SVOLGIMENTI PROVA SCRITTA di ANALISI 1 – 8/10/2025

1)
$$\frac{1-n\ln\left(1+\frac{1}{n}\right)}{n^\alpha} = \frac{1-n\left[\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right]}{n^\alpha}$$

$$= \frac{1}{n^\alpha}\left[\cancel{1-1}+\frac{1}{2n}+o\left(\frac{1}{n}\right)\right] \sim \frac{1}{2n^{\alpha+1}}$$

Pertanto
$$\sum a_n \approx \frac{1}{2}\sum \frac{1}{n^{\alpha+1}}$$

$$\Rightarrow \begin{cases} \text{converge per } \alpha>0 \\ \text{diverge per } \alpha\le 0. \end{cases}$$

2)
$$I_{def}: \begin{cases} 1-x^2\ge 0 \\ -1\le\sqrt{1-x^2}\le 1 \end{cases} \Rightarrow \begin{cases} |x|\le 1 \\ 0\le\sqrt{1-x^2}\le 1 \end{cases}$$

$$\Rightarrow \begin{cases} -1\le x\le 1 \\ 1-x^2\le 1 \leftarrow \forall x\in\mathbb{R} \end{cases} \Rightarrow I_{def}=[-1,1].$$

$$f(-x)=\arcsin\left(\sqrt{1-(-x)^2}\right)=\arcsin\left(\sqrt{1-x^2}\right)=f(x)$$

f PARI.

INTERSEZIONI CON ASSI: $f(0)=\arcsin 1 = \frac{\pi}{2}$

$$f(x)=0 \iff \arcsin\left(\sqrt{1-x^2}\right)=0$$
$$\iff \sqrt{1-x^2}=0 \iff x=\pm 1.$$

Poiché $\arcsin t\ge 0 \iff 0\le t\le 1$,
allora $f(x)\ge 0 \quad \forall x\in I_{def}$.

$f\in C^0(I_{def})$.

$$f'(x) = \frac{1}{\sqrt{1-(1-x^2)}} \cdot \left[\frac{-2x}{2\sqrt{1-x^2}}\right]$$

$$= \frac{-x}{\sqrt{x^2}\,\sqrt{1-x^2}} = \frac{-x}{|x|\sqrt{1-x^2}}$$

$$= \begin{cases} \dfrac{-1}{\sqrt{1-x^2}} < 0, & 0 < x < 1 \\ \dfrac{1}{\sqrt{1-x^2}} > 0 & -1 < x < 0 \end{cases}$$

f cresce in $[-1, 0)$ e decresce in $(0, 1]$.

f non è derivabile in $x = \pm 1$:

$$\lim_{x \to 1^-} f'(x) = \frac{-1}{0^+} = -\infty$$

$$\lim_{x \to -1^+} f'(x) = \frac{1}{0^+} = +\infty$$

f non è derivabile in $x = 0$:

$$\lim_{x \to 0^\pm} f'(x) = \mp 1. \quad \text{(PUNTO ANGOLOSO)}$$

$x = 0$ punto di MAX. ASS. $\quad f(0) = \frac{\pi}{2}$

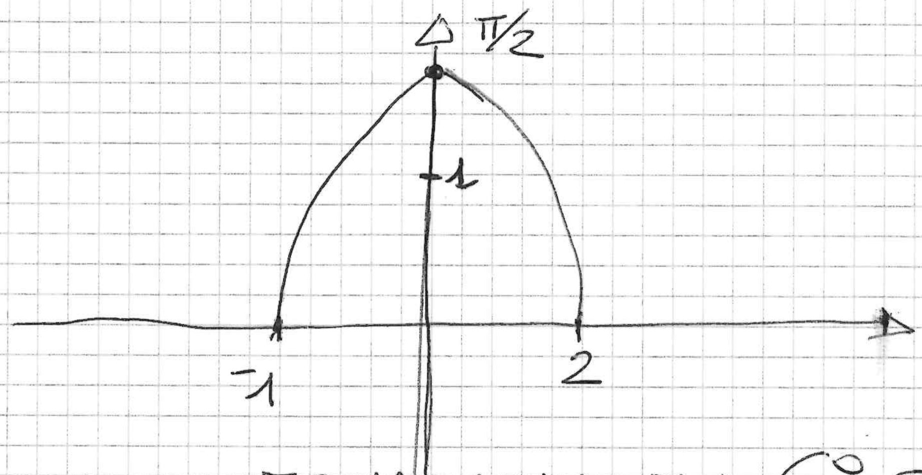
$x = \pm 1$ punti di MIN. ASS. $\quad f(\pm 1) = 0.$

