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On Some Almost-Periodicity Problems in Various Metrics

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Abstract. The Bohr-type and the Bochner-type definitions for almost periodic functions are examined in various metrics (Stepanov, Weyl and Besicovitch). The correct definitions of Besicovitch-like multifunctions are given. Weak almost-periodic solutions are proved for differential equations and inclusions. This problem is also discussed as a fixed-point problem in function spaces.

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1. Introduction

The theory of almost-periodic (a.p.) functions was initiated in the 20's through the pioneering work of the Danish mathematician Harald Bohr. Since that time, this theory and the theory of a.p. oscillations have made much progress (see the monographs [AP, Bo, Bes, C1, Fa, Fi, L, LZ, M, Kh, KBK, P, SY, Y]). The classical results and the central notions are mostly related to the names of H. Bohr, A. S. Besicovitch, S. Bochner, V. V. Stepanov, H. Weyl, and B. M. Levitan.

Since the very beginning, applications to differential equations have been of great interest, within the framework of the nonlinear oscillations theory, but there are much fewer results than for a closely related subclass of periodic oscillations. Especially, those for nonuniformly a.p. solutions are rather rare (see, e.g., [A1, A2, BG1, BG2, C3, H, ZL], and the references therein).

In [A1, A2], applications to differential inclusions are given for obtaining Weyl-like a.p. solutions, which are required to deal with a.p. multifunctions. These have

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been considered so far in the sense of H. Bohr (see [BVVL]), V. V. Stepanov (see [D1, D2, DS]) and H. Weyl (see [A1, A2]), but not in the sense of B. S. Besicovitch, whose concept has been recently investigated systematically in the single-valued case (see, e.g., [ABI, P]).

The aim of this paper is to examine the Bohr-type and the Bochner-type definitions of a.p. functions in various metrics, in order to clarify the hierarchy of a.p. function spaces. In the multivalued case, special attention is paid to the correct definitions of Besicovitch-like multifunctions. All of these are studied from the point of view of suitable applications to differential equations and inclusions. Since Weyl-like a.p. solutions have been already treated in [A1, A2], we restrict ourselves to the other classes. Thus, under a Stepanov a.p. forcing, a.p. solutions are studied nontraditionally as a fixed-point problem and Besicovitch-like a.p. solutions are deduced, under appropriate restrictions, from bounded solutions. In the concluding remarks, some open problems are posed.

2. Remarks on a.p. Functions in the Sense of H. Bohr and V. V. Stepanov

H. Bohr has defined *uniformly almost-periodic* (a.p.) *functions* $f \in C(\mathbb{R}, \mathbb{R}^n)$ as:

$$\forall \varepsilon > 0\ \exists k > 0\ \forall a \in \mathbb{R}\ \exists \tau \in [a, a+k] : \|f(t+\tau) - f(t)\| < \varepsilon.$$

It is well known (see, e.g., [L]) that the space of uniformly a.p. functions endowed with the norm $\|\cdot\| := \sup_{t\in\mathbb{R}} |\cdot|$ is Banach. The same is true (see, e.g., [AP, C1, L]) when this norm is replaced by the *Stepanov metric* (with an arbitrary l).

$$D_{S_l}^p(f, g) := \left[\sup_{b\in\mathbb{R}} \frac{1}{l} \int_b^{b+l} |f(t) - g(t)|^p \mathrm{d}t\right]^{1/p}, \quad p \geqslant 1.$$

Then one speaks about *Stepanov a.p.* (S-a.p.) *functions* $f \in L_{\mathrm{loc}}^p(\mathbb{R}, \mathbb{R}^n)$.

On the other hand, the completeness is lost, e.g., in the Weyl metric (see Section 4) or if the space of (continuous) uniformly a.p. functions is only endowed with the topology of the uniform convergence on compact subintervals of $\mathbb{R}$ (cf. [AK]). Nevertheless, as we will now see, a suitably chosen subset can become closed and subsequently complete because $C(\mathbb{R}, \mathbb{R}^n)$ is Fréchet in such a topology.

Defining, for a given uniformly a.p. function $f \in C(\mathbb{R}, \mathbb{R}^n)$, the sets

$$\begin{aligned}\Omega_f &:= \{(\varepsilon, k, a, \tau) \in \mathbb{R}^4 \mid \forall t \in \mathbb{R}\ \tau \in [a, a+k] \\ &\qquad \text{and } |f(t+\tau) - f(t)| < \varepsilon\}, \\ M_f &:= \{g \in C(\mathbb{R}, \mathbb{R}^n) \mid (\varepsilon, k, a, \tau) \in \Omega_f \Rightarrow \forall t \in \mathbb{R}\ |g(t+\tau) - g(t)| < \varepsilon\},\end{aligned}$$

we can state the following three lemmas, which will be employed in the next section.

LEMMA 1. *M_f is a convex set.*

Proof. Letting $u, v \in M_f, \lambda \in [0,1], (\varepsilon, k, a, \tau) \in \Omega_f$, we have, for each $t \in \mathbb{R}$:

$$\begin{aligned}&|(\lambda u + (1-\lambda)v)(t+\tau) - (\lambda u + (1-\lambda)v)(t)|\\&\quad\leqslant \lambda|u(t+\tau) - u(t)| + (1-\lambda)|v(t+\tau) - v(t)|\\&\quad\leqslant \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon.\end{aligned}$$ □

LEMMA 2. *M_f is a closed subset* (*in the topology of the uniform convergence on compact subintervals of* $\mathbb{R}$) *of* $C(\mathbb{R}, \mathbb{R}^n)$.

Proof. Let $M_f \ni g_l \stackrel{\text{loc}}{\to} g$ hold on $\mathbb{R}$, by which $g \in C(\mathbb{R}, \mathbb{R}^n)$. Assume that $(\varepsilon, k, a, \tau) \in \Omega_f$, $\forall t \in \mathbb{R}$. Then, for each $\delta > 0$, there exists l_0 such that, for all $l > l_0$, we have

$$|g(t+\tau) - g_l(t+\tau)| < \frac{\delta}{2} \quad \text{and} \quad |g(t) - g_l(t)| < \frac{\delta}{2}.$$

It follows from the inequality

$$\begin{aligned}&|g(t+\tau) - g(t)|\\&\quad\leqslant |g(t+\tau) - g_l(t+\tau)| + |g_l(t+\tau) - g_l(t)| + |g_l(t) - g(t)|\end{aligned}$$

that

$$l > l_0 : |g(t+\tau) - g(t)| < \delta + \varepsilon.$$

Since δ can be chosen arbitrarily small, we arrive at $|g(t+\tau) - g(t)| < \varepsilon$. □

LEMMA 3. *Let $\{g_l\}_{l=1}^{\infty}$ be a sequence in M_f such that* $\lim_{l\to\infty} g_l = g$. *Then we have*

$$\lim_{l\to\infty} \{|g_l(t) - g(t)| : t \in \mathbb{R}\} = 0.$$

Proof. We would like to prove that

$$\forall \varepsilon > 0\ \exists l_0 \in \mathbb{R}\ l > l_0\ \forall a \in \mathbb{R} : |g_l(a) - g(a)| < \varepsilon.$$

