COMPITO B

B1

1) $\left(1-\frac{1}{i^3}\right)^8 = \left(1+\frac{1}{i}\right)^8 = (1-i)^8$

$= \left[\sqrt{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]\right]^8 = 16\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]^8$

$= 16\left[\cos(-2\pi) + i\sin(-2\pi)\right] = 16e^{-i2\pi} = 16.$

2) Equazione definita in tutto $\mathbb{R}$.

$$y(x) = e^{-\int\frac{x}{x^2+4}dx}\left[\int e^{\int\frac{x}{x^2+4}dx}\frac{1}{\sqrt{x^2+4}}dx + C\right]$$

$$= e^{-\frac{1}{2}\ln(x^2+4)}\left[\int e^{\frac{1}{2}\ln(x^2+4)}\frac{1}{\sqrt{x^2+4}}dx + C\right]$$

$$= \frac{1}{\sqrt{x^2+4}}\left[\int\frac{\sqrt{x^2+4}}{\sqrt{x^2+4}}dx + C\right] = \frac{x+C}{\sqrt{x^2+4}}$$

Poichè $\lim_{x\to-\infty}\frac{x+C}{\sqrt{x^2+4}} = \lim_{x\to-\infty}\frac{x}{\sqrt{x^2}} =$

$= \lim_{x\to-\infty}\frac{x}{|x|} = -1,$

~~non~~ nessuna soluzione particolare soddisfa la condizione al limite proposta.

(B₂)

3) $$a_n = \frac{1}{n}\left[\cancel{\frac{1}{\sqrt{n+1}}} + \frac{1}{2(n+1)} + o\left(\frac{1}{n+1}\right) - \cancel{\frac{1}{\sqrt{n+1}}} + \frac{1}{6(n+1)^{3/2}} + o\left(\frac{1}{(n+1)^{3/2}}\right)\right] \operatorname{arctg}(n^2)$$

$$= \frac{1}{n}\left[\frac{1}{2(n+1)} + o\left(\frac{1}{n+1}\right)\right] \operatorname{arctg}(n^2)$$

$$\sim \frac{1}{2n(n+1)} \cdot \frac{\pi}{2} \sim \frac{\pi}{4n^2}$$

La serie converge, perché converge la serie

$$\sum \frac{\pi}{4n^2} = \frac{\pi}{4}\sum \frac{1}{n^2}.$$

4) $D = \mathbb{R}$

$f(x) = 0 \iff x_1 = 0;\ x_2 = 1.$

$f(x) > 0 \iff x(x-1) > 0 \iff x < 0;\ x > 1$

$$f'(x) = \frac{1}{3(x^2-x)^{2/3}}(2x-1) = 0 \iff x = \frac{1}{2}$$

$f'(x) > 0 \iff x > \frac{1}{2}$

f decresce in $\left(-\infty, \frac{1}{2}\right)$, cresce in $\left(\frac{1}{2}, +\infty\right)$;

$x = \frac{1}{2}$ punto di MIN. REL. e ASS.; $f\left(\frac{1}{2}\right) = -\sqrt[3]{\frac{1}{4}}$

Poiché $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \sqrt[3]{x^2} = +\infty$,

NON ESISTONO MAX. REL. O ASS.

Poichè $f \in C^0(\mathbb{R})$, $f(\mathbb{R}) = \left[-\sqrt[3]{\frac{1}{4}}, +\infty\right)$.

N.B.: $f'(x)$ NON è definita in x_1 e x_2:

$$\left.\begin{aligned} &\lim_{x\to 0^\pm} f'(x) = \frac{-1}{3\cdot 0^+} = -\infty \\ &\lim_{x\to 1^\pm} f'(x) = \frac{1}{3\cdot 0^+} = +\infty \end{aligned}\right\} \text{PUNTI A TANGENTE VERTICALE}$$

$$f''(x) = \frac{1}{3}\left[\frac{2(x^2-x)^{2/3} - (2x-1)\frac{2}{3}(x^2-x)^{-\frac{1}{3}}(2x-1)}{(x^2-x)^{4/3}}\right]$$

