SVOLGIMENTI PROVA SCRITTA di ANALISI del 9/2/2021

1) $f(x) = e^{\frac{(x+1)^2}{x}}$

$I_{def} = \{x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$.

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{x} = +\infty$

ANDAMENTO SUPERLINEARE $\Rightarrow \nexists$ ASINTOTO OBLIQUO per $x \to +\infty$.

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^{x} = 0^+$ AS. ORIZZONTALE $y = 0$.

$\lim_{x \to 0^+} f(x) = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty$ AS. VERTICALE da DX $x = 0$.

$\lim_{x \to 0^-} f(x) = e^{\frac{1}{0^-}} = e^{-\infty} = 0$

$\nexists f(0)$. NO $\cap$ asse y.

$f(x) > 0 \quad \forall x \in I_{def} \Rightarrow$ NO $\cap$ asse x.

$$f'(x) = e^{\frac{(x+1)^2}{x}} \left[\frac{2(x+1)x - (x+1)^2}{(x^2)} \right]$$

$$= e^{\frac{(x+1)^2}{x}} \frac{(x+1)}{x^2} (2x - x - 1) = \frac{e^{\frac{(x+1)^2}{x}}}{x^2} (x^2 - 1)$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (x^2-1)\cdot \lim_{x\to 0^-} \frac{e^{\frac{(x+1)^2}{x}}}{x^2}$$ ②

$$= -1\cdot \lim_{x\to 0^-} e^{\frac{x^2+2x}{x}} \cdot \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{x^2}$$

$$= -\lim_{x\to 0^-} e^{x+2} \cdot \lim_{t\to -\infty} e^{+t}t^2 \qquad t = +\frac{1}{x}$$

$$= -e^2 \cdot \lim_{t\to -\infty} \frac{t^2}{e^{-t}} = 0^-$$ (per gli ordini di infinito.

$f'(x) > 0 \iff x<-1 \vee x>+1$

f cresce in $(-\infty,-1)$; decresce in $(-1,0)$;

decresce in $(0,1)$; cresce in $(1,+\infty)$.