$$f''(x) = \begin{cases} -\left[(1-x^2)^{-\frac{1}{2}}\right]' & \text{se } 0 < x < 1 \\ \left[(1-x^2)^{-\frac{1}{2}}\right]' & \text{se } -1 < x < 0. \end{cases}$$

$$= \begin{cases} \frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = \frac{-x}{(1-x^2)^{3/2}} < 0 & \text{se } 0<x<1 \\ -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = \frac{x}{(1-x^2)^{3/2}} < 0 & \text{se } -1<x<0 \end{cases}$$

f è CONCAVA in $[-1, 1]$.

Grafico:



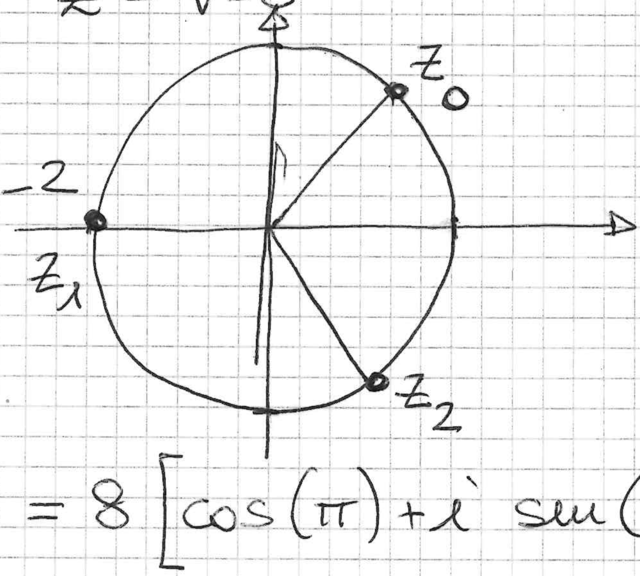
EQUAZIONE DI 6° GRADO
$\Rightarrow$ 6 SOLUZIONI

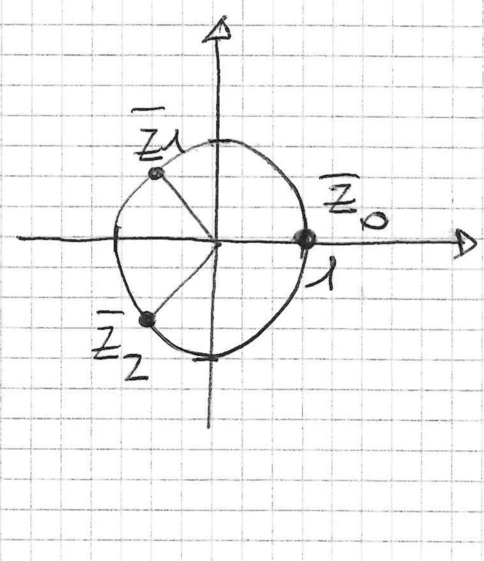
3) $z^3 = t$

$$t^2 + 7t - 8 = 0 \Rightarrow (t+8)(t-1) = 0$$

$$\Rightarrow z^3 = -8 \quad \vee \quad z^3 = 1$$

$$z = \sqrt[3]{-8}$$





$$-8 = 8\left[\cos(\pi) + i\,\sin(\pi)\right]$$

$$\Rightarrow \sqrt[3]{-8} = 2\left[\cos\left(\frac{\pi+2k\pi}{3}\right) + i\,\sin\left(\frac{\pi+2k\pi}{3}\right)\right]$$

$$z_0 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$ (1)

