SVOLGIMENTI PROVA SCRITTA di ANALISI 2
dell' 8/2/2021

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1) la serie è a termini positivi
$\Rightarrow$ CONV. ASS. $\equiv$ CONV. SEMPLICE

Criterio della radice

$$\sqrt[n]{\varphi_n(x)} = \left(\sin^2 x + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \sin^2 x < 1$$

$$\Leftrightarrow x \neq \frac{\pi}{2} + k\pi$$

Per $x = \frac{\pi}{2} + k\pi$, $\sin^2 x = 1$

$$\Rightarrow \sum \varphi_n\left(\frac{\pi}{2} + k\pi\right) = \sum \left(1 + \frac{1}{n}\right)^n$$

Ma $\varphi_n = \left(1 + \frac{1}{n}\right)^n \rightarrow e \neq 0$

$\Rightarrow$ NO CONV.

La serie converge ass. e sempl. $\forall x \neq \frac{\pi}{2} + k\pi$, ovvero in ogni intervallo della forma

$$I_k = \left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right).$$

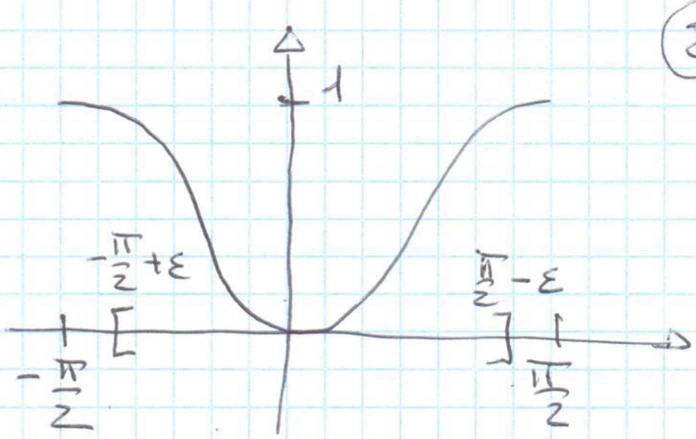
Poichè in ognuno di tali intervalli

$$\sup |\varphi_n(x)| = \left(1 + \frac{1}{n}\right)^n \rightarrow e \neq 0$$

$\Rightarrow$ NO CONV. TOTALE in I_k.

Se però consideriamo $U_k = \left[\frac{\pi}{2} + k\pi + \varepsilon, \frac{\pi}{2} + (k+1)\pi - \varepsilon\right]$

$\Rightarrow$



$\sup|\varphi_n(x)|$

$= \max\left(\sin^2 x + \frac{1}{n}\right)^n$

$= \left[\sin^2\left(\frac{\pi}{2}+k\pi+\varepsilon\right)+\frac{1}{n}\right]^n$

Per il criterio della radice

$$\sqrt[n]{\sup|\varphi_n|} = \sin^2\left(\frac{\pi}{2}+k\pi+\varepsilon\right)+\frac{1}{n}$$

$$\to \sin^2\left(\frac{\pi}{2}+k\pi+\varepsilon\right) < 1$$

$\Rightarrow$ CONV. TOTALE (e UNIFORME) in ogni U_k.

2)

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} \left|\frac{x\sin(y^2)}{\sqrt{x^2+y^2}}\right|$$

$$= \lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{\sqrt{x^2+y^2}}\right| = \lim_{\rho\to 0}\left|\frac{\rho^{\cancel{3}}\cos\vartheta\sin^2\vartheta}{\cancel{\rho}}\right|$$

$$\leq \lim_{\rho\to 0}\rho^2 = 0 \quad \text{UNIF. rispetto a } \vartheta$$

$$\Rightarrow f\in C^0((0,0)).$$

$$f(0,y) = f(x,0) = 0 \Rightarrow f_x(0,0) = f_y(0,0) = 0$$

$$\frac{df}{d\vec{v}}(0,0) = \lim_{t\to 0} \frac{t^{\cancel{3}\,2}\alpha\beta^2}{|t|\cancel{\,}} = \lim_{t\to 0} \frac{|t|^2\alpha\beta^2}{|t|}$$ (3)

