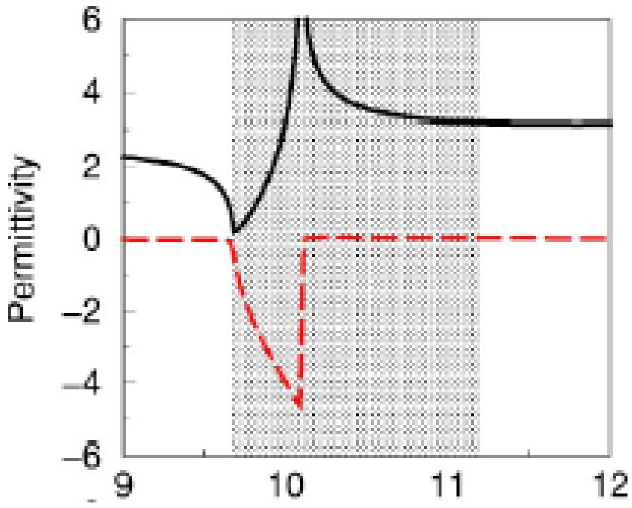
**PRB, 65, 195103 (2002)**

# Representative results and ambiguities

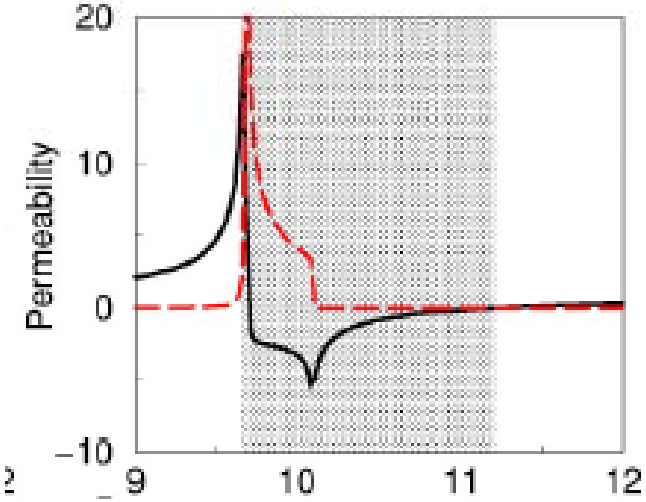
Refractive index *n*




Permittivity ε




**Permeability μ**

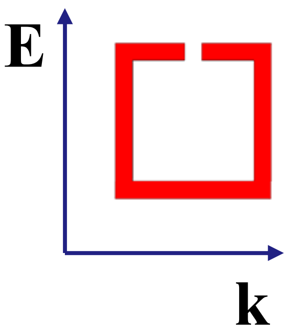



Real ————

Imag - - - - - - -

**"Paradoxes-Anomalies"**

- **Distorted/truncated resonances**
- **Resonance-antiresonance coupling**
- **Wrong sign of imaginary parts**
- **etc**



