SVOLGIMENTI PROVA SCRITTA di ANALISI 1 - 3/9/2020

1) $I_{def} = \mathbb{R}$. $f \in C^0(\mathbb{R})$.

Nè pari, nè dispari

$$f(x) = \begin{cases} x - 2\operatorname{arctg}(x-1) & \text{se } x \geq 1 \\ x + 2\operatorname{arctg}(x-1) & \text{se } x < 1 \end{cases}$$

$$f(0) = -2|\operatorname{arctg}(-1)| = -\frac{\pi}{2}.$$

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

ASINTOTO OBLIQUO:

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left[1 - \frac{2|\operatorname{arctg}(x-1)|}{x}\right] = 1 = m$$

$$\lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty} \left[-2|\operatorname{arctg}(x-1)|\right]$$

$$= -2\frac{\pi}{2} = -\pi$$

AS. OBL. a $\pm\infty$: $y = x - \pi$.

N.B.: $f(x) > x - \pi \quad \forall x \in \mathbb{R}$

Infatti $\cancel{x} - 2|\text{arctg}(x-1)| > \cancel{x} - \pi$

$\iff |\text{arctg}(x-1)| < \frac{\pi}{2}$ sempre vero.

f giace al di sopra dell'asintoto.

$$f'(x) = \begin{cases} 1 - \dfrac{2}{1+(x-1)^2} & x > 1 \\ 1 + \dfrac{2}{1+(x-1)^2} > 0 & x < 1 \end{cases}$$

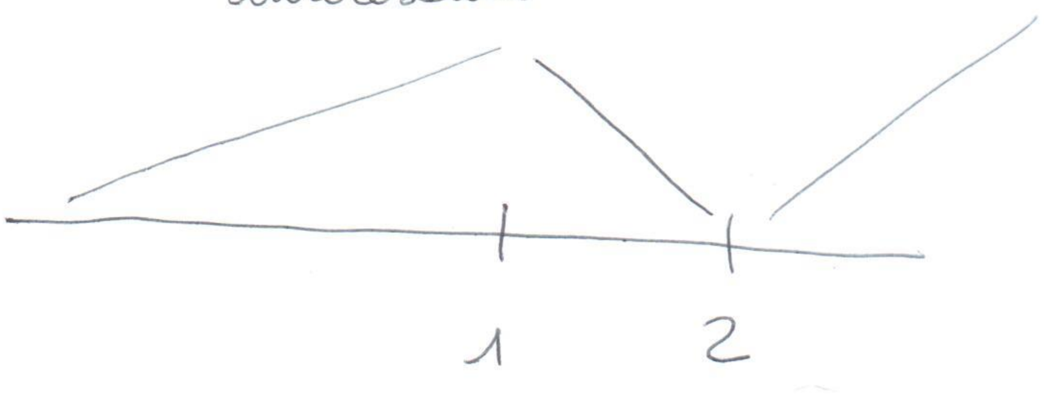
$$= \begin{cases} \dfrac{(x-1)^2 - 1}{1+(x-1)^2} & \text{se } x > 1 \\ 1 + \dfrac{2}{1+(x-1)^2} & \text{se } x < 1 \end{cases}$$

<u>Per $x>1$</u>: $f'(x) > 0 \iff (x-1)^2 > 1$

$\iff x - 1 < -1 \quad \vee \quad x - 1 > 1$

$\iff x < 0 \quad \vee \quad x > 2$

non interessante



f cresce in $(-\infty, 1)$; decresce in $(1, 2)$; cresce in $(2, +\infty)$. ③

MIN. REL.

$f(1) = 1$; $f(2) = 2 - 2\,\mathrm{arctg}\, 1 = 2 - \frac{\pi}{2}$

MAX. REL

$0 < f(2) < 1$.

Poichè $f(0) = -\frac{\pi}{2} < 0$; $f(1) = 1 > 0$,

f si annulla in un punto $0 < x_1 < 1$

Poichè f cresce in $(-\infty, 1)$, in tale intervallo ha zero solo in x_1.

Poichè in $[1, +\infty)$ f ammette MIN. REL. in $x = 2$ e $f(2) > 0$ $\Rightarrow$ non vi sono ulteriori zeri.