$$= \lim_{t\to 0} |t|\alpha\beta^2 = 0 \quad \forall(\alpha,\beta).$$

DIFFERENZIABILITÀ:

$$\lim_{(h,k)\to(0,0)} \left| \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} \right|$$

$$= \lim_{(h,k)\to(0,0)} \left| \left[\frac{hk^2}{\sqrt{h^2+k^2}}\right] \frac{1}{\sqrt{h^2+k^2}} \right| = \lim_{\rho\to 0} \left| \frac{\rho^{\cancel{3}}\cos\vartheta\sin^2\vartheta}{\cancel{\rho^2}} \right|$$

$$\leq \lim_{\rho\to 0} \rho\,|\cos\vartheta\sin^2\vartheta| \leq \lim_{\rho\to 0} \rho = 0$$

UNIFORM. rispetto a ϑ.

Ovviamente, se si fosse verificata subito la differenziabilità, avremmo ottenuto, senza conti, la continuità di f in $(0,0)$ e, per la formula del gradiente, $\frac{df}{d\vec{v}}(0,0) = 0 \quad \forall \vec{v}$.

Negli altri punti del dominio (che non è tutto $\mathbb{R}^2$, perchè devo imporre $1+x\sin(y^2)>0$)

$$f_x(x,y) = \left[\frac{\sin(y^2)}{1+x\sin(y^2)} \cdot \sqrt{x^2+y^2} - \log(1+x\sin(y^2)) \cdot \frac{x}{\sqrt{x^2+y^2}} \right] \cdot \frac{1}{(x^2+y^2)}$$

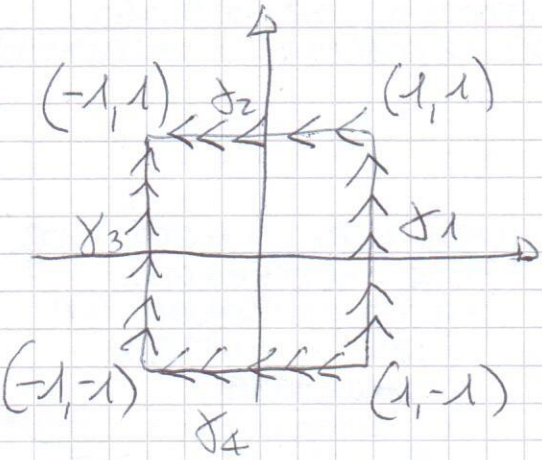
$$= \frac{1}{(x^2+y^2)^{3/2}} \left[\frac{\sin(y^2)}{(1+x\sin(y^2)} (x^2+y^2) - x \log(\quad) \right]$$

$$f_y(x,y) = \left[\frac{x\cos(y^2)2y}{1+x\sin(y^2)} \sqrt{x^2+y^2} - \log(\quad) \frac{y}{\sqrt{x^2+y^2}} \right] \frac{1}{x^2+y^2}$$

$$= \frac{1}{(x^2+y^2)^{3/2}} \left[\frac{2xy\cos(y^2)}{1+x\sin(y^2)} (x^2+y^2) - y \log(\quad) \right]$$

3) $f_x(x,y) = -\left[e^{-x^2-x^4}\right] < 0$

$f_y(x,y) = e^{-y^2-y^4} > 0$ $\Rightarrow \nexists$ punti stazionari.