In $x=-1$ punto di MAX. REL.; $f(-1)=1$

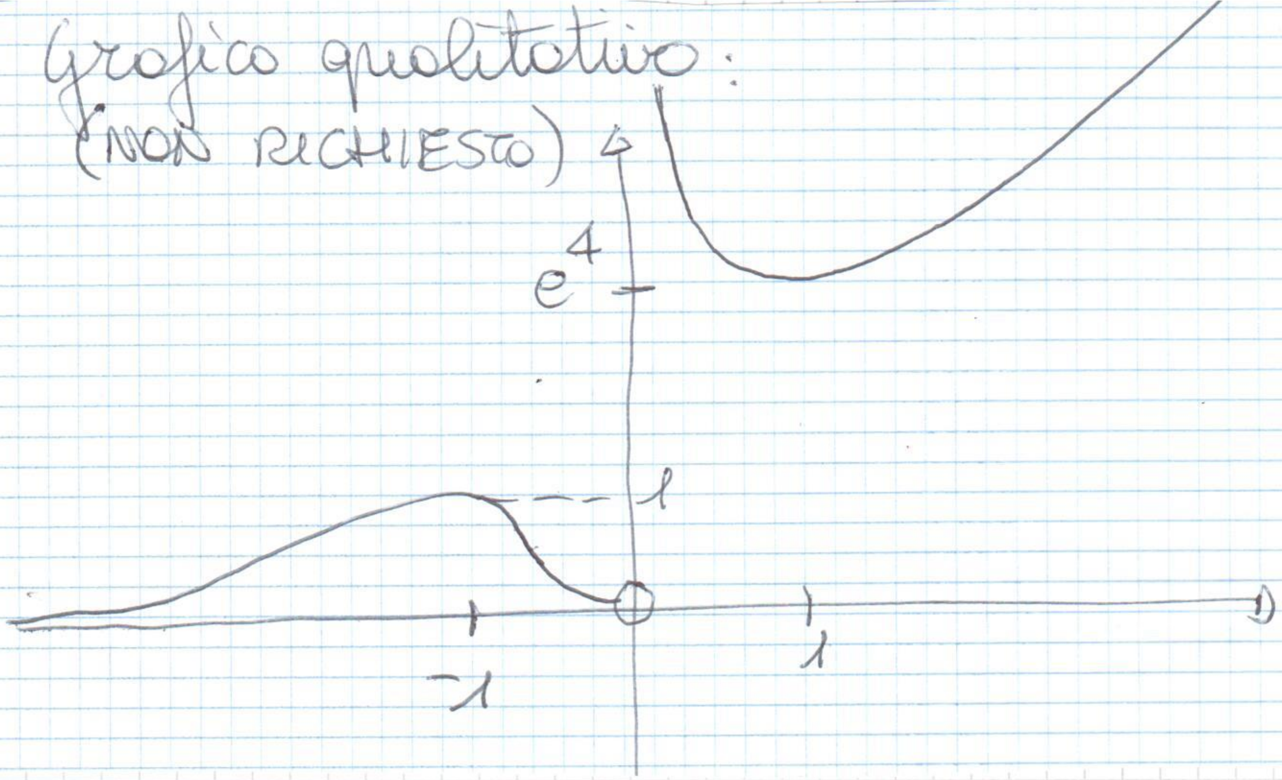
In $x=1$ punto di MIN. REL.; $f(1)=e^4$

Poiché $\lim_{x\to 0^-} f(x)=0$ e $f(x)>0$

$\Rightarrow \nexists$ MIN. ASS., ma $\exists$ inf $(f)=0$.

Poiché $\lim_{x\to +\infty} f(x)=+\infty \Rightarrow \nexists$ MAX. ASS.

Grafico qualitativo:
(NON RICHIESTO)



3

2) $\lim_{x\to 0^-} f(x) = f(0) = b$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{c}{\sqrt{x}}\left[\sqrt{x+1}-\sqrt{1-x}\right]$$

$$= c\lim_{x\to 0^+} \frac{1}{\sqrt{x}}\left[\frac{(x+1)-(1-x)}{\sqrt{x+1}+\sqrt{1-x}}\right]$$

$$= c\lim_{x\to 0^+} \frac{1}{\sqrt{x}}\left[\frac{2x}{\sqrt{x+1}+\sqrt{1-x}}\right]$$

$$= 2c\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{1-x}} = 0 \qquad \forall c\in\mathbb{R}$$

$\Rightarrow$ f è CONTINUA in $x=0$ se $b=0$;
$\forall a, c\in\mathbb{R}$

4

$$f'(x) = \begin{cases} a & \text{se } x < 0 \\ c\left[-\frac{1}{x^2}\left[\frac{1}{2\sqrt{1+\frac{1}{x}}} - \frac{1}{2\sqrt{\frac{1}{x}-1}}\right]\right] & \text{se } x > 0 \end{cases}$$

$$-\frac{c}{2x^2}\sqrt{x}\left[\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{1-x}}\right]$$

$$= \frac{-c}{2x^{3/2}}\left[\frac{\sqrt{1-x} - \sqrt{x+1}}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \lim_{x\to 0^+} f'(x) = \frac{-c}{2} \lim_{x\to 0^+} \frac{1}{\sqrt{1-x^2}} \cdot \lim_{x\to 0^+} \frac{\sqrt{1-x} - \sqrt{x+1}}{x^{3/2}}$$

$$= \frac{-c}{2} \lim_{x\to 0^+} \frac{(1-x)-(x+1)}{x^{3/2}\left[\sqrt{1-x} + \sqrt{x+1}\right]}$$

$$= \frac{-c}{2\cdot\cancel{2}} \lim_{x\to 0^+} \frac{-\cancel{2}x}{x^{3/2}} = \frac{c}{2} \lim_{x\to 0^+} \frac{1}{\sqrt{x}}$$

$$\Rightarrow \lim_{x\to 0^-} f'(x) = a = \lim_{x\to 0^+} \frac{c}{2}\frac{1}{\sqrt{x}} \in \mathbb{R}$$

$$\Leftrightarrow a = c = 0$$

$\Rightarrow$ f è derivabile in $x = 0$

$$\Leftrightarrow a = b = c = 0$$

cioè solo nel caso in cui $f(x) \equiv 0$.