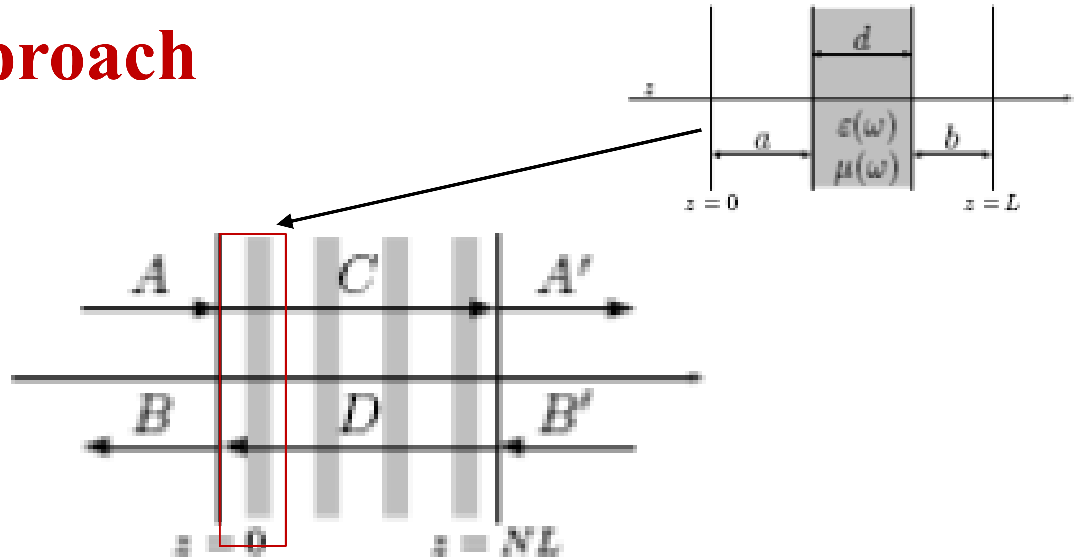

**Observed for**

$$q_{eff}(\omega) = \frac{\omega}{c} n_{eff}(\omega) \geq \frac{\pi}{a}$$

# Alternative retrieval approaches

## Periodic effective medium approach

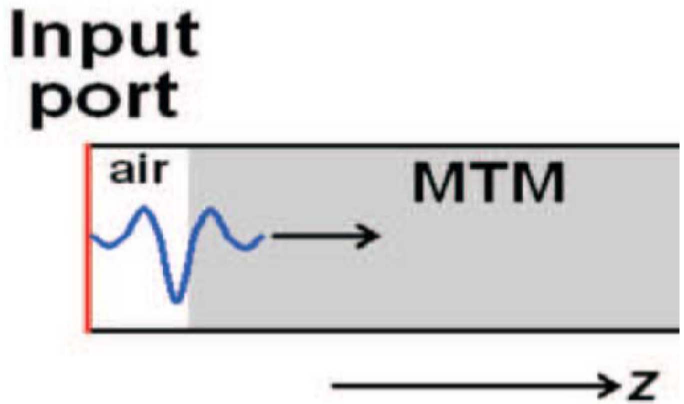
Treatment of metamaterial as a periodic medium made of alternating resonant and air slabs



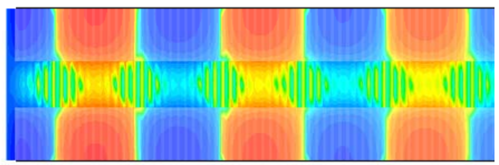

Th. Koschny, et al, *Phys. Rev. B* 71, 245105 (2005)

## Wave propagation approach

Impedance from single interface
Refractive index from modulation of propagating field




Andryieuski et al, *Phys. Rev. B* 80, 193101 (2009)



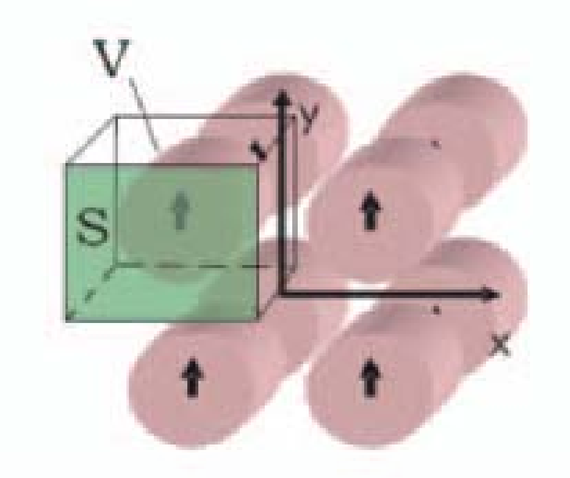
**Many more!!! Also for anisotropic and bianisotropic materials**

# Discrepancies between retrieved and averaged effective params

**Wave (retrieved) parameters often do not obey**

$$<\mathbf{D}> = \varepsilon_{eff} <\mathbf{E}>$$

$$<\mathbf{H}> = (1/\mu_{eff}) <\mathbf{B}>$$




**Averaged parameters often do not give correct scattering properties**

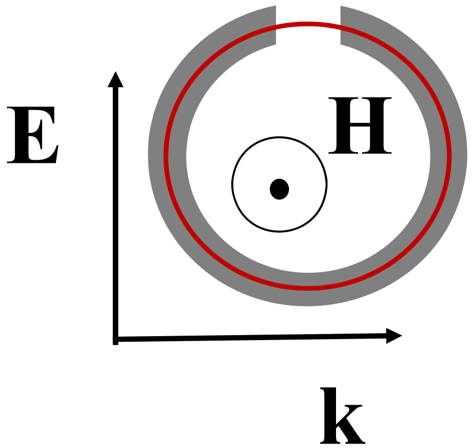
## Reason?