Fix $\varepsilon > 0$. Since $g \in M_f$, $g_l \in M_f$, for all $l \in \mathbb{N}$, there is a number $k > 0$ such that

$$\begin{aligned}&\forall a \in \mathbb{R}\ \exists \tau \in [a, a+k]\ \forall t \in \mathbb{R}\ \forall l \in N : |g(t+\tau) - g(t)| \leqslant \frac{\varepsilon}{4}\\&\qquad \text{and} \quad |g_l(t+\tau) - g_l(t)| \leqslant \frac{\varepsilon}{4}.\end{aligned}$$

Let $l_0 \in \mathbb{N}$ be a number such that, for all $l > l_0$, the following is true:

$$\forall t \in [-k, k] : |g(t) - g_l(t)| \leqslant \frac{\varepsilon}{4}.$$

Then, for all $a \in \mathbb{R}$, $l > l_0$, we have

$$\begin{aligned} |g(a) - g_l(a)| &\leqslant |g((a-\tau)+\tau) - g(a-\tau)| + |g(a-\tau) - g_l(a-\tau)| + \\ &\quad + |g_l(a-\tau) - g_l((a-\tau)+\tau)| \\ &\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

□

Now, let us turn to Stepanov a.p. functions with $\|f - g\|_{S^p} = D^p_{S_1}(f, g)$, i.e.

$$\|f\|_{S^p} := \left[\sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s)|^p \mathrm{d}s\right]^{1/p}, \quad p \geqslant 1,$$

in order to derive an analogical result to Lemma 2.

Defining the Banach space

$$BS^p := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid \|f\|_{S^p} < \infty\},$$

we have obviously $S^p \subset BS^p$, where S^p denotes the space of S^p-a.p. functions.

The *Bochner transform* (cf., e.g., [P])

$$f^b(t) = f(t + \eta), \quad \eta \in [0, 1], \ t \in \mathbb{R},$$

associates, to each $t \in \mathbb{R}$, a function defined on [0, 1].

Thus, if $f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, then $f^b \in L^p_{\mathrm{loc}}(\mathbb{R}, L^p([0, 1]))$.

Consequently,

$$BS^p(\mathbb{R}) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid f^b \in L^\infty(\mathbb{R}, L^p([0, 1]))\},$$

because $\|f\|^p_{S^p} = \|f^b\|_{L^\infty}$:

$$\|f^b\|_{L^\infty} = \sup_{t\in\mathbb{R}} \mathrm{ess}\, \|f^b\|_{L^p([0,1])} = \sup_{t\in\mathbb{R}} \mathrm{ess} \left[\int_0^1 |f(t+\eta)|^p \mathrm{d}\eta\right]^{1/p}.$$

Moreover, if

$$\forall f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) : f^b \in C(\mathbb{R}, L^p([0, 1])),$$

then

$$BS^p(\mathbb{R}) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid f^b \in \mathrm{BC}(\mathbb{R}, L^p([0, 1]))\},$$

where BC denotes the spaces of bounded continuous functions.

S. Bochner has shown (cf., e.g., [AP], pp. 76–78) that

$$S^p(\mathbb{R}) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid f^b \in C_{\mathrm{ap}}(\mathbb{R}, L^p([0, 1]))\},$$

where S^p and C_{ap} denote the Stepanov and the Bohr spaces of a.p. functions, respectively.

Obviously,

$$\|f\|_{S^p}^p = \|f^b\|_{\mathrm{BC}(\mathbb{R}, L^p([0,1]))},$$

and $f_n \to f$ in $S^p \Leftrightarrow f_n^b \to f^b$ in $\mathrm{BC}(\mathbb{R}, L^p([0, 1]))$.

Since $L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ is a Fréchet space ($S^p \subset BS^p \subset L^p_{\mathrm{loc}}$), endowed with the topology of the uniform convergence on compact subintervals of $\mathbb{R}$, namely

$$\|f\|_{S^p_{\mathrm{loc}}} := \left[\sup_{t\in K} \int_t^{t+1} |f(s)|^p \mathrm{d}s\right]^{1/p}, \quad K \subset \mathbb{R},\ K \text{ is compact},$$

we have that (cf. [P], p. 25)

$$f_n \to f \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \Leftrightarrow f_n^b \to f^b \text{ in } C(\mathbb{R}, L^p([0, 1])),$$

uniformly on compacts.

After all, we are in position to formulate an analogous statement to Lemma 2.

Hence, taking this time

$$\begin{aligned}\widetilde{\Omega}_f &:= \{(\varepsilon, k, a, \tau) \in \mathbb{R}^4 \mid \forall t \in \mathbb{R}\ \tau \in [a, a+k] \\ &\qquad \text{and } \|f^b(t+\tau) - f^b(t)\| < \varepsilon\}, \\ \widetilde{M}_f &:= \{g \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \mid (\varepsilon, k, a, \tau) \in \widetilde{\Omega}_f \\ &\qquad \Rightarrow \forall t \in \mathbb{R} \|g^b(t+\tau) - g^b(t)\| < \varepsilon\},\end{aligned}$$

and realizing that $g \in \widetilde{M}_f$ implies $g \in S^p(\mathbb{R})$, we can give

LEMMA 4. *$\widetilde{M}_f$ is a closed subset of $L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Since the proof of Lemma 2 can be repeated in terms of the Bochner transforms, in view of the above arguments, we are done. □

We conclude this part by recalling the definition of an a.p. multifunction in the sense of V. V. Stepanov (for $p = 1$) which will be used later.

An essentially bounded multifunction $\varphi\colon \mathbb{R} \to 2^{\mathbb{R}^n}\setminus\{\emptyset\}$ (briefly, $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}^n$) with nonempty closed values, which is measurable (i.e. if, for any open $U \subset \mathbb{R}$, the set $\{t \in \mathbb{R} \mid \varphi(t) \cap U \neq \emptyset\}$ is measurable), is called (cf. [A1]) *Stepanov a.p.* if, for every $\varepsilon > 0$, there exists a positive number $k = k(\varepsilon)$ such that, in each interval of the length k, there is at least one number τ satisfying

$$\sup_{a\in\mathbb{R}} \int_a^{a+1} d_H(\varphi(t), \varphi(t+\tau))\, \mathrm{d}t < \varepsilon,$$

where $d_H(\cdot, \cdot)$ stands for the Hausdorff metric.

3. A.P. Problem as a Fixed-Point Problem for Differential Equations

This rather unusual approach is based either on the Banach contraction principle or on the Schauder–Tikhonov fixed-point theorem. Similarly as for (e.g., periodic) boundary-value problems, a.p. solutions should correspond to fixed-points of the associated operators in suitable function spaces.

Consider the linear system

$$X' + AX = P(t), \tag{1}$$

where A is a constant $(n \times n)$-matrix, which is hyperbolic (i.e. it has no purely imaginary associated eigenvalues), and $P \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ is essentially bounded and S-a.p.