Su ∂D:

$\gamma_1: \begin{cases} x=1 \\ y\in[-1,1] \end{cases}$; $\gamma_2 = \begin{cases} x\in[-1,1] \\ y=1 \end{cases}$

$\gamma_3: \begin{cases} x=-1 \\ y\in[-1,1] \end{cases}$; $\gamma_4 = \begin{cases} x\in[-1,1] \\ y=-1 \end{cases}$

$$f|_{\gamma_1} = f(1,y) = \int_0^y e^{-t^2-t^4}\,dt - \underbrace{\int_0^1 e^{-t^2-t^4}\,dt}_{\text{COSTANTE} < 0}$$

$$\left(f|_{\gamma_1}\right)' = e^{-y^2-y^4} > 0 \quad \text{crescente}$$

N.B.: $f(1,1) = \int_0^1 - \int_0^1 = 0.$

$$f|_{\gamma_2} = f(x,1) = \underbrace{\int_0^1 e^{-t^2-t^4}\,dt}_{\text{COSTANTE} > 0} - \int_0^x e^{-t^2-t^4}\,dt$$

$$\left(f|_{\gamma_2}\right)' = -\left(e^{-x^2-x^4}\right) < 0 \quad \text{decrescente}$$

$$f|_{\gamma_3} = f(-1,y) = \int_0^y e^{-t^2-t^4}\,dt - \underbrace{\int_0^{-1} e^{-t^2-t^4}\,dt}_{\text{COSTANTE} > 0}$$

$$\left(f|_{\gamma_3}\right)' = e^{-y^2-y^4} > 0 \quad \text{crescente}$$

$$f|_{\gamma_4} = f(x,-1) = \underbrace{\int_0^{-1} e^{-t^2-t^4}\,dt}_{\text{COSTANTE} < 0} - \int_0^x e^{-t^2-t^4}\,dt$$

$$\Rightarrow \left(f|_{\gamma_4}\right)' = -\left(e^{-x^2-x^4}\right) < 0 \quad \text{decrescente}$$

N.B.: $f(-1,-1) = \int_0^{-1} - \int_0^{-1} = 0.$

$\Rightarrow$ (dal grafico) $P_1 \equiv (1,-1)$ di MIN. ASS. ; $P_3 \equiv (-1,1)$ di MAX. ASS.

Infatti: $f(1,-1) = \int_0^{-1} e^{-t^2-t^4}\,dt - \int_0^{1} e^{-t^2-t^4}\,dt$

$$= -\int_{-1}^{1} e^{-t^2-t^4}\,dt < 0$$

$$f(-1,1) = \int_0^{1} e^{-t^2-t^4}\,dt - \int_0^{-1} e^{-t^2-t^4}\,dt$$

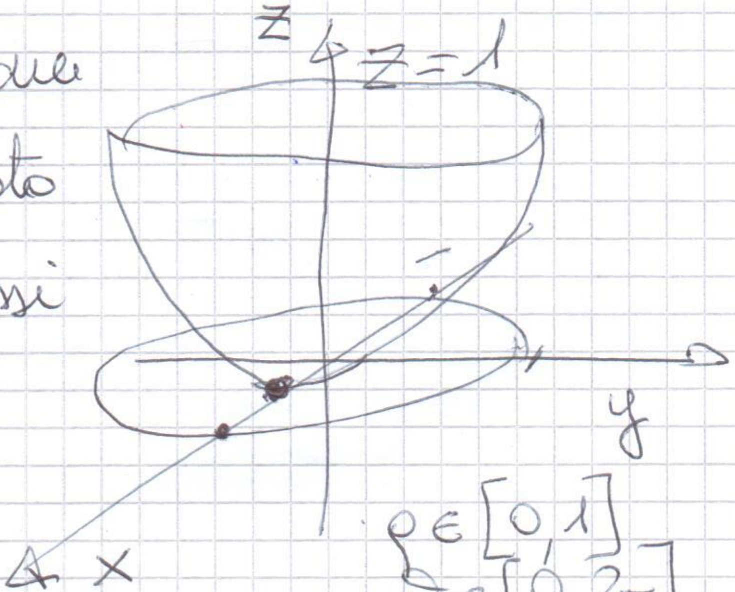
$$= \int_{-1}^{1} e^{-t^2-t^4}\,dt > 0.$$

$$\| \quad -f(1,-1)$$

4) $\operatorname{div}\vec{F} = 1+1-1 = \underline{1}$

$$\Rightarrow \underline{\Phi}_{\partial E}(\vec{F}) = \mathrm{vol}(E).$$

Paraboloide a sezione ellissoidale, centrato in $(1,0)$, di semiassi $a=3$, $b=4$.



