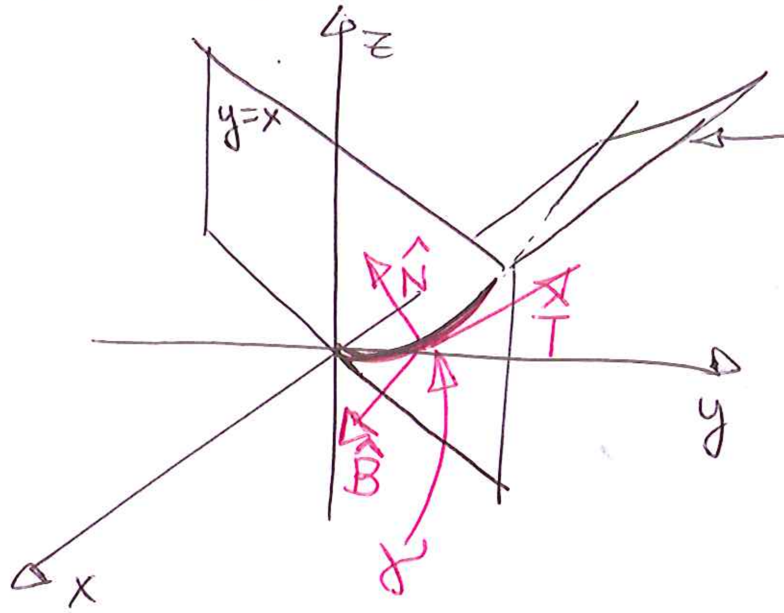
SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 12/4/2019

1) $\vec{r}\,'(t) = \begin{cases} x'(t) = \sqrt{2}t \\ y'(t) = \sqrt{2}t \\ z'(t) = 3t^2 \end{cases}$; $\vec{r}\,''(t) = \begin{cases} x''(t) = \sqrt{2} \\ y''(t) = \sqrt{2} \\ z''(t) = 6t \end{cases}$

la curva è piana, giacendo sul piano verticale $y = x$.
Poichè $t, x, y, z \geq 0$, ottengo anche che

$$z = t^3 = (\sqrt{2}x)^{\frac{3}{2}}$$

Quindi la curva è l'intersezione fra il piano $y = x$ e la superficie $z = (\sqrt{2}x)^{3/2}$



$\vec{r}(0) = (0, 0, 0)$

~~$\vec{r}(2) = (2\sqrt{2}, 2\sqrt{2},$~~

$\vec{r}(2) = (2\sqrt{2}, 2\sqrt{2}, 8)$

CURVA NON CHIUSA

CURVA SEMPLICE, PERCHÈ x, y, z sono CRESCENTI AL CRESCERE DI t.

N.B.: $\vec{r}$ NON È REGOLARE per $t = 0$.

$$v(t)=\sqrt{2(\sqrt{2}t)^2+9t^4}=\sqrt{4t^2+9t^4}$$

$$\ell(\gamma)=\int_0^2 |t|\sqrt{4+9t^2}\,dt=\int_0^2 t\sqrt{4+9t^2}\,dt$$

$$=\frac{1}{18}\int_0^2 18t\sqrt{4+9t^2}\,dt=\frac{1}{27}\left[(4+9t^2)^{\frac{3}{2}}\right]_0^2=$$

$$=\frac{1}{27}\left[(40)^{\frac{3}{2}}-4^{\frac{3}{2}}\right]=\frac{1}{27}\left[8(10)^{\frac{3}{2}}-8\right]$$

$$=\frac{8}{27}\left[(10)^{\frac{3}{2}}-1\right].$$

$$\cancel{k(t)=}\ \cancel{\|}\vec{r}\,'(t)\wedge\vec{r}\,''(t)\cancel{\|}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \sqrt{2}t & \sqrt{2}t & 3t^2\\ \sqrt{2} & \sqrt{2} & 6t\end{vmatrix}$$

$$=3\sqrt{2}t^2\,\vec{i}-3\sqrt{2}t^2\,\vec{j}$$

$$\Rightarrow\quad k(t)=\frac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{v^3(t)}=\frac{6t^2}{t^3\left(\sqrt{4+9t^2}\right)^3}$$

$$=\frac{6}{t(4+9t^2)^{3/2}}\quad\left(\text{formula valida } \forall t\neq 0\right)$$