It follows from the Bohr–Neugebauer-type theorem (see, e.g., [C2, C3]) that (1) possesses, under the above assumptions, a unique entirely bounded solution $X(t)$ which is uniformly-a.p. This solution takes the form (see, e.g., [AK])

$$X(t) = \int_{-\infty}^{\infty} G(t-s)P(s)\,\mathrm{d}s \tag{2}$$

with

$$\begin{aligned} \sup_{t\in\mathbb{R}} |X(t)| &\leqslant \operatorname*{sup\,ess}_{t\in\mathbb{R}} |P(t)| \left(\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |G(t-s)|\,\mathrm{d}s \right) \\ &\leqslant C(A) \operatorname*{sup\,ess}_{t\in\mathbb{R}} |P(t)|, \end{aligned} \tag{3}$$

where $C(A)$ is a real constant depending only on A.

Let us still consider a one-parameter family of linear systems

$$X' + AX = f(q(t)) + p(t), \quad q \in Q, \tag{4}$$

where A is same as above, $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ has at most a linear growth, $p \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ is essentially bounded and S-a.p., and Q is a subset of S-a.p. functions.

Since the composed function $f(q(t))$ becomes, for every $q \in Q$, S-a.p. (see, e.g., corollary to Lemma 3 in [D2] and the references therein), system (4) possesses, under the above assumptions, a unique uniformly-a.p. solution of the form (2) with (3), where $P(t) := f(q(t)) + p(t)$.

Therefore, denoting

$$T(q) := \int_{-\infty}^{\infty} G(t-s)[f(q(s)) + p(s)]\,\mathrm{d}s, \quad q \in Q, \tag{5}$$

one can already discuss the possibility of applying the fixed-point theorem.

For the Banach principle, it is required that

(i) Q is complete,
(ii) $T\colon Q \to Q$ is contractible.

Hence, taking Q as the Banach space of uniformly-a.p. (= uniformly continuous S-a.p.) functions, only (ii) remains to be verified, i.e.

$$\|T(q_1) - T(q_2)\| \leqslant L_1 \|q_1 - q_2\|, \quad \text{for all } q_1, q_2 \in Q,$$

where $L_1 \in [0, 1)$ and $\|\cdot\| = \sup_{t\in\mathbb{R}} |\cdot|$.

Assuming, additionally, that f is Lipschitzian with a constant $L_2 > 0$, we obtain (cf. (3))

$$\begin{aligned}&\|T(q_1) - T(q_2)\|\\&\quad = \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{\infty} G(t-s)[f(q_1(s)) - f(q_2(s))]\,\mathrm{d}s\right|\\&\quad \leqslant L_2 \sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |G(t-s)||q_1(s) - q_2(s)|\,\mathrm{d}s \leqslant L_2 C(A)\|q_1(t) - q_2(t)\|.\end{aligned}$$

Obviously, for a sufficiently small L_2, concretely for $L_2 < 1/C(A)$, we get the desired contraction $L_1 := L_2 C(A) < 1$. □

The following statement is based on the application of the Schauder–Tikhonov fixed-point theorem (for a more general version, see [A2]).

PROPOSITION 1. *Consider the problem*

$$X' + AX = f(X) + p(t), \quad X \in Q, \tag{6}$$

where A, f, p *are as above, but* Q *is assumed, additionally, to be a* (*nonempty*), *closed, convex, and bounded subset of the Fréchet space* $C(\mathbb{R}, \mathbb{R}^n)$ *endowed with the topology of the uniform convergence on compact subintervals of* $\mathbb{R}$.

If the associated system (4), *for each* $q \in Q$, *has a unique solution* $X(t) \in Q$, *then problem* (6) *is solvable.*

It immediately follows from the above investigations that Proposition 1 applies in a desired manner, when

(i) Q with the above properties is also a subset of the Banach space of Stepanov (or, in particular, uniformly) a.p. functions,
(ii) $T(Q) \subset Q$, for T defined in (5).

Defining, for a given uniformly-a.p. function $\xi \in C(\mathbb{R}, \mathbb{R}^n)$, the sets ($\Delta \gg 1$)

$$\begin{aligned}\Omega_\xi &:= \left\{(\varepsilon, k, a, \tau) \in \mathbb{R}^4 \mid \forall t \in \mathbb{R}\tau \in [a, a+k] \text{ and } |\xi(t+\tau) - \xi(t)| < \frac{\varepsilon}{\Delta}\right\},\\ M_\xi &:= \{\varphi \in C(\mathbb{R}, \mathbb{R}^n) \mid (\varepsilon, k, a, \tau) \in \Omega_\xi \Rightarrow \forall t \in \mathbb{R}|\varphi(t+\tau) - \varphi(t)| < \varepsilon\},\end{aligned}$$

it is self-evident that every $\varphi \in M_\xi$ must be an a.p. function as well.

Since M_ξ has been already proved to be a convex and closed subset (in the given topology) of $C(\mathbb{R}, \mathbb{R}^n)$, the same is true for $M_\xi \cap Q_C$, where

$$Q_C := \left\{\varphi \in C(\mathbb{R}, \mathbb{R}^n) \mid \sup_{t\in\mathbb{R}} |\varphi(t)| \leqslant C\right\},$$

because, for every $\alpha \in [-1, 1]$, $\beta \in \mathbb{R}$, we have that $\alpha M_\xi + \beta \subset M_\xi$.

Therefore, taking this time $Q := M_\xi \cap Q_C$, it remains to verify (ii). More precisely, since the inclusion $T(Q) \subset Q_C$ can be easily obtained with a suitable C, provided the coefficient of at most linear growth of f is sufficiently small (see [A1, AK]), it is enough to show that $T(Q) \subset M_\xi$.

Hence, let $\xi(t)$ be a unique uniformly-a.p. solution of the equation

$$X' + AX = p(t)$$

with an almost-period τ_ξ (for a given ε/Δ, $\Delta \gg 1$). Assuming, additionally, the Lipschitzianity of f with a sufficiently small constant $L > 0$, we obtain for a unique uniformly-a.p. solution $X(t)$ of (4) that

$$\begin{aligned}
&|X(t+\tau_\xi) - X(t)| \\
&\quad = \left| \int_{-\infty}^{\infty} G(t-s)[f(q(s+\tau_\xi)) - f(q(s))]\,\mathrm{d}s + [\xi(t+\tau_\xi) - \xi(t)] \right| \\
&\quad \leqslant \frac{\varepsilon}{\Delta} + \int_{-\infty}^{\infty} |G(t-s)|[|f(q(s+\tau_\xi)) - f(q(s))|]\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{\Delta} + L\int_{-\infty}^{\infty} |G(t-s)|[|q(s+\tau_\xi) - q(s)|]\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{\Delta} + L\sup_{t\in\mathbb{R}} |q(t+\tau_\xi) - q(t)| \cdot \int_{-\infty}^{\infty} |G(t)|\,\mathrm{d}t \\
&\quad \leqslant \frac{\varepsilon}{\Delta} + LC(A)\sup_{t\in\mathbb{R}} |q(t+\tau_\xi) - q(t)| \leqslant \frac{\varepsilon}{\Delta} + \varepsilon LC(A).
\end{aligned}$$

If L is sufficiently small, concretely $L < 1/C(A)$, we arrive at

$$\sup_{t\in\mathbb{R}} |X(t+\tau_\xi) - X(t)| \leqslant \varepsilon\left(\frac{1}{\Delta} + LC(A)\right) < \varepsilon,$$

as required. □

Remark 1. The same can be reformulated in terms of the Bochner transforms, by means of Lemma 4.