$$\begin{cases} x = 1+3\rho\cos\theta \\ y = 4\rho\sin\theta \end{cases} \qquad \rho\in[0,1], \quad \theta\in[0,2\pi]$$

$$= \iint_{D(x,y)} dx\,dy \left[1 - \frac{(x-1)^2}{9} - \frac{y^2}{16}\right] =$$

$$\int_0^{2\pi} d\theta \int_0^1 \left[1-\rho^2\right] 12\rho\, d\rho = 24\pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4}\right]_0^1$$

$$= 24\pi \left[\frac{1}{2} - \frac{1}{4}\right] = 6\pi.$$

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5)
$$\gamma'(t) : \begin{cases} x'(t) = 2e^{2t} \\ y'(t) = 2\sqrt{2} \\ z'(t) = -2e^{-2t} \end{cases}$$

$$v(t) = \sqrt{4e^{4t} + 8 + 4e^{-4t}} = 2\sqrt{(e^{2t}+e^{-2t})^2}$$

$$= 2\left|e^{2t}+e^{-2t}\right| = 2\left(e^{2t}+e^{-2t}\right)$$

$$= 4\,\mathrm{Ch}(2t)$$

$$\Rightarrow \ell(\gamma) = \int_0^{\ln 2} 4\,\mathrm{Ch}(2t)\,dt = 2\,\mathrm{Sh}(2t)\Big|_0^{\ln 2}$$

$$= \left(e^{2t} - e^{-2t}\right)\Big|_0^{\ln 2} = \left[e^{2\ln 2} - e^{-2\ln 2}\right]$$

$$= 4 - \frac{1}{4} = \frac{15}{4}.$$

$$s(t) = \int_0^t 4\,\mathrm{Ch}(2\tau)\,d\tau = 2\,\mathrm{Sh}(2\tau)\Big|_0^t$$

$$= 2\,\mathrm{Sh}(2t)$$

$$\Rightarrow \quad t = \frac{1}{2} \, \mathrm{SettSh}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{s}{2} + \sqrt{\frac{s^2}{4}+1}\right)$$

$$\Rightarrow \quad \gamma(s): \begin{cases} x = x(s) = e^{\ln\left(\frac{s}{2}+\sqrt{\frac{s^2}{4}+1}\right)} = \frac{s}{2} + \sqrt{\frac{s^2}{4}+1} \\ y = y(s) = \sqrt{2}\,\ln\left(\frac{s}{2}+\sqrt{\frac{s^2}{4}+1}\right) \\ z = z(s) = e^{-\ln\left(\frac{s}{2}+\sqrt{\frac{s^2}{4}+1}\right)} = \dfrac{1}{\left[\frac{s}{2}+\sqrt{\frac{s^2}{4}+1}\right]} \end{cases}$$

Anche lungo e laborioso, è possibile effettuare il calcolo per verificare, come da teoria, che $v(s) = \|\vec{\gamma}'(s)\| = 1$.

6) La forma ω, definita in $\mathbb{R}^3$, NON è CHIUSA: $X_y = 1 \neq Y_x = -1$

$\Rightarrow$ non ammette potenziale $\Rightarrow$ L'integrale DIPENDE dalla curva scelta