**Impedance obtained through R/T-parameter retrieval implies surface averaged fields**

**Impedance obtained through averaging procedure implies volume averaged fields**

$$z = <E>/<H>$$

# RLC circuit description (1)




$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \mathbf{B}\cdot \mathrm{d}\mathbf{S}$$

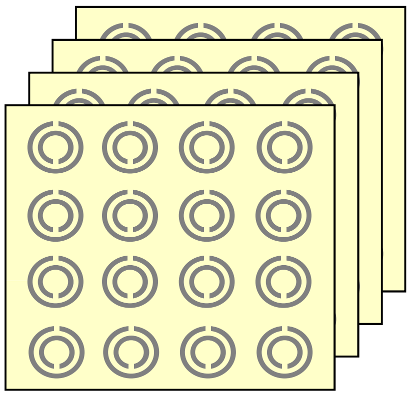
***Faraday law***

$$RI + \frac{1}{C}\int I\,\mathrm{d}t$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\iint \mathbf{B}_{\mathrm{inc}}\cdot \mathrm{d}\mathbf{S} - \frac{\mathrm{d}}{\mathrm{d}t}\iint \mathbf{B}_{\mathrm{circuit}}\cdot \mathrm{d}\mathbf{S}.$$

$$\mu_0 HA \qquad -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = -\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mu_0 H_0 A e^{-i\omega t}\right) \rightarrow I$$



SRR Medium

$$m = IA \qquad M = m / V_{uc} = (\mu / \mu_0 - 1)H$$

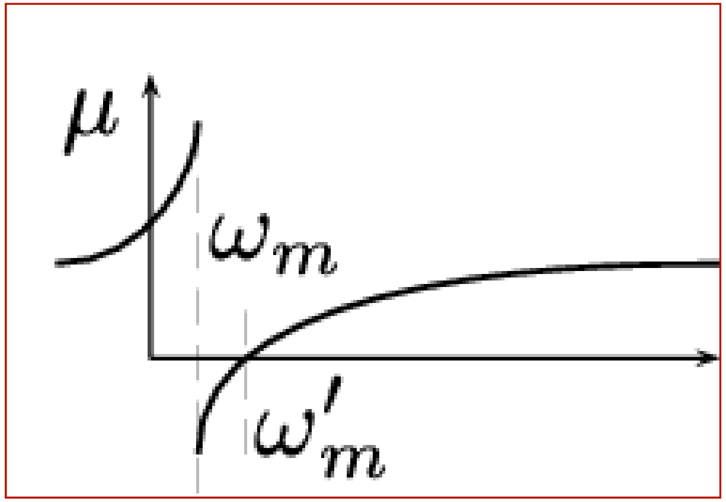
# RLC circuit description (2)

***F*** **~ volume fraction of the resonator within unit cell**

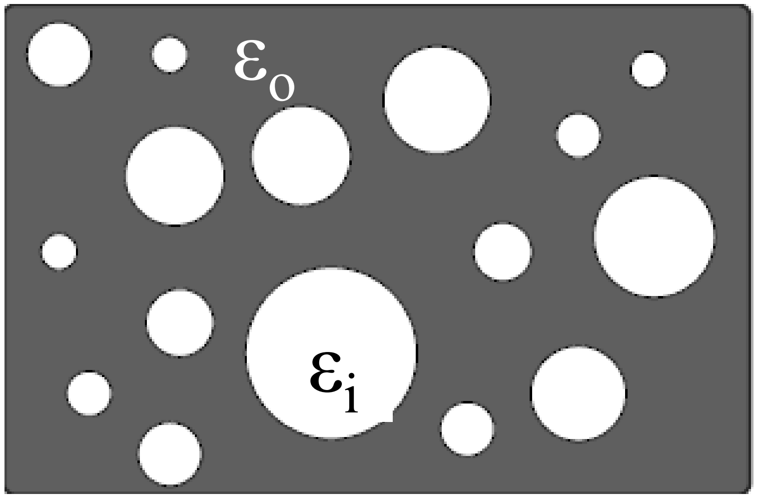
$$\mu(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2 + i\omega\gamma}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\gamma = \frac{R}{L}$$