Since a Lipschitz-continuous multifunction F with a sufficiently small Lipschitz constant possesses a Lipschitz-continuous selection $f \subset F$ with (not necessarily same, but) a sufficiently small Lipschitz constant (see [AC], p. 77) and an essentially bounded a.p. multifunction P with nonempty, closed values possesses an essentially bounded S-a.p. selection $p \subset P$ (see [DS]), we can summarize our investigations as follows.

THEOREM 1. *The system*

$$X' + AX \in F(X) + P(t), \tag{7}$$

where A is a constant hyperbolic $(n \times n)$-matrix, $F\colon \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Lipschitz-continuous (multivalued) function with a sufficiently small Lipschitz constant and $P\colon \mathbb{R} \rightsquigarrow \mathbb{R}^n$ is an essentially bounded Stepanov-a.p. (multivalued) function with nonempty closed values, admits a uniformly-a.p. solution.

Remark 2. In the single-valued case, results of this type have been obtained, with a uniformly-a.p. functions, by several authors (see, e.g., [AK] and the references therein), when applying more standard methods. Besides the existence, the uniqueness result is guaranteed under the same conditions (see [AK]).

Remark 3. If $A = (a_{ij})$ satisfies the Gershgorin-type inequalities (cf. condition (2) in [C3]),

$$\text{either } \Bigg(a_{ii} - \sum_{\binom{j=1}{j\neq i}}^{n} |a_{ij}| > 0 \text{ or } a_{ii} + \sum_{\binom{j=1}{j\neq i}}^{n} |a_{ij}| < 0 \text{ for } i = 1, \ldots, n\Bigg)$$
$$\text{or } \Bigg(a_{jj} - \sum_{\binom{i=1}{i\neq j}}^{n} |a_{ij}| > 0 \text{ or } a_{jj} + \sum_{\binom{i=1}{i\neq j}}^{n} |a_{ij}| < 0 \text{ for } j = 1, \ldots, n\Bigg),$$

by which the real parts of the associated eigenvalues of A are either all negative or positive, then (at least, in the single-valued case) the constant $C(A)$ in (3) can be expressed explicitly in terms of the entries a_{ij} (see [AK]). Moreover, $\mathbb{R}^n$ can be replaced everywhere by a real Banach space (cf. [C3]).

4. Remarks on A.P. Functions in the Sense of H. Weyl

We make the following conventions and notations in this section. All functions are considered on a real line $\mathbb{R}$ (with their usual natural structures, e.g., metrics) into some normed spaces, say $(E, \|\cdot\|)$. These functions will be denoted by f, g. In the family of (Bochner) locally integrable functions, we introduce two 'metrics':

$$D_{S_l}(f, g) := \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} \|f(t) - g(t)\| \,\mathrm{d}t \quad \text{(Stepanov)},$$
$$D_W(f, g) := \lim_{l\to\infty} D_{S_l}(f, g) \quad \text{(Weyl)},$$

for $f, g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$. We do not exclude the infinity ∞ as a value of D_{S_l}, D_W. The limit $D_W = \lim_{l\to\infty} D_{S_l}$ exists by the simple argumentation in [L], pp. 221–222, or in [Bes], pp. 72–73.

For our convenience, we write

$$D_{S_l}(f) = D_{S_l}(f, 0), \qquad D_W(f) = D_W(f, 0).$$

Hence,

$$D_{S_l}(f, g) = D_{S_l}(f - g), \qquad D_W(f, g) = D_W(f - g).$$

The distances D_{S_l} and D_W are the ‘usual’ (i.e. finite) metrics on the subspace

$$\{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E) \mid D_{S_l}(f) < \infty\} = \{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E) \mid D_W(f) < \infty\}$$

of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$ (cf. [BF], p. 37).

To a function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$, we associate the translation $f^h \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$, $f^h(t) = f(t + h) \forall t \in \mathbb{R}$. Thus, we have the *translation operator* $\mathbb{R} \ni h \mapsto f^h \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$, denoted by $\mathcal{T}_f$.

The Stepanov and Weyl metrics have some nice properties we shall need later.

PROPOSITION 2. *Assume that* $l, l_1, l_2 > 0$, $l_1 < l_2$, $h \in \mathbb{R}$ *and* $f, g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$.

(i) (*Equivalence*)

$$D_{S_{l_1}}(f, g) \leqslant \frac{l_2}{l_1} \cdot D_{S_{l_2}}(f, g),$$

$$D_{S_{l_2}}(f, g) \leqslant \left(1 + \frac{l_1}{l_2}\right) \cdot D_{S_{l_1}}(f, g).$$

(ii) (*Shift invariance*)

$$D_{S_l}(f^h, g^h) = D_{S_l}(f, g), \qquad D_W(f^h, g^h) = D_W(f, g).$$

Recall that a set $A \subset \mathbb{R}$ is *relatively dense* (*r.d.*) if there exists a number k (a constant of relative density) such that each (closed) interval I of length k contains some element of A (cf. [ZL], p. 7). Note that the closedness of intervals I can be replaced by their openness.

It is possible to give alternative definitions for the spaces of a.p. functions. The central notions are collected below.

DEFINITION 1. A function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$ is said

(i) *Stepanov almost periodic* (*S-a.p.*) if

$$\forall \varepsilon > 0 \text{ the set } \{\tau \in \mathbb{R} \mid D_{S_l}(f^\tau, f) < \varepsilon\} \text{ is r.d.},$$

(ii) *equi-Weyl almost periodic* (*equi-W-a.p.*) if

$$\forall \varepsilon > 0 \quad \exists l > 0 \text{ such that the set } \{\tau \in \mathbb{R} \mid D_{S_l}(f^\tau, f) < \varepsilon\} \text{ is r.d.},$$

(iii) *Weyl almost periodic* (*W-a.p.*) if

$$\forall \varepsilon > 0 \text{ the set } \{\tau \in \mathbb{R} \mid D_W(f^\tau, f) < \varepsilon\} \text{ is r.d.}$$

Let us add some remarks. The notions (i) and (ii) are studied, e.g., in [L, BF]. In fact, the equi-Weyl almost periodicity is there called ‘Weyl almost periodicity’. On the other hand, it seems that a similar concept to (iii) has been investigated for the first time by A. S. Kovanko in [K] (unfortunately, we have at our disposal only its French translation with no proofs). Recently, interest has been again growing in this matter (see, [A1, A2]).

In view of Proposition 2(i), Definition 1(i) does not depend on the choice of l. Definitions 1(i)–(iii) do also make sense without a finiteness assumption⋆ (i.e. $D_{S_l}(f) < \infty$, $D_W(f) < \infty$). However, it can be proved that S-a.p. functions as well as equi-W-a.p. functions must satisfy this condition (see [L], pp. 200–201, pp. 222–223).

The numbers τ in the above definitions are often called as D_{S_l}, *ε-almost periods* and D_W, *ε-almost periods*, respectively. In what follows, we restrict ourselves to (iii).

It can be easily seen that if $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$ is W-a.p., then the same is true for $f^h - f$, for any $h \in \mathbb{R}$. If, in addition, $D_W(f) < \infty$, then also

$$D_W(f^h - f) \leqslant D_W(f) + D_W(f^h) = 2D_W(f) < \infty.$$

Obviously, each essentially bounded function f satisfies $D_W(f) < \infty$.