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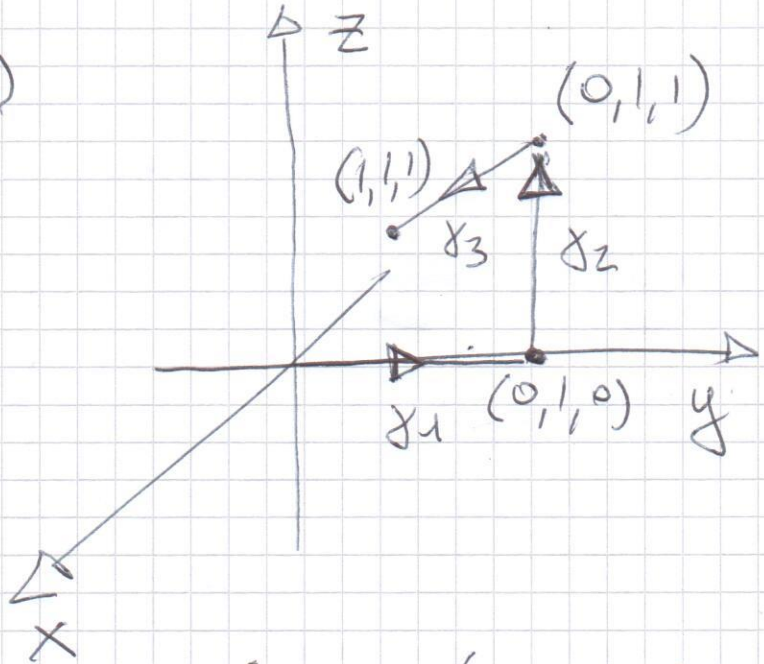
a)



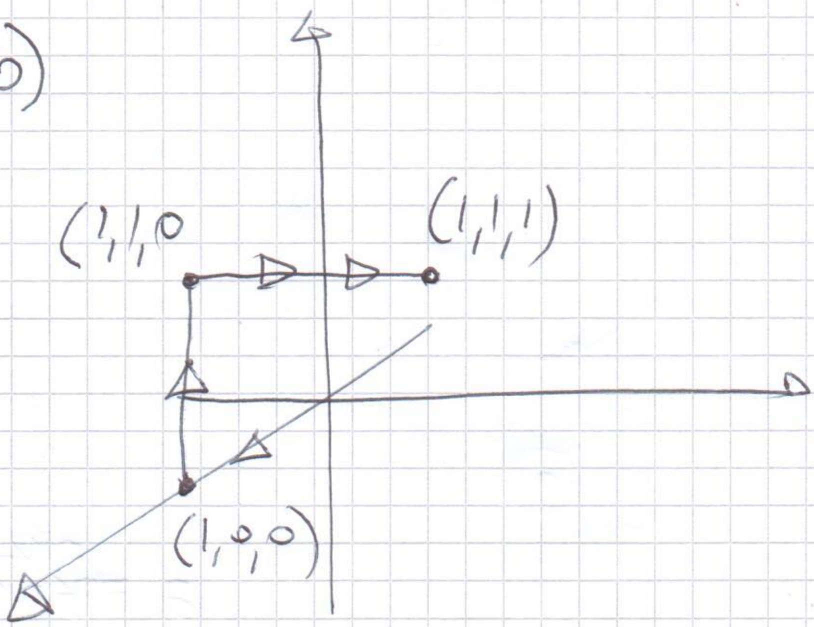
$$\gamma_1: \begin{cases} x=0 \\ y\in[0,1] \\ z=0 \end{cases} \qquad \gamma_2: \begin{cases} x=0 \\ y=1 \\ z\in[0,1] \end{cases} \qquad \gamma_3: \begin{cases} x\in[0,1] \\ y=1 \\ z=1 \end{cases}$$

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega =$$

$$= \int_0^1 y\,dy + \int_0^1 z\,dz + \int_0^1 2\,dx$$

$$= \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{z^2}{2}\right]_0^1 + 2x\Big|_0^1 = \frac{1}{2} + \frac{1}{2} + 2 = 3$$

b)



$$\gamma_1: \begin{cases} x\in[0,1] \\ y=0 \\ z=0 \end{cases} \qquad \gamma_2: \begin{cases} x=1 \\ y\in[0,1] \\ z=0 \end{cases} \qquad \gamma_3: \begin{cases} x=1 \\ y=1 \\ z\in[0,1] \end{cases}$$

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$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega =$$

$$= \int_0^1 0\,dx + \int_0^1 (y-1)\,dy + \int_0^1 (z-1)\,dz$$

$$= \frac{(y-1)^2}{2}\Big|_0^1 + \frac{(z-1)^2}{2}\Big|_0^1 = -\frac{1}{2} - \frac{1}{2} = -1.$$