$$= 1 + i\sqrt{3}$$

$$z_1 = 2\left[\cos(\pi) + i\,\text{sen}(\pi)\right] = -2$$

$$z_2 = 2\left[\cos\left(\frac{5}{3}\pi\right) + i\,\text{sen}\left(\frac{5}{3}\pi\right)\right] = 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= 1 - i\sqrt{3}$$

$$z = \sqrt[3]{1}$$

$$1 = \cos(0) + i\,\text{sen}(0)$$

$$\Rightarrow \sqrt[3]{1} = \cos\left(\frac{2k\pi}{3}\right) + i\,\text{sen}\left(\frac{2k\pi}{3}\right)$$

$$\bar{z}_0 = 1$$

$$\bar{z}_1 = \cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\bar{z}_2 = \cos\left(\frac{4\pi}{3}\right) + i\,\text{sen}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

4) Equazione lineare del 1° ordine

$$a(x) = \frac{1}{x} \in C^\infty\big((-\infty, 0) \cup (0, +\infty)\big)$$

$$f(x) = \sqrt{2 - x^2} \in C^0\big([-\sqrt{2}, \sqrt{2}]\big)$$

Poiché $x_0 = 1 \Rightarrow$ il problema è definito in $(0, \sqrt{2}]$.

$\exists!$ sol. GLOBALE $y \in C^1\big((0, \sqrt{2}]\big)$.

$$y(x) = e^{-\int_1^x \frac{1}{t}\,dt}\left[-\frac{1}{3} + \int_1^x e^{\int_1^t \frac{1}{s}\,ds}\sqrt{2-t^2}\,dt\right]$$

$$= e^{-\ln|t|\big|_1^x}\left[-\frac{1}{3} + \int_1^x e^{\ln|s|\big|_1^t}\sqrt{2-t^2}\,dt\right]$$

Ma $t, x > 0 \Rightarrow |t| = t;\ |x| = x$

$$y(x) = e^{-\ln x + \overset{0}{\overbrace{\ln 1}}}\left[-\frac{1}{3} + \int_1^x e^{\ln t - \overset{0}{\overbrace{\ln 1}}}\sqrt{2-t^2}\,dt\right]$$

$$= \frac{1}{x}\left[-\frac{1}{3} + \int_1^x t\sqrt{2-t^2}\,dt\right]$$

$$= \frac{1}{x}\left[-\frac{1}{3} - \frac{1}{3}(2-t^2)^{\frac{3}{2}}\Big|_1^x\right]$$

$$= \frac{1}{x}\left[\cancel{-\frac{1}{3}} - \frac{1}{3}(2-x^2)^{\frac{3}{2}} \cancel{+\frac{1}{3}}\right]$$

$$= -\frac{1}{3x}(2-x^2)^{\frac{3}{2}}$$

Effettivamente, $y(x)$ è definito in $(0, \sqrt{2}]$.

5) Pongo $\cos x = t$; $dt = -\sin x\,dx$

$t(0) = 1$; $t\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\int_0^{\pi/4} \frac{\sin x(\cos x - 2)}{1+\cos^2 x}\,dx = -\int_1^{\sqrt{2}/2} \frac{(t-2)}{t^2+1}\,dt$$

$$= \int_{\sqrt{2}/2}^1 \left[\frac{t}{t^2+1} - \frac{2}{t^2+1}\right]dt$$

$$= \left[\frac{1}{2}\ln(t^2+1) - 2\,\mathrm{arctg}\, t\right]_{\sqrt{2}/2}^{1}$$

$$= \frac{1}{2}\left(\ln 2 - \ln\left(\frac{3}{2}\right)\right) - 2\,\mathrm{arctg}\, 1 + 2\,\mathrm{arctg}\,\frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}\ln\left(\frac{4}{3}\right) - 2\,\frac{\pi}{4} + 2\,\mathrm{arctg}\,\frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}\ln\left(\frac{4}{3}\right) - \frac{\pi}{2} + 2\,\mathrm{arctg}\left(\frac{\sqrt{2}}{2}\right).$$