Thus, we can prove the continuity of $\mathcal{T}_f$, in the metric D_W, provided

HYPOTHESIS. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$ with $D_W(f) < \infty$ be uniformly continuous in the mean, i.e.*

$$\forall \frac{\varepsilon}{3} > 0\ \exists \delta > 0\ \forall |h| < \delta : \frac{1}{l}\int_0^l \|f^h(t) - f(t)\|\,\mathrm{d}t < \frac{\varepsilon}{3},$$

uniformly w.r.t. $l \in (0, \infty)$.

The proof of the following lemma is quite analogous to the one for the equi-W.-a.p. class (cf. [L], pp. 223–224).

LEMMA 5. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$ be a W-a.p. function satisfying the Hypothesis. Then the translation operator $h \overset{\mathcal{T}_f}{\longmapsto} f^h$ is continuous in the Weyl metric D_W.*

Proof. Consider the (triangle) inequality

$$\begin{aligned} D_W(f^h, f) &\leqslant D_W(f^h, f^{h+\tau}) + D_W(f^{h+\tau}, f^\tau) + D_W(f^\tau, f) \\ &= 2D_W(f, f^\tau) + D_W(f^{h+\tau}, f^\tau). \end{aligned} \tag{8}$$

Fix $\varepsilon > 0$. Since f is W-a.p., we have

$$\forall a \in \mathbb{R} \quad \exists \tau \in [-a, -a+k] : D_W(f, f^\tau) < \varepsilon/3, \tag{9}$$

where k is a constant of r.d. to the set $\{\tau \mid D_W(f, f^\tau) < \varepsilon/3\}$.

⋆ Finiteness in the Weyl and the Stepanov metrics is the same (see [BF], p. 37).

If $\tau \in [-a, -a+k]$, $a \leqslant t \leqslant a+l$, then

$$0 \leqslant a+\tau \leqslant t+\tau \leqslant a+\tau+l \leqslant k+l.$$

Hence,

$$\frac{1}{l}\int_a^{a+l} \|f^{h+\tau}(t) - f^{\tau}(t)\| \,\mathrm{d}t \leqslant \frac{1}{l}\int_0^{k+l} \|f^h(t) - f(t)\| \,\mathrm{d}t \quad \forall a \in \mathbb{R},$$

and subsequently

$$\sup_{a\in\mathbb{R}} \frac{1}{l}\int_a^{a+l} \|f^{h+\tau}(t) - f^{\tau}(t)\| \,\mathrm{d}t \leqslant \frac{1}{l}\int_0^{k+l} \|f^h(t) - f(t)\| \,\mathrm{d}t.$$

This inequality gives us

$$\begin{aligned} &D_W(f^{h+\tau}, f^{\tau}) \\ &\quad = \limsup_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l}\int_a^{a+l} \|f^{h+\tau}(t) - f^{\tau}(t)\| \,\mathrm{d}t \\ &\quad \leqslant \limsup_{l\to\infty} \frac{1}{l}\int_0^{k+l} \|f^h(t) - f(t)\| \,\mathrm{d}t. \end{aligned} \tag{10}$$

Moreover (by the hypothesis),

$$\exists \delta > 0\ \forall |h| < \delta : \limsup_{l\to\infty} \frac{1}{l}\int_0^{k+l} \|f^h(t) - f(t)\| \,\mathrm{d}t \leqslant \frac{\varepsilon}{3}. \tag{11}$$

After all (cf. (8)–(11)), we obtain that

$$\exists \delta > 0 \forall\ |h| < \delta : D_W(f^h, f) < 2\cdot\varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which completes the proof. □

Now, we would like to examine the Bochner-type criterion for W-a.p. functions. A similar result to the theorem we are going to present, has been developed by A. S. Kovanko [K], in 1944. However, as we have already pointed out, since we have not to our disposal the original proofs, we rediscover in fact this theorem characterizing W-a.p. functions in a Bochner manner, i.e. in terms of precompactness of a family of translates w.r.t. the Weyl metric.

Usually, the Bochner criterion for S-a.p. and uniformly (i.e. Bohr) a.p. functions is expressed in terms of compactness. But since the spaces of S-a.p. and uniformly a.p. functions are complete, the precompactness is equivalent to compactness (see [L], pp. 23–27, 199–200, 216–220, [ZL], pp. 10–11, 38, [BF], pp. 51–53). Surprisingly, the space of equi-W-a.p. functions is incomplete with respect to D_W (see [L], pp. 242–247, [BF], pp. 58–61).

The space of equi-W-a.p. functions is D_W-closed subspace of D_W-finite functions and the latter one is also incomplete (see [BF], p. 39, 58–61). Consequently, the between-lying space of D_W-finite W-a.p. functions,

$$\{f \mid f \text{ is equi-W-a.p.}\} \\ \subset \{f \mid f \text{ is W-a.p.}, D_W(f) < \infty\} \subset \{f \mid D_W(f) < \infty\},$$

is also D_W-incomplete. Indeed. The set $\{f \mid f \text{ is equi-W-a.p.}\}$ is D_W-closed, so it is D_W-closed as well w.r.t. the subspace $\{f \mid f \text{is W-a.p.}, D_W(f) < \infty\}$. Therefore, the space of D_W-finite W-a.p. functions cannot be complete.

Omitting the problems about completeness of spaces of a.p. functions (especially, W-a.p. functions), we formulate our Bochner type criterion just in terms of precompactness (which is certainly a weaker condition than the strict compactness). The proof is performed by ε-nets (for more details, see [L]), because the precompactness is nothing else but the complete boundedness (as the classical Hausdorff theorem states).

THEOREM 2. *A function $f \in L^1_{loc}(\mathbb{R}, E)$, satisfying the Hypothesis, is W-a.p. if and only if the family $\{f^h \mid h \in \mathbb{R}\}$ of shifts is D_W-precompact* (*i.e. precompact w.r.t. the Weyl metric*).

Proof. 'If' part. Fix $\varepsilon > 0$. Since $\{f^h \mid h \in \mathbb{R}\}$ is D_W-precompact,

$$\exists f^{h_1}, \dots, f^{h_n} \ \forall h \in \mathbb{R} \\ \exists j = 1, \dots, n : D_W(f^{h-h_j}, f) = D_W(f^{h_j}, f^h) < \varepsilon. \tag{12}$$

Thus, the numbers $\tau = h - h_j$ are D_W, ε-almost periods.

We show r.d. for

$$\{\tau \mid \tau = h - h_j, h \in \mathbb{R}, j = 1, \dots, n; D_W(f^{h_j}, f^h) < \varepsilon\}. \tag{13}$$

Take

$$k = \max_{j=1,\dots,n} |h_j| \tag{14}$$

and let $a \in \mathbb{R}$ be arbitrary. If $h = a + k$ and h_j satisfy (12), we obtain, in view of (14), that $h - h_j \in [a, a + 2k]$. Each interval of length $2k$ contains a D_W, ε-almost period of f. The number $2k$ is so a constant of r.d. to the set (13).

