SVOLGIMENTI COMPITO ANALISI 1 dell'11/6/2020.

1) $$e^{-x^2}-1-\alpha x^2 = \left(\cancel{1}-x^2+\frac{x^4}{2}-\cancel{1}-\alpha x^2+o(x^4)\right)$$

$$= (-1-\alpha)x^2+\frac{x^4}{2}+o(x^4)$$

$$\Rightarrow \quad f(x) = \frac{(-1-\alpha)x^2+\frac{x^4}{2}+o(x^4)}{x^4} \quad \text{se } x\neq 0$$

$$\lim_{x\to 0} f(x) = \begin{cases} \frac{1}{2} & \text{se } \alpha=-1 \\ (-1-\alpha)(+\infty) & \text{se } \alpha\neq -1. \end{cases}$$

$$\Rightarrow \quad f\in C^\infty(\mathbb{R}) \iff \alpha=-1$$

Per $\alpha=-1$:

$$f(x) = \begin{cases} \frac{e^{-x^2}-1+x^2}{x^4} & \text{se } x\neq 0 \\ \frac{1}{2} & \text{se } x=0 \end{cases}$$

$$f'(0) = \lim_{x\to 0} \frac{1}{x}\left[\frac{e^{-x^2}-1+x^2}{x^4}-\frac{1}{2}\right]$$

$$= \lim_{x\to 0} \frac{1}{2x^5}\left[2(e^{-x^2}-1+x^2)-x^4\right]$$

$$= \lim_{x\to 0} \frac{1}{2x^5}\left[2\left(\cancel{1} - \cancel{x^2} + \cancel{\frac{x^4}{2}} - \frac{x^6}{3!} + o(x^6)\right) - \cancel{1} + \cancel{x^2} - \cancel{x^4} \right]$$

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$$= \lim_{x\to 0} \frac{-x^6}{6x^5} = \lim_{x\to 0}\left(\frac{-x}{6}\right) = 0$$

$\Rightarrow$ f derivabile in $x=0$.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{x^4} = 0$$

AS. ORIZZONTALE a $\pm\infty$ $y=0$.

2) In forma normale:

$$y' = \frac{(e^y - 1)}{x}$$

eq. a var. sep., definita $\forall x \neq 0$.

Poiché $x_0 = 1 \Rightarrow$ cerchiamo la sol. in $(0,+\infty)$. $b(y) = e^y - 1 \in C^\infty(\mathbb{R})$; $a(x) = \frac{1}{x} \in C^\infty(0,+\infty)$

$\exists!$ sol. $y \in C^1(I(1))$ SOL. LOCALE.

$y \equiv 0$ sol. singolare. Non è sol. del Pb. di Cauchy.

$$\Rightarrow \int \frac{dy}{e^y - 1} = \int \frac{1}{x}\,dx$$

$$e^y = t \Rightarrow dt = e^y dy \Rightarrow dy = \frac{dt}{t}$$ ③

$$\Rightarrow \int \frac{dy}{e^y - 1} = \int \frac{dt}{t(t-1)} = \int \left[\frac{1}{t-1} - \frac{1}{t} \right] dt$$

$$= \ln \left| \frac{t-1}{t} \right| = \ln \left| \frac{e^y - 1}{e^y} \right|$$

Poiché $y_0 = \ln 2 > 0 \Rightarrow y > 0 \Rightarrow e^y - 1 > 0$

$$\Rightarrow \ln \left(\frac{e^y - 1}{e^y} \right) = \ln |x| + C$$

$$= \ln x + C$$

$$\Rightarrow \frac{e^y - 1}{e^y} = e^C x = Kx \qquad (K > 0)$$

C.I.: $y(1) = \ln 2$

$$\Rightarrow \frac{2-1}{2} = K \Rightarrow K = \frac{1}{2}$$

$$\Rightarrow \frac{e^y - 1}{e^y} = \frac{1}{2} x \Rightarrow e^y - 1 = e^y \frac{1}{2} x$$

$$\Rightarrow e^y \left(1 - \frac{1}{2} x \right) = 1$$

$$\Rightarrow y = -\ln \left(1 - \frac{1}{2} x \right)$$

~~Si osserva che~~ La località della soluzione si osserva dal fatto

che $y = -\ln\left(1-\frac{1}{2}x\right)$ è definita

~~$\forall x < 2$~~ solo per $x<2$, benché

$a(x) = \frac{1}{x} \in C^{\infty}(0,+\infty)$.

3) $(x+iy)^2 + (x-iy)^2 + 2(x^2+y^2)$

$+i(\cancel{x}+iy - (\cancel{x}-iy)) + 2(x+iy) = -i$

$2x^2 \cancel{-2y^2} \cancel{+2ixy} \cancel{-2ixy} + 2x^2 \cancel{+2y^2} - 2y$

$+2x+2iy = -i$

$$\begin{cases} 4x^2+2x-2y=0 \\ y=-\frac{1}{2} \end{cases} \Rightarrow \begin{cases} 4x^2+2x+1=0 \\ y=-\frac{1}{2} \end{cases}$$

$\Delta < 0$

$\nexists\, x \in \mathbb{R}$

$\Rightarrow \nexists$ soluzioni

4) Criterio della radice:

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n^{\sqrt{n}}}}{e} = \frac{n^{\frac{1}{\sqrt{n}}}}{e} = \frac{\left[(\sqrt{n})^2\right]^{\frac{1}{\sqrt{n}}}}{e}$$

$$= \frac{\left(\sqrt{n}^{\frac{1}{\sqrt{n}}}\right)^2}{e}$$

Poichè $n^{\frac{1}{n}} \xrightarrow[n \to +\infty]{} 1$

$$\Rightarrow \lim_{n \to \infty} \sqrt[n]{a_n} = \frac{(1)^2}{e} = \frac{1}{e} < 1$$

La serie converge.

Poichè $\sum a_n$ converge $\Rightarrow a_n \xrightarrow[n \to \infty]{} 0$

5) Poniamo $\sqrt{x} = t \Rightarrow dt = \frac{1}{2\sqrt{x}} dx$

$\Rightarrow dx = 2t\, dt$; $x = t^2$;

$t(0) = 0$; $t(1) = 1$;

$$\Rightarrow \int_0^1 \frac{2t^2\, dt}{1+t^2} = 2 \int_0^1 \left[1 - \frac{1}{1+t^2}\right] dt$$

$$= 2\left[t - \operatorname{arctg} t\right]_0^1 = 2\left[1 - \operatorname{arctg} 1\right]$$

$$= 2\left[1 - \frac{\pi}{4}\right] = \frac{4-\pi}{2}.$$

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