Essendo la curva piana, $\tau(t) \equiv 0$.

$$\vec{B}(t) = \frac{\vec{r}\,'(t) \wedge \vec{r}\,''(t)}{\|\vec{r}\,'(t) \wedge \vec{r}\,''(t)\|} = \frac{1}{6t^2}\left(3\sqrt{2}\,t^2, -3\sqrt{2}\,t^2, 0\right)$$

$$= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \qquad \forall t \neq 0$$

come previsto dal fatto che, giacendo sul piano $y = x$ $\Rightarrow$ $\hat{B} = \pm \frac{(1,-1,0)}{\|(1,-1,0)\|}$

$$= \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

e che, noto dal grafico il verso di crescita di γ, occorreva considerare il versore con segno $+$.

2) Come noto, $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$ $\forall t \in \mathbb{R}$

$$\Rightarrow x e^{x^2} = \sum_{k=0}^{+\infty} \frac{x \cdot x^{2k}}{k!} = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{k!}$$

La serie, per le proprietà delle serie di potenze e di Taylor, ha raggio $R = +\infty$ e converge puntualmente a $f(x)$.

~~3)~~ Si può anche affermare che la serie converge anche assolutamente in tutto $\mathbb{R}$, mentre converge TOTALMENTE e UNIFORMEMENTE in ogni intervallo chiuso e limitato.

3) La funzione è α-~~POS~~OMOGENEA:

$$f_\alpha(xt, yt) = (xt)^\alpha \sin\left(\frac{\pi \cancel{t^2} xy}{\cancel{t^2}(x^2+y^2)}\right)$$

$$= t^\alpha \left[x^\alpha \sin\left(\frac{\pi xy}{x^2+y^2}\right)\right] = t^\alpha f_\alpha(x,y)$$

Se $\alpha > 0 \Rightarrow f_\alpha$ è prolungabile per continuità in $(0,0)$ e $\tilde{f}_\alpha(0,0) = 0$.

Se $\alpha \leq 0 \Rightarrow f_\alpha$ NON ammette limite e NON è prolungabile.

<u>Per $\alpha > 0$:</u>

Poiché $\tilde{f}_\alpha(0,y) = \tilde{f}_\alpha(x,0) = 0$

$\Rightarrow \vec{\nabla} \tilde{f}_\alpha(0,0) = \vec{0}$.

DIFFERENZIABILITÀ:

$f_\alpha(x,y)$ è $C^\infty(\mathbb{R}^2-\{(0,0)\})$, quindi differenziabile.

In $(0,0)$ f_α risulta differenziabile se $\alpha > 1$.

4) $\begin{cases} f_x = -12x^3 - 12y = -12(x^3+y) = 0 \\ f_y = -12x - 12y^5 = -12(x+y^5) = 0 \end{cases}$

$\begin{cases} y = -x^3 \\ x - x^{15} = x(1-x^{14}) = 0 \end{cases}$

$\Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \cup \begin{cases} x=1 \\ y=-1 \end{cases} \cup \begin{cases} x=-1 \\ y=1 \end{cases}$

$f_{xx} = -36x^2$; $f_{xy} = f_{yx} = -12$; $f_{yy} = -60y^4$

$H_f(0,0) = \begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}$

$|H_f(0,0)| = -144 < 0$ punto di sella

$f(0,0) = 10$

Infatti, ad esempio, lungo $y=x$

$$\Delta f\Big|_{y=x} = -3x^4-12x^2-2x^6 = -x^2\left(12+3x^2+2x^4\right)<0$$

lungo $y=-x$, per valori di $x\sim 0$,

$$\Delta f\Big|_{y=-x} = -3x^4+12x^2-2x^6 \sim 12x^2>0$$

$$H_f(\pm1,\mp1) = \begin{pmatrix} \overset{<0}{-36} & -12 \\ -12 & -60 \end{pmatrix} = 144\left[3\cdot5-1\right] = 14\cdot144>0$$

$P_1=(1,-1)$ e $P_2=(-1,1)$

$\Rightarrow$ punti di MAX. REL.