'Only if' part. Assume that f is W-a.p. and fix $\varepsilon > 0$. Because of a D_W-continuity of the translation operator $hr \mapsto f^h$ (see Lemma 5), we have

$$\exists \delta > 0 \ \forall |w| < \delta : D_W(f, f^w) < \varepsilon/2. \tag{15}$$

Let k be a constant of r.d. to the set $\{\tau \mid D_W(f, f^w) < \varepsilon/2\}$, i.e.

$$\forall I \text{ – an interval of length } k \ \exists \tau \in I : D_W(f^\tau, f) < \varepsilon/2. \tag{16}$$

To these k and δ, we associate a positive integer n such that

$$n\delta \leqslant k < (n+1)\delta \tag{17}$$

and put $h_j = j \cdot \delta$ ($j = 1, \dots, n$).

Take any $h \in \mathbb{R}$. In the interval $[-h, -h+k]$ of length k, we find by means of (16) some D_W, $\varepsilon/2$-almost period τ, i.e.

$$D_W(f^\tau, f) < \varepsilon/2. \tag{18}$$

Furthermore, we match h and τ with $j = 1, \dots, n$ in such a way that

$$|h + \tau - h_j| < \delta. \tag{19}$$

It is possible because of (17) and the fact that $\tau \in [-h, -h+k]$.

By means of (15), (18) and (19), we can arrive at

$$\begin{aligned} D_W(f^h, f^{h_j}) &\leqslant D_W(f^h, f^{h+\tau}) + D_W(f^{h+\tau}, f^{h_j}) \\ &= D_W(f, f^\tau) + D_W(f^{h+\tau-h_j}, f) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This shows that $\{f^{h_j} \mid j = 1, \dots, n\}$ is a finite ε-net to $\{f^h \mid h \in \mathbb{R}\}$, in the sense of the Weyl metric D_W, as required. □

Let us note that the above proof is based partly on [L], pp. 219–220, and [S].

It is well known (see, e.g., [BF]) that

$$\text{S-a.p.} \subsetneq \text{equi-W-a.p.}$$

and the inclusion is strict.

The following example shows that the class W-a.p. is more general than the one of equi-W-a.p. functions.

EXAMPLE 1. Consider the step function $f\colon \mathbb{R} \to \mathbb{R}$:

$$f(t) = \begin{cases} 0, & \text{whenever } t \leqslant 0, \\ 1, & \text{whenever } t > 0. \end{cases}$$

Obviously, $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$.

A relative density of the set $\{\tau \mid D_{S_l}(f, f^\tau) < \varepsilon\}$, for some l, requires arbitrary large values of τ's in this set (we can extract some sequences of $\tau_n \to \infty$ with $n \to \infty$). If we demand that l is (perhaps large but) constant for all τ's (we fix ε), then for most of them we get $\tau > l$, and subsequently

$$D_{S_l}(f^\tau, f) = \frac{1}{l} \cdot l = 1 = \text{const}.$$

So, $D_{S_l}(f^\tau, f) < \varepsilon$ is impossible (for all τ's, simultaneously), by which f is not equi-W-a.p.

On the other hand, we have

$$D_W(f^\tau, f) = \lim_{l\to\infty} \frac{1}{l} \cdot \tau = 0$$

(we can assume that $\tau < l$, since $l \to \infty$), by which f is W-a.p.

Therefore, equi-W-a.p. $\subseteq$ W-a.p. and we can conclude by extending our hierarchy, namely

S-a.p. $\subsetneq$ equi-W-a.p. $\subsetneq$ W-a.p.,

where the above inclusions are strict.

Instead of a normed space E in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$, we can consider any metric space (X, d). Although we do not have at our disposal the theory of Bochner integrals, it is still reasonable to define a class of locally integrable functions with values in X. One can introduce

$$\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, X) = \left\{ f\colon \mathbb{R} \to X \mid \forall a < b : \int_a^b d(f(t), x_0)\,\mathrm{d}t < \infty \right\}$$

with fixing some point $x_0 \in X$. Observe that this definition is independent of the choice of x_0. In fact, we only need to integrate real-valued nonnegative functions (the distances, in fact). Such an approach permits us to investigate multifunctions (cf. [A1, A2]).

Those in the sense of V. V. Stepanov and H. Weyl have been already treated in [A1, A2, D1, D2, DS]. Therefore, we will restrict ourselves to a.p. multifunctions in the sense of A. S. Besicovitch.

5. Remarks on Besicovitch-Like a.p. Multifunctions

Unlike in the single-valued case, a.p. multifunctions have been studied rarely (see [A1, A2, BVVL, D1, D2, DS] and the references therein). Roughly speaking, the vector norms have been replaced by the Hausdorff metrics in the related definitions for measurable multifunctions. It seems that Bohr-like a.p. multifunctions have been investigated for the first time in [BVVL] w.r.t. the existence of a (continuous) Bohr a.p. selection. It is rather surprising that this problem was answered there negatively, in general.

On the other hand, for the Stepanov-like a.p. multifunctions (measurable) Stepanov a.p. selections do exist (see [DS, D1, D2]). A natural question therefore arises (posed already in [A1, A2]), whether or not measurable Weyl a.p. selections exist for (equi-) Weyl-like a.p. multifunctions. Since this question remains open, we considered in [A1, A2] a special subclass of selectional Weyl-like a.p. multifunctions for applications to differential inclusions. In fact, selectional Weyl-like a.p. multifunctions take the particular form of a Stepanov-like a.p. multifunction plus a (single-valued) Weyl a.p. function, which are Weyl-like selectionable because of the Stepanov-like selectionability of a multi-part mentioned above.

Perhaps the most delicate situation appears for measurable multifunctions in the Besicovitch case. As far as we know, no appropriate definition has yet been given. Hence, our next remarks will be devoted to this problem. In the single-valued case, there are several definitions of Besicovitch-like a.p. functions (cf. [ABI, BG1, Bes, P]). For our convenience, first we recall here those in [Bes] which are transparently sketched in Appendix 4 of [Ber]. Namely,

$$C_{B^p}\{\text{u.a.p.}\} = C_{B^p}(A) = \{B^p\text{-a.p.}\}, \quad p \geqslant 1,$$

where

$$C_{B^p}\{\text{u.a.p.}\} := \Bigg\{\text{The closure, under the norm}$$

$$\|f\|^p_{B^p} := \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^p \,\mathrm{d}t, \tag{20}$$

$$\text{of the set of all uniformly (Bohr-like) a.p. functions}\Bigg\},$$

$C_{B^p}(A)$:= {The closure, under the norm (20), of the set A of all trigonometric polynomials of the form $\sum a_j \mathrm{e}^{i\lambda_j t}$, where the coefficients a_j as well as the exponents λ_j are real numbers and the sum contains only a finite number of terms},

$\{B^p\text{-a.p.}\} := \big\{f(t) \in L^p_{\mathrm{loc}}(\mathbb{R}),\ \forall\varepsilon > 0 \exists$ a 'satisfactorily uniform' set of numbers

$$\ldots \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 \ldots,$$

i.e. such that

(i) there is $l > 0$ with

$$2 > \frac{\max_{t\in(-\infty,\infty)}(\text{the number of terms in } \{\{\tau_0, \tau_{\pm1}, \tau_{\pm2}, \ldots\} \cap [t, t+l]\})}{\min_{t\in(-\infty,\infty)}(\text{the number of terms in } \{\{\tau_0, \tau_{\pm1}, \tau_{\pm2}, \ldots\} \cap [t, t+l]\})}$$

(ii) for each j,

$$\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t+\tau_j) - f(t)|^p \,\mathrm{d}t < \varepsilon^p,$$

(iii) for every $C > 0$,

$$\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left\{\limsup_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^{n} \frac{1}{C}\int_{t}^{t+C} |f(s+\tau_j) - f(s)|^p \,\mathrm{d}s\right\} \mathrm{d}t < \varepsilon^p\Big\}.$$

For brevity, all these classes will be denoted, as usual, by B^p_{ap}, i.e.

$$B^p_{\mathrm{ap}} := C_{B^p}\{\text{u.a.p.}\} = C_{B^p}(A) = \{B^p\text{-a.p.}\}, \qquad p \geqslant 1.$$

The reason why the definition of $\{B^p\text{-a.p.}\}$-spaces takes this complicated form is strictly related to the proof of the above equivalence (see [Bes], pp. 91–104).