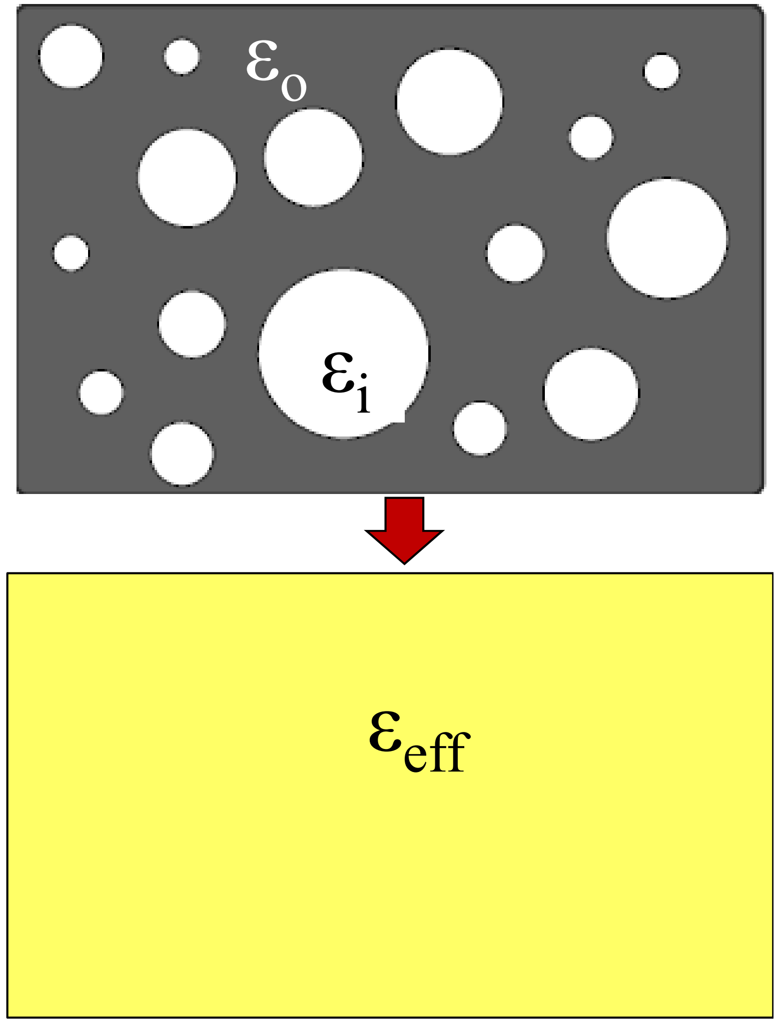
# Mixing rules




A. Sihvola, *Electromagnetic mixing formulas and applications, 1999*

# Quasi-static effective medium approaches: Mixing rules

**Give effective permittivity (or permeability) of a system of scatterers in the long wavelength limit**




**Valid for wavelengths $\lambda_{host} >> r$, $\lambda_{scat} >> r$**

**Offer**

- **Quick approach to assess metamaterial properties**
- **Path to tailor the metamaterial response**

**Most simple and widely used**

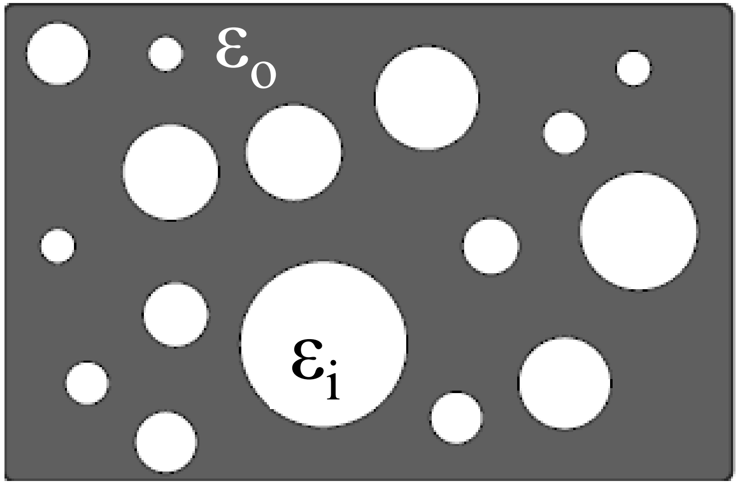
- **Maxwell-Garnett (or Claussious Mossoti)**
- **Bruggeman**