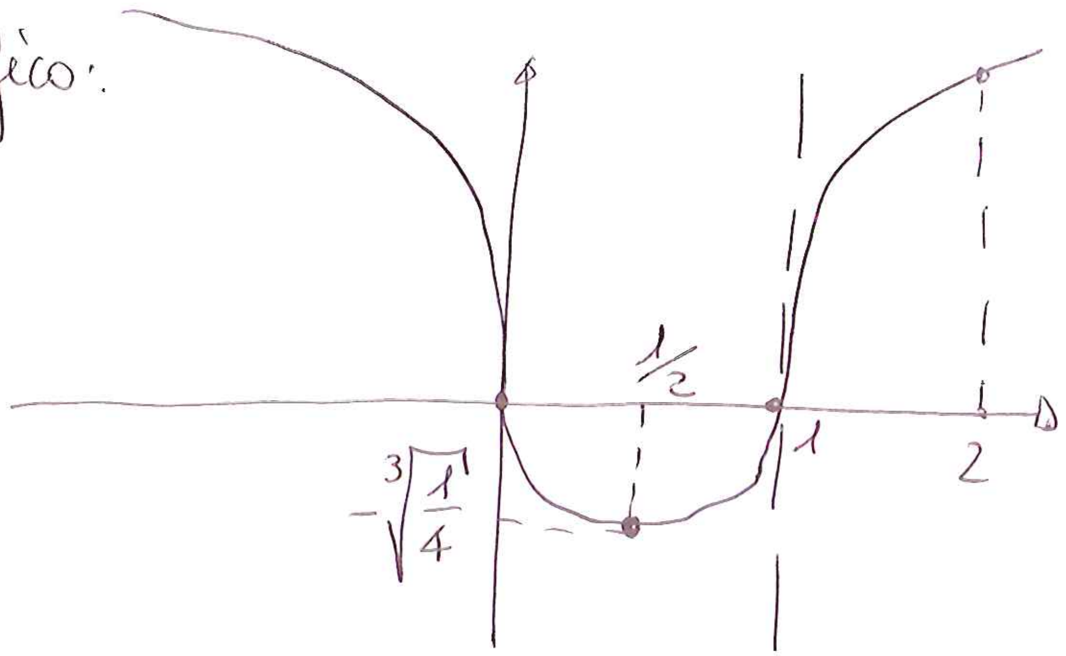
$$= \frac{2}{9}\left[\frac{3(x^2-x) - (2x-1)^2}{(x^2-x)^{5/3}}\right] = \frac{2}{9}\left[\frac{3x^2-3x-4x^2+4x-1}{(x^2-x)^{5/3}}\right]$$

$$= \frac{2}{9}\left[\frac{-x^2+x-1}{(x^2-x)^{5/3}}\right] = -\frac{2}{9}\left[\frac{x^2-x+1}{(x^2-x)^{5/3}}\right] > 0$$

$$\iff (x^2-x) < 0 \iff 0 < x < 1.$$

f è concava in $(-\infty, 0)$, convessa in $(0, 1)$, concava in $(1, +\infty)$.

Grafico:



In $\left[0,2\right]$, f decresce in $\left[0,\frac{1}{2}\right)$, cresce in $\left(\frac{1}{2},2\right]$.

$x=0$ punto di MAX. REL. $f(0)=0$;

$x=2$ punto di MAX. REL.: $f(2)=\sqrt[3]{2}$

$\Rightarrow x=2$ punto di MAX. ASS.

$x=\frac{1}{2}$ è punto di MIN. ASS.

5) $\dfrac{x^2}{4+x^6} \underset{x\to+\infty}{\sim} \dfrac{1}{x^4}$, che è integrabile a $+\infty$.

Quindi l'integrale converge.

$$\int_0^{+\infty} \frac{x^2}{4+x^6}\,dx = \int_0^{+\infty} \frac{x^2}{4+(x^3)^2}\,dx \;=\;$$

B5

$x^3 = t$; $dt = 3x^2 dx$

$\Rightarrow x^2 dx = \frac{dt}{3}$

$t(0) = 0$; $t(+\infty) = +\infty$

$$= \frac{1}{3}\int_0^{+\infty} \frac{dt}{4+t^2} = \frac{1}{6} \operatorname{arctg}\left(\frac{t}{2}\right)\Big|_0^{+\infty}$$

$$= \frac{1}{6}\cdot\frac{\pi}{2} = \frac{\pi}{12}.$$