Furthermore, it is known (see [BG1]) that every $f \in B^p_{\text{ap}}$, $p \geqslant 1$, satisfying

$$\forall \varepsilon > 0 : \left\{ \tau \in \mathbb{R} \lim_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{T} |f(t+\tau) - f(t)|^p \, \mathrm{d}t \right]^{1/p} < \varepsilon \right\}$$

is relatively dense. Thus, B^p_{ap} is involved in the set of all $f \in L^p_{\text{loc}}(\mathbb{R})$ with the above property. Moreover, the last definition seems to be the most convenient for applications to differential equations, within the framework of the theory of nonlinear oscillations. However, we do not know whether $f \in L^p_{\text{loc}}(\mathbb{R})$, whose set of shifts $\{f^h \mid h \in \mathbb{R}\}$, where $f^h(t) = f(t+h)\ \forall t \in \mathbb{R}$, is precompact in the above metric, satisfies the above property, or even $f \in B^p_{\text{ap}}$.

Hence, taking into account the class $C_{B^p}(A)$, for $p = 1$, we can introduce the definition of Besicovitch-like a.p. measurable multifunctions as follows:

DEFINITION 2. An essentially bounded measurable multifunction φ: $\mathbb{R} \rightsquigarrow \mathbb{R}^n$ (i.e. φ: $\mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$) with nonempty closed values is said to be *a.p. in the sense of Besicovitch* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite trigonometric polynomials with real coefficients such that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d_H(\varphi(t), P_n(t)) \, \mathrm{d}t = 0,$$

where $d_H(\cdot,\cdot)$ stands for the Hausdorff metric (cf., e.g., [A1, A2]).

Let us also recall that φ is *measurable* if, for any open $U \subset \mathbb{R}$, the set $\{t \in \mathbb{R} \mid \varphi(t) \cap U \neq \emptyset\}$ is measurable.

Since any measurable multifunction φ with nonempty closed values is well known (cf. [A1, A2, DS]) to be representable in the sense of Castaing, namely

$$\varphi(t) = \overline{\bigcup_{n\in\mathbb{N}} \varphi_n(t)},$$

where $\{\varphi_n(t)\}_{n=1}^{\infty}$ is a suitable sequence of its measurable selections, Definitions 2 is correct. It is really so because the problem of measurability of $d_H(\varphi(t), P_n(t))$ then turns out to be equivalent to the one for $d_H(\overline{\bigcup_{n\in\mathbb{N}} \varphi_n(t)}, P_n(t))$, which is obviously measurable by standards single-valued arguments (cf. [A1, A2]). The local convergence of the Lebesgue integral in Definition 2 follows by means of the well-known Lebesgue-dominated theorem.

One can easily extend Definition 2, analogously as in the single-valued case (cf. [ABI]), to essentially bounded measurable vector multifunctions φ: $\mathbb{R} \rightsquigarrow \mathbb{R}^n$. Nevertheless, in view of mentioned applications (nonlinear oscillations for differential inclusions), we still prefer to give the following definition when involving the notion of almost-periods:

DEFINITION 3. An essentially bounded measurable multifunction $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}^n$ with nonempty closed values is called *a.p. in the generalized sense of Besicovitch*, if for every $\varepsilon > 0$, there exists a positive number $k = k(\varepsilon)$ such that, in each interval of length k, there is at least one number τ satisfying

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} d_H(\varphi(t+\tau), \varphi(t))\,\mathrm{d}t < \varepsilon.$$

By the same reasons as above, Definition 3 is correct. Of course, one can also correctly extend the definitions of the classes C_{B^1} {u.a.p.}, or {B^1-a.p.} to the vector multivalued case, when just replacing the vector norms by the Hausdorff metric.

Similarly as for (equi-) Weyl-like a.p. multifunctions, a natural question arises whether or not a measurable selection $f \in B^1_{\mathrm{ap}}$ exists to an a.p. multifunction φ, in the sense of Definition 3.

6. Besicovitch-Like a.p. Solutions to Differential Systems

In this section, we establish the sufficient conditions for the existence of a.p. solutions in the generalized sense of Besicovitch (as above) to the Carathéodory systems

$$X' + AX = f(t, X), \quad \text{where } f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n, \tag{21}$$

whose solutions $X(t)$ are understood in the sense of Carathéodory, namely $X(t) \in AC_{\mathrm{loc}}(\mathbb{R})$, satisfying (21) a.e.

DEFINITION 4. A Carathéodory function $f(t, X) \in \mathbb{R}^n$ (i.e. measurable in t, for all $X \in \mathbb{R}^n$, and continuous in X, for a.a. $t \in \mathbb{R}$) is called *a.p. in t in the generalized sense of Besicovitch, uniformly w.r.t.* $X \in \mathbb{R}^n$, if for every $\varepsilon > 0$ and every $D > 0$ there exists a positive number $k = k(\varepsilon, D)$ such that, in each interval of the length k, there is at least one number τ satisfying

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t+\tau, X) - f(t, X)|\,\mathrm{d}t < \varepsilon, \tag{22}$$

for $|X| \leqslant D$.

THEOREM 3. *Let the following assumptions be satisfied:*

(i) *a constant $(n \times n)$-matrix A is hyperbolic, i.e. all the associated eigenvalues have nonzero real parts*;
(ii) *a Carathéodory function $f(t, X)\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is essentially bounded in t and Lipschitz-continuous in X, for a.a. $t \in \mathbb{R}$, with a constant $L < \lambda/2k$, where the constants λ and k are specified below*;
(iii) *$f(t, X)$ is a.p. in t in the generalized sense of Besicovitch, uniformly w.r.t. $X \in R^n$ (see Definition* 4*).*

Then Equation (21) *admits an a.p. solution in the generalized sense of Besicovitch.*

Proof. It follows from the investigations in [A2] that conditions (i), (ii) represent the assumptions implying the existence of an entirely bounded solution $X(t)$ of (21). This solution takes the form

$$X(t) = \int_{-\infty}^{\infty} G(t,s) f(s, X(s))\,\mathrm{d}s, \tag{23}$$

where the Green function $G(t,s)$ satisfies

$$|G(t,s)| \leqslant k\exp(-\lambda(t-s)) \quad \text{for } t \geqslant s,$$
$$|G(t,s)| \leqslant k\exp(-\lambda(s-t)) \quad \text{for } t \leqslant s.$$