3) Poniamo $e^z = w$

$$\Rightarrow w^2 + 8iw - 16 = 0$$

$$\Rightarrow (w + 4i)^2 = 0 \Rightarrow w = -4i$$

con m.a. $= 2$

$$\Rightarrow e^z = -4i \Rightarrow e^x \cdot e^{iy} = 4 \cdot (-i)$$

$$\Rightarrow \begin{cases} e^x = 4 \\ e^{iy} = e^{i\left(-\frac{\pi}{2} + 2k\pi\right)} \end{cases} ; \ k \in \mathbb{Z}$$

$$\Rightarrow \begin{cases} x = \ln 4 \\ y = -\frac{\pi}{2} + 2k\pi \end{cases} ; \ k \in \mathbb{Z}$$

INFINITE SOLUZIONI

4)
$$a_n = \frac{\frac{1}{n^2}\left[1 - \frac{1}{2n^4} + o\left(\frac{1}{n^4}\right)\right] - \left(\frac{1}{n^2} - \frac{1}{2n^4} + o\left(\frac{1}{n^4}\right)\right)}{\cancel{\frac{1}{n}} - \frac{1}{3n^3} - \cancel{\frac{1}{n}} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)}$$

$$= \frac{\cancel{\frac{1}{n^2}} - \cancel{\frac{1}{n^2}} + \frac{1}{2n^4} + o\left(\frac{1}{n^4}\right)}{-\frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)} \sim \frac{-3}{n}$$

$\Rightarrow$ Per il criterio del confronto asintotico la serie DIVERGE NEGATIVAMENTE.

5) Poniamo $e^x = t$ $\quad dt = e^x dx \Rightarrow dx = \frac{dt}{t}$

$t(\ln 4) = 4$; $t(+\infty) = +\infty$

(6)

$$\Rightarrow \int_4^{+\infty} \frac{1}{t(t^2+t-2)}\,dt = \int_4^{+\infty} \frac{1}{t(t-1)(t+2)}\,dt$$

$$\frac{1}{t(t-1)(t+2)} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{t+2} = \cancel{\frac{A(t-1)(t+2)}{t(t-1)(t+2)}}$$

$$= \frac{A(t-1)(t+2) + Bt(t+2) + Ct(t-1)}{t(t-1)(t+2)}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ A+2B-C=0 \\ -2A=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{2} \\ B+C=\frac{1}{2} \\ 2B-C=\frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{3} \\ C=\frac{1}{2}-\frac{1}{3}=\frac{1}{6} \end{cases}$$

$$\Rightarrow \int_4^{+\infty} \left[-\frac{1}{2t} + \frac{1}{3(t-1)} + \frac{1}{6(t+2)}\right] dt$$

$$= \left[-\frac{1}{2}\ln(|t|) + \frac{1}{3}\ln(|t-1|) + \frac{1}{6}\ln(|t+2|) \right]_4^{+\infty}$$

$$= \left[\ln\left(\frac{(t-1)^{\frac{1}{3}}(t+2)^{\frac{1}{6}}}{t^{\frac{1}{2}}} \right) \right]_4^{+\infty}$$

7

$$= \lim_{t \to +\infty} \ln\left(\frac{t^{\frac{1}{3}} \cdot t^{\frac{1}{6}}}{t^{1/2}} \right) - \ln\left(\frac{3^{\frac{1}{3}} \cdot 6^{\frac{1}{6}}}{2} \right)$$

$$= \lim_{t \to +\infty} \ln\left(\frac{t^{\frac{1}{2}}}{t^{\frac{1}{2}}} \right) - \ln\left(\frac{(9 \cdot 6)^{\frac{1}{6}}}{2} \right)$$

$$= -\ln\left[\frac{\sqrt[6]{54}}{2} \right] = \frac{1}{6}\ln\left(\frac{32}{27}\right) > 0.$$

6) $1 + x^2 \neq 0 \quad \forall x \in \mathbb{R}$

$$\Rightarrow \begin{cases} y' - \frac{2x}{1+x^2} y = (1+x^2) \\ y(0) = 1 \end{cases}$$

$$y(x) = e^{\int_0^x \frac{2t}{1+t^2}dt} \left[1 + \int_0^x e^{-\int_0^t \frac{2s}{1+s^2}ds} (1+t^2)\,dt \right]$$

$$= e^{\ln(1+t^2)\big|_0^x} \left[1 + \int_0^x e^{-\ln(1+s^2)\big|_0^t} (1+t^2)\,dt \right]$$

$$= e^{\ln(1+x^2)}\left[1+\int_0^x e^{-\ln(1+t^2)}(1+t^2)\,dt\right]$$

$$= (1+x^2)\left[1+\int_0^x \frac{\cancel{(1+t^2)}}{\cancel{(1+t^2)}}\,dt\right]$$

$$= (1+x^2)\left[1+x\right].$$