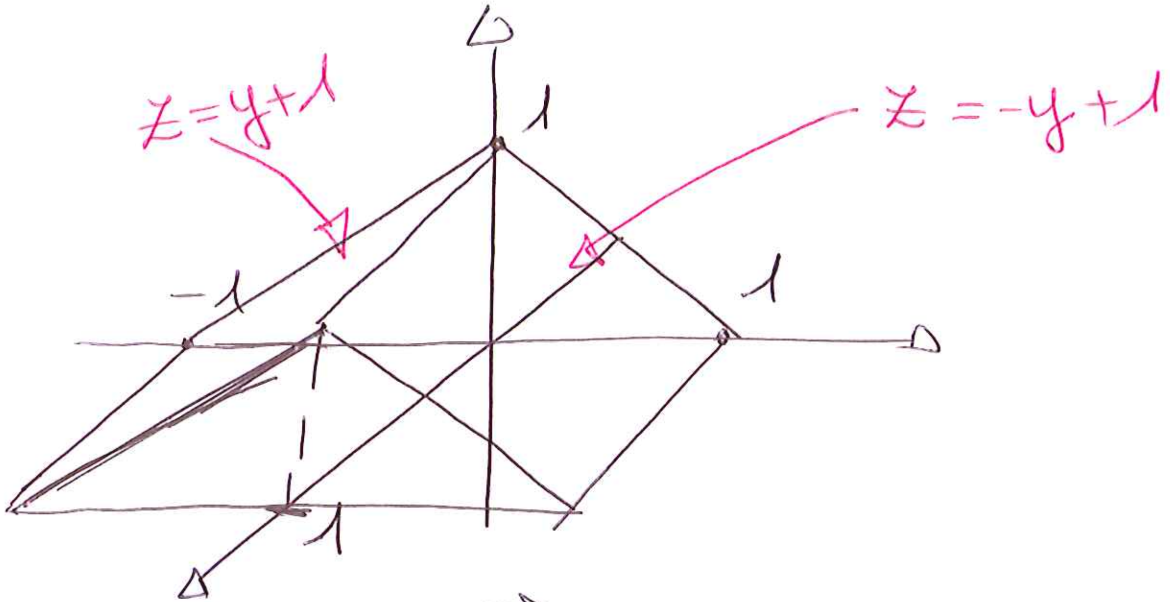
$$f(\pm1,\mp1) = -3+12-2+10=17.$$

La funzione è ILLIMITATA INF.MENTE, perché, ad esempio, lungo $y=x$

$$f(x,x) = -3x^4-12x^2-2\cancel{x}^6+10 \xrightarrow[x\to\pm\infty]{} -\infty$$

Si osserve che f è a SIMMETRIA RADIALE, perché $f(-x,-y)=f(x,y)$.

5) Il dominio è dato da



$$\Phi_{\vec{F}}(\partial D) = \iiint_D \operatorname{div} \vec{F} \, dx\,dy\,dz =$$

$$= \iiint_D \left[\pi \cos\left(\pi(x-1)\right) + 2z\right] dx\,dy\,dz$$

$$D = D_1 \cup D_2 = \left\{ 0 \le x \le 1 ;\ -1 \le y \le 0 ;\ 0 \le z \le y+1 \right\}$$

$$\cup \left\{ 0 \le x \le 1 ;\ 0 \le y \le 1 ;\ 0 \le z \le 1-y \right\}$$

oppure

$$= \left\{ 0 \le x \le 1 ;\ 0 \le z \le 1 ;\ -1+z \le y \le 1-z \right\}$$

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$$\iiint_D \left[\pi\cos(\pi(x-1)) + 2z\right] dx\,dy\,dz =$$

$$\int_0^1 dx \int_0^1 dz \int_{-1+z}^{1-z} dy \left[\pi\cos(\pi(x-1)) + 2z\right].$$

$$= \int_0^1 dx \int_0^1 dz \left\{\left[\pi\cos(\pi(x-1)) + 2z\right]\left(1-z-(-1+z)\right)\right\}$$