By the hypothesis, for every $\varepsilon > 0$ and every $D > 0$, there exists a positive number $k = k(\varepsilon, D)$ such that, in each interval of the length k, there is at least one number τ satisfying (22), for $|x| \leqslant D$. Since for $X(t)$ in (23), we also have

$$\begin{aligned} X(t+\tau) &= \int_{-\infty}^{\infty} G(t+\tau, s) f(s, X(s))\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} G(t,s) f(s+\tau, X(s+\tau))\,\mathrm{d}s, \end{aligned}$$

one can deal with $\int_{-T}^{T} |X(t+\tau) - X(t)|\,\mathrm{d}t$, to prove the almost-periodicity of $X(t)$ in the Besicovitch metric used in (22). Hence, applying (after several steps) the well-known Fubini theorem and using the norm $\|\cdot\| = \sup_{t\in\mathbb{R}} |\cdot|$ and compatible vector and matrix norms, we successively obtain

$$\begin{aligned} &\int_{-T}^{T} |X(t+\tau) - X(t)|\,\mathrm{d}t \\ &\quad = \int_{-T}^{T} \left| \int_{-\infty}^{\infty} G(t,s)[f(s+\tau, x(s+\tau)) - f(s, X(s))]\,\mathrm{d}s \right| \mathrm{d}t \\ &\quad \leqslant \int_{-T}^{T} \mathrm{d}t \int_{-\infty}^{\infty} |G(t,s)||f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \\ &\quad \leqslant \int_{-T}^{T} \mathrm{d}t \bigg[\int_{-\infty}^{t} k\,\mathrm{e}^{-\lambda(t-s)} |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s + \\ &\qquad + \int_{t}^{\infty} k\,\mathrm{e}^{\lambda(t-s)} |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \bigg] \\ &\quad = k \int_{-\infty}^{-T} |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \int_{-T}^{T} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}t + \\ &\qquad + k \int_{-T}^{T} |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \int_{s}^{T} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}t + \\ &\qquad + k \int_{-T}^{T} |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \int_{-T}^{s} \mathrm{e}^{\lambda(t-s)}\,\mathrm{d}t + \end{aligned}$$

$$
\begin{aligned}
&+k\int_T^\infty |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \int_{-T}^T \mathrm{e}^{\lambda(t-s)}\,\mathrm{d}t \\
&= \frac{k}{\lambda}\big(\mathrm{e}^{\lambda T} - \mathrm{e}^{-\lambda T}\big)\int_{-\infty}^T \mathrm{e}^{\lambda s}|f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s + \\
&+\frac{k}{\lambda}\int_{-T}^T (1-\mathrm{e}^{-\lambda(T-s)})|f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s + \\
&+\frac{k}{\lambda}\int_{-T}^T (1-\mathrm{e}^{-\lambda(T+s)})|f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s + \\
&+\frac{k}{\lambda}\big(\mathrm{e}^{\lambda T} - \mathrm{e}^{-\lambda T}\big)\int_T^\infty \mathrm{e}^{-\lambda s}|f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s \\
&\leqslant 2\frac{k}{\lambda}\int_{-T}^T |f(s+\tau, X(s+\tau)) - f(s, X(s))|\,\mathrm{d}s + \\
&+2\frac{k}{\lambda^2}\|f(t+\tau, X(t+\tau)) - f(t, X(t))\|.
\end{aligned}
$$

Since $X(t)$ is bounded and subsequently $\|f(t, X(t))\| < \infty$, the last term vanishes in the Besicovitch metric used in (22). Thus, furthermore, we get by means of (22) and the Lipschitz property, that

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^T |X(t+\tau) - X(t)|\,\mathrm{d}t\right] \\
&\quad\leqslant 2\frac{k}{\lambda}\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^T |f(t+\tau, X(t+\tau)) - f(t, X(t))|\,\mathrm{d}t\right] \\
&\quad\leqslant 2\frac{k}{\lambda}\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^T |f(t+\tau, X(t+\tau)) - f(t+\tau, X(t))| + \right. \\
&\qquad\qquad \left. + |f(t+\tau, X(t)) - f(t, X(t))|\,\mathrm{d}t\right] \\
&\quad< 2\varepsilon\frac{k}{\lambda} + 2L\frac{k}{\lambda}\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^T |X(t+\tau) - X(t)|\,\mathrm{d}t\right],
\end{aligned}
$$

where L is sufficiently small.

After all, we arrive at

$$
\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^T |X(t+\tau) - X(t)|\,\mathrm{d}t\right] < \frac{2\varepsilon k}{\lambda - 2Lk},
$$

as far as $L < \lambda/2k$, which already verifies the desired almost-periodicity of $X(t)$. So, the proof is complete. □

It is also known (cf. [A1] and the references therein) that a multivalued function $[F(X) + P(t) + p(t)]$, where F: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is Lipschitz-continuous (in the Hausdorff metric), $P(t)$: $\mathbb{R} \rightsquigarrow \mathbb{R}^n$ is essentially bounded and Stepanov-like a.p. (cf. [A1]), and $p(t)$: $\mathbb{R} \rightsquigarrow \mathbb{R}^n$ is a single-valued essentially bounded a.p. function in a generalized sense of Besicovitch, possesses a Carathéodory selection $f(t, X) \subset [F(X) + P(t) + p(t)]$ which is essentially bounded and a.p. in t, in the generalized sense of Besicovitch, uniformly w.r.t. $X \in \mathbb{R}^n$ (cf. Definition 4), and Lipschitzian in X, for a.a. $t \in \mathbb{R}$. Moreover, if a Lipschitz constant to F is sufficiently small, then so is (but not necessarily the same) the one to f (see [AC], p. 77).

Thus, we can immediately give the corollary to Theorem 3 for the Carathéodory differential inclusion

$$X' + AX \in F(X) + P(t) + p(t), \tag{24}$$

where the right-hand side is as above.

COROLLARY 1. *Let A be a (single-valued) hyperbolic $(n \times n)$-matrix and $[F(X) + P(t) + p(t)]$ be a Carathéodory function, as above, with a sufficiently small Lipschitz constant to F. Then inclusion* (24) *admits an a.p. solution in the generalized sense of Besicovitch.*

Remark 4. An analogous result has been obtained for Weyl-like a.p. solutions in [A2] (cf. [A1]).

7. Concluding Remarks (Open Problems)

As one can see, there are still many related open problems which can be sketched in the following way:

- Can the Hypothesis be verified for essentially bounded on $\mathbb{R}$ functions $L^1_{loc}(\mathbb{R}, E)$?
- Can the Weyl D_W-metric be replaced by the Besicovitch one (20) in Theorem 2?
- Do there exist measurable Weyl or Besicovitch a.p. selections of Weyl or Besicovitch a.p. multifunctions, respectively, in any sense of related definitions?
- Can an a.p. problem for Weyl or Besicovitch a.p. solutions of differential equations be considered as a fixed-point problem, again in any sense of related definitions?
- Can a Lipschitzian restriction imposed on the right-hand sides of differential equations be avoided, for any sort of a.p. solutions?

Some of these problems will be tackled by ourselves elsewhere.

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