A. Sihvola, *Electromagnetic mixing formulas and applications, 1999*

# Maxwell-Garnett approach

$$\frac{\varepsilon_{eff}-\varepsilon_o}{\varepsilon_{eff}+2\varepsilon_o} = f\,\frac{\varepsilon_i-\varepsilon_o}{\varepsilon_i+2\varepsilon_o}$$

**Also Reyleigh formulation**




***f*: filling ratio, *ε*: dielectric function**

$$\frac{\varepsilon_{eff}-\varepsilon_o}{\varepsilon_{eff}+2\varepsilon_o} = \frac{na}{3\varepsilon_i} \qquad (1)$$

**C-M formulation**

**Suitable for non-symmetric composites**

$$\mathbf{p} = a\mathbf{E}_{loc} \rightarrow \mathbf{P} = \varepsilon_0(\varepsilon_{eff}-1)\mathbf{E}$$

**Uses polarizability (*a*) in the static limit & interaction among scatterers**

**Extended Maxwell-Garnett: Eq. (1) with *a* non-static polarizability**

**Predicts magnetic response from non-magnetic composites**

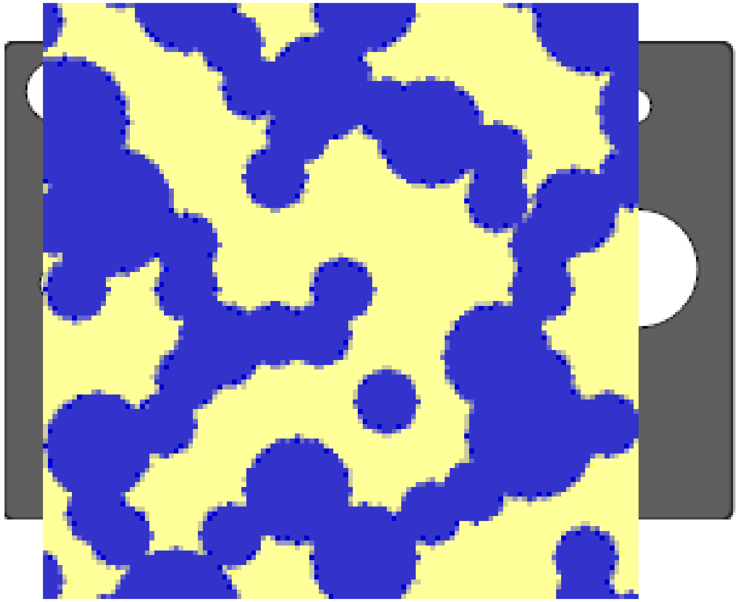
**Rupin, Tretyakov, Yannopapas, …**

# Other approaches

## Bruggeman

**Requires vanishing of averaged polarization of the actual medium relative to the effective medium**

$$f\frac{\varepsilon_i-\varepsilon_{eff}}{\varepsilon_i+\varepsilon_{eff}}+(1-f)\frac{\varepsilon_o-\varepsilon_{eff}}{\varepsilon_o+\varepsilon_{eff}}=0$$

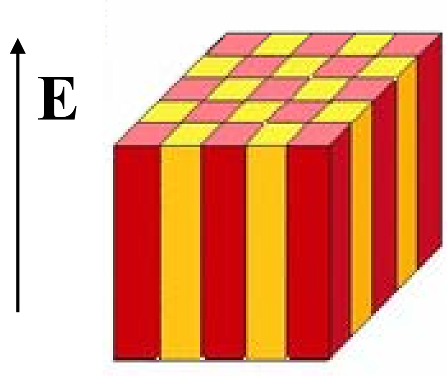


**Suitable for symmetric composites – (interconnected phases)**

## Averaging

$$\varepsilon_{eff}=f\varepsilon_i+(1-f)\varepsilon_o$$

**If the applied field is parallel to the interfaces**



# The end

*Thank you*