$$= 2\int_0^1 dx \int_0^1 dz\,(1-z)\left[\pi\cos(\pi(x-1)) + 2z\right]$$

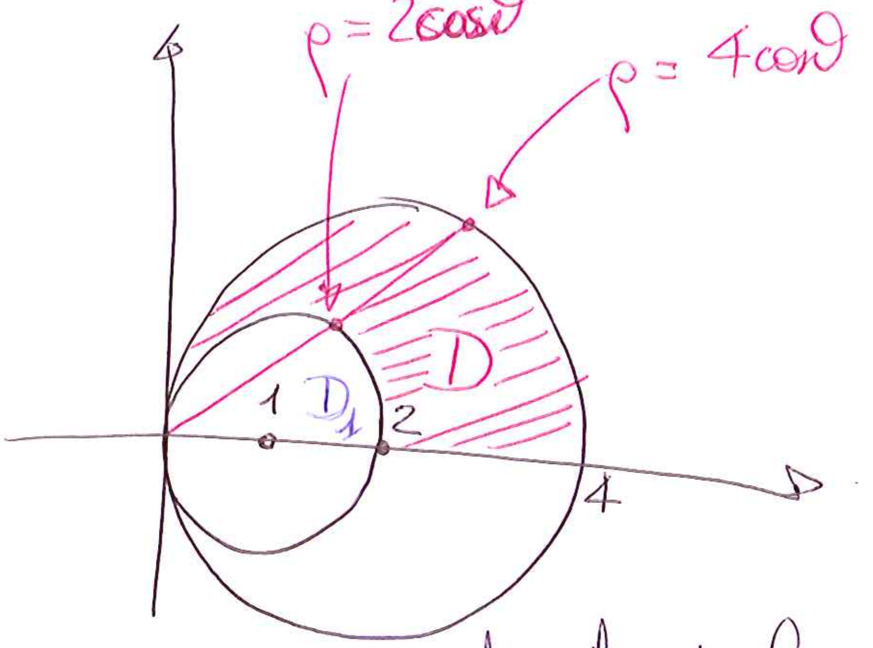
$$= 2\int_0^1 dz\,(1-z)\left[\sin(\pi(x-1))\right]_0^1$$

$$+ 2\int_0^1 dx \left[z^2 - \frac{2}{3}z^3\right]_0^1$$

$$= 2\left[z - \frac{z^2}{2}\right]_0^1 \underbrace{\left[-\sin(-\pi)\right]}_{=0} + 2\left[1 - \frac{2}{3}\right]$$

$$= \frac{2}{3}.$$

6)



Poniamoci in coordinate polari:

$$\begin{cases} (\rho\cos\vartheta - 2)^2 + \rho^2\sin^2\vartheta \leq 4 \\ (\rho\cos\vartheta - 1)^2 + \rho^2\sin^2\vartheta \geq 1 \\ \rho\sin\vartheta \geq 0 \end{cases}$$

$$\begin{cases} \rho^2 - 4\rho\cos\vartheta \leq 0 \\ \rho^2 - 2\rho\cos\vartheta \geq 0 \\ \sin\vartheta \geq 0 \end{cases} \Longrightarrow \vartheta \in [0, \pi]$$

$$\begin{cases} \rho \leq 4\cos\vartheta \\ \rho \geq 2\cos\vartheta \\ \vartheta \in [0, \pi] \end{cases}$$

ne segue che $\cos\vartheta \geq 0$

$$\begin{cases} 2\cos\vartheta \leq \rho \leq 4\cos\vartheta \\ \vartheta \in \left[0, \frac{\pi}{2}\right] \end{cases}$$

$$\Rightarrow \iint_D (x^2y+1)\,dx\,dy =$$

$$\int_0^{\frac{\pi}{2}} d\vartheta \int_{2\cos\vartheta}^{4\cos\vartheta} \rho\left[\rho^3\cos^2\vartheta\,\sin\vartheta+1\right]d\rho$$

$$= \int_0^{\frac{\pi}{2}} d\vartheta \left[\frac{\rho^5}{5}\cos^2\vartheta\,\sin\vartheta + \frac{\rho^2}{2}\right]_{2\cos\vartheta}^{4\cos\vartheta}$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{(4^5-2^5)}{5}\cos^7\vartheta\,\sin\vartheta + \frac{(16-4)}{2}\cos^2\vartheta\right]d\vartheta$$