$\lim_{x \to 1^+} f'(x) = -1$; $\lim_{x \to 1^-} f'(x) = 3$

In $x = 1$ punto angoloso.

$$f''(x) = \begin{cases} \dfrac{2}{\left[1 + (x-1)^2\right]^2}\, 2(x-1) & x > 1 \\[2ex] \dfrac{-2}{\left[1 + (x-1)^2\right]}\, 2(x-1) & x < 1 \end{cases}$$

In $x<1$ $\quad f''(x) > 0$

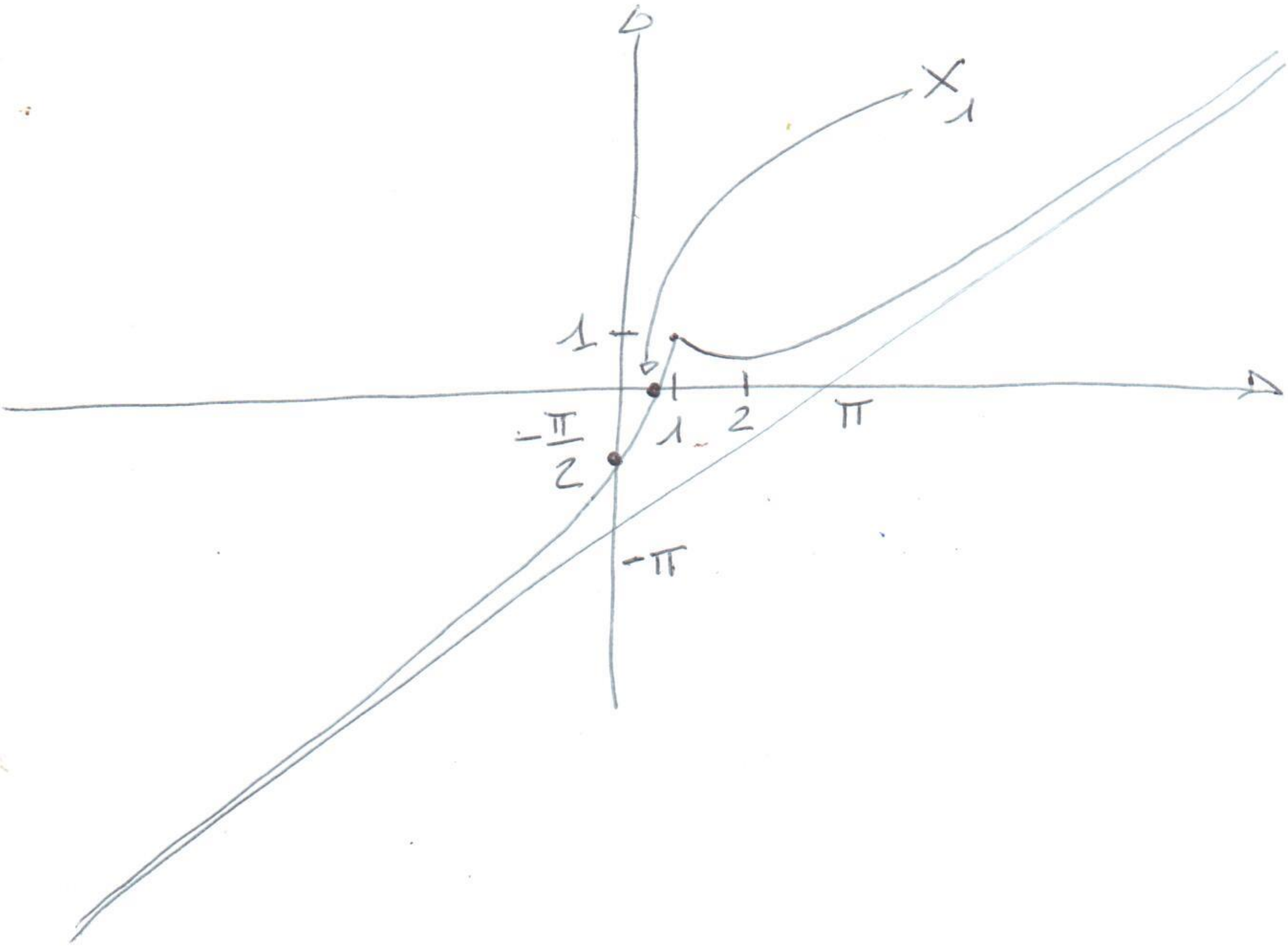
In $x>1$ $\quad f''(x) > 0$

$\Rightarrow f$ CONVESSA in $(-\infty, 1)$ e in $(1, +\infty)$.

$x = 1$ punto di MAX. REL.

$x = 2$ punto di MIN. REL.

Poichè $\lim_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow \nexists$ MAX. MIN. ASS.