$$\cancel{= \int_0^{\frac{\pi}{2}}\left[\frac{32\cdot 31}{5}\right.} = -\frac{\overset{4}{\cancel{32}}\cdot 31}{5}\cdot\frac{1}{\underset{1}{\cancel{8}}}\cos^8\vartheta\Big|_0^{\frac{\pi}{2}}$$

$$+6\int_0^{\frac{\pi}{2}}\cos^2\vartheta\,d\vartheta$$

$$= -\frac{124}{5}(-1) + 6\int_0^{\frac{\pi}{2}}\frac{1}{2}\left[\cos 2\vartheta+1\right]d\vartheta$$

$$= \frac{124}{5} + 3\left[\frac{1}{2}\sin 2\vartheta+\vartheta\right]_0^{\frac{\pi}{2}} = \frac{124}{5}+\frac{3}{2}\pi\,.$$

In alternativa, si sarebbe potuto calcolare l'integrale su D come differenza fra (11)
l'integrale sul semicerchio D_1 di centro $(0,2)$ e quello sul semicerchio D_2 di centro $(0,1)$, utilizzando per ogni integrale le coordinate decentrate

$$\begin{cases} x = 2+\rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} ;\ \vartheta\in[0,\pi]\ ;\ \begin{cases} x = 1+\rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \vartheta\in[0,\pi]$$

rispettivamente.

Consideriamo inoltre che

$$\iint_D 1\,dx\,dy = \text{Area } D = \text{Area } D_1 - \text{Area } D_2$$
$$= 2\pi - \frac{\pi}{2} = \frac{3}{2}\pi.$$

Inoltre,

$$\iint_D x^2 y\,dx\,dy = \iint_{D_1} x^2 y\,dx\,dy + \iint_{D_2} x^2 y\,dx\,dy$$

$$= \int_0^{\pi} d\vartheta \int_0^{2} \rho\left[(2+\rho\cos\vartheta)^2\right]\rho\sin\vartheta\,d\rho + \int_0^{\pi} d\vartheta \int_0^{1} \rho\left[(1+\rho\cos\vartheta)^2\right] \cdot\rho\sin\vartheta\,d\rho$$

$$= \int_0^\pi d\vartheta \int_0^2 \left[4\rho^2 \sin\vartheta + 4\rho^3 \cos\vartheta \sin\vartheta + \rho^4 \cos^2\vartheta \sin\vartheta\right] d\rho$$ 12

$$- \int_0^\pi d\vartheta \int_0^1 \left[\rho^2 \sin\vartheta + 2\rho^3 \cos\vartheta \sin\vartheta + \rho^4 \cos^2\vartheta \sin\vartheta\right] d\rho$$

$$= \int_0^\pi d\vartheta \left\{ \left[\frac{4}{3}\rho^3 \sin\vartheta + \rho^4 \cos\vartheta \sin\vartheta + \frac{\rho^5}{5} \cos^2\vartheta \sin\vartheta\right]_0^2 - \left[\frac{1}{3}\rho^3 \sin\vartheta + \frac{1}{2}\rho^4 \cos\vartheta \sin\vartheta + \frac{\rho^5}{5} \cos^2\vartheta \sin\vartheta\right]_0^1 \right\}$$

$$= \int_0^\pi d\vartheta \left[\left(\frac{32}{3} - \frac{1}{3}\right) \sin\vartheta + \left(16 - \frac{1}{2}\right) \cos\vartheta \sin\vartheta + \frac{1}{5}(32-1) \cos^2\vartheta \sin\vartheta\right]$$

$$= \left[-\frac{31}{3} \cos\vartheta \mp \frac{31}{4} \cos^2\vartheta - \frac{31}{15} \cos^3\vartheta\right]_0^\pi$$

$$= \frac{31}{3} \cdot 2 + \frac{31}{15} \cdot 2 = 62\left(\frac{1}{3} + \frac{1}{15}\right) = \frac{62 \cdot 6}{15} = \frac{124}{5}$$

$$\Rightarrow \iint_D (x^2 y + 1)\, dx\, dy = \frac{124}{5} + \frac{3}{2}\pi.$$