2) L'equazione è definita per

$$\begin{cases} x \neq 0 \\ \ln^2 x \neq 1 \\ x > 0 \end{cases} \Longrightarrow \begin{cases} x > 0 \\ x \neq e, \frac{1}{e} \end{cases}.$$

Poichè $x_0 = 1 \Rightarrow$ risolviamo il Pb. di Cauchy in $\left(\frac{1}{e}, e\right)$.

$$y(x) - y(1) = \int_1^x \frac{\ln t}{t(1-\ln^2 t)}\, dt$$

Poniamo $\ln t = s \Rightarrow ds = \frac{1}{t} dt$

$$\Rightarrow y(x) = 1 + \int_0^{\ln x} \frac{s}{1-s^2}\, ds$$

$$= 1 - \frac{1}{2} \ln(1-s^2) \Big|_0^{\ln x}$$

$$= 1 - \frac{1}{2} \ln(1-\ln^2 x).$$

La soluzione è definita per $1-\ln^2 x > 0$ cioè per $x \in \left(\frac{1}{e}, e\right)$.

N.B.: $\lim_{x \to e^-} y(x) = \lim_{x \to \frac{1}{e}^+} y(x) = +\infty.$

3) Poniamo $z^2 - 1 = w$

$$w^2 + w + 1 = 0$$

$$\Rightarrow w_{1,2} = \frac{-1 + \sqrt{-3}}{2} = \begin{cases} -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i \\[2ex] -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i \end{cases}$$

$\Rightarrow$ a) $z^2 - 1 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i$

b) $z^2 - 1 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i$

a): $$z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$$

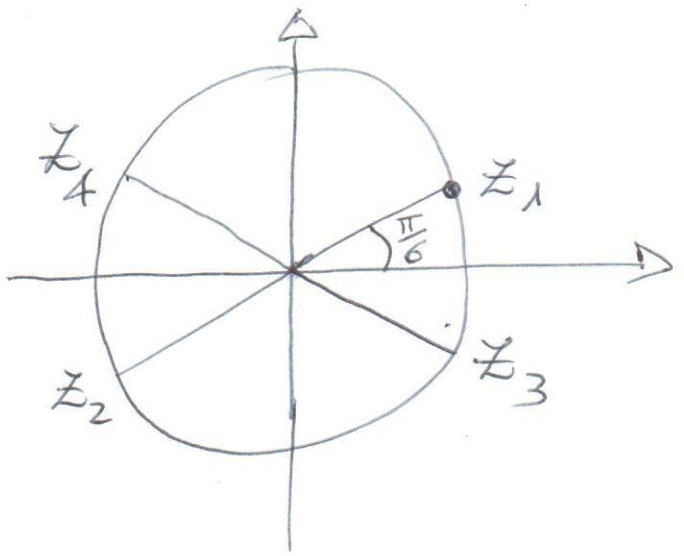
$$\Rightarrow z_1 = \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + i \frac{1}{2}$$

$$z_2 = -z_1 = \frac{-\sqrt{3}}{2} - i \frac{1}{2}$$

b) $$z^2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow z_3 = \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - i \frac{1}{2}$$

$$z_4 = -z_3 = \frac{-\sqrt{3}}{2} + i \frac{1}{2}.$$



4) L'arcoseno è definito se $\left|\frac{1}{n^\alpha}\right| \leq 1$

$\Rightarrow \alpha \geq 0$.

Per $\alpha = 0$: $a_n = \arcsin 1 = \frac{\pi}{2}$

convergente a $\frac{\pi}{2}$.

Per $\alpha > 0$: $a_n \sim \frac{1}{n^\alpha} \xrightarrow[n \to +\infty]{} 0 \quad \forall \alpha > 0$

Serie associata:

Per $\alpha = 0$ la serie diverge a $+\infty$.

Per $\alpha > 0$ $\sum a_n \approx \sum \frac{1}{n^\alpha}$

La serie converge $\forall \alpha > 1$

diverge $\forall \alpha \in [0, 1]$

5) (8)

$$f(x) = \frac{\ln\left(1 - \frac{x^2}{2} + o(x^2)\right) - \ln\left(1 + \cancel{x} + \frac{x^2}{2} \cancel{- x} + o(x^2)\right)}{x\left(x + o(x)\right)}$$

$$= \frac{-\frac{x^2}{2} - \frac{x^2}{2} + o(x^2)}{x^2 + o(x^2)} \underset{x \to 0}{\sim} \frac{-x^2}{x^2} \xrightarrow[x \to 0]{} -1.$$