SVOLGIMENTO PROVA SCRITTA di ANALISI MAT. 2 del 12/9/2019

(1)

1) f funzione pari $\Rightarrow b_k = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx =$$

$$= \frac{2}{\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{4}{\pi}$$

$$\int_0^{\pi} |\sin x| \cos kx\,dx = \int_0^{\pi} \sin x \cdot \cos kx\,dx$$

$$= \left[-\cos x \cos(kx)\Big|_0^{\pi} - k\int_0^{\pi} \cos x \sin(kx)\,dx \right]$$

$$= \left[\cos(k\pi) + 1 - k\left[\cancel{\sin x \sin(kx)\Big|_0^{\pi}} - k\int_0^{\pi} \sin x \cos(kx)\,dx \right]\right.$$

$$= \left[(-1)^k + 1 + k^2 \int_0^{\pi} \sin x \cos(kx)\,dx \right]$$

$$\Rightarrow \int_0^{\pi} \sin x \cos(kx)\,dx = \frac{\left[(-1)^k + 1\right]}{1-k^2}$$

$$\Rightarrow a_k = \frac{2}{\pi}\int_0^{\pi} \sin x \cos(kx)\,dx =$$

$$= \begin{cases} 0 & \text{se } k = 2n+1 \\ \dfrac{4}{\pi\left[1-(2n)^2\right]} & \text{se } k = 2n \end{cases}$$

$$\Rightarrow f(x) \sim \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty} \frac{2}{\left[1-(2n)^2\right]} \cos(2nx)\right]$$

$f \in C^0(\mathbb{R})$ e regolare a tratti. Pertanto CONVERGENZA TOTALE in $\mathbb{R}$. La somma è, punto per punto, pari a $f(x) = |\sin x|$.

$$\Rightarrow |\sin 0| = \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty} \frac{2}{(1-(2n)^2)} \cos(0)\right]$$

$$\Rightarrow 1 + \sum_{n=1}^{\infty} \frac{2}{1-(2n)^2} = 0 \qquad \Rightarrow \underline{\sum_{n=1}^{\infty} \frac{1}{(2n)^2-1} = \frac{1}{2}}$$

2) $D = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0;\ y \neq 0\}$

Studiamo il limite per $(x,y) \to (0, y_0)$ (generico punto dell'asse y):

$$\lim_{(x,y)\to(0,y_0)} \frac{\sin(xy)}{xy} = \lim_{(x,y)\to(0,y_0)} \frac{xy}{xy} = 1$$

(in quanto xy è infinitesimo).

Analogamente, per il $\lim_{(x,y)\to(x_0,0)} f(x,y)$.

f è prolungabile su tutto $\mathbb{R}^2$:

$$\tilde{f}(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{se } x \in \underbrace{\mathbb{R}^2 - \left\{\{x=0\} \cup \{y=0\}\right\}}_{I} \\ 1 & \text{se } x=0 \text{ oppure } y=0. \end{cases}$$

Ovviamente $\tilde{f} \in C^0(\mathbb{R}^2)$

Derivabilità: in I $\tilde{f}$ è ovviamente derivabile e differenziabile.

$$\frac{\partial \tilde{f}}{\partial x} = \frac{y\cos(xy)\cdot xy - \sin(xy)y}{x^2y^2} = \frac{xy\cos(xy) - \sin(xy)}{x^2 y}$$

Per simmetria,

$$\frac{\partial \tilde{f}}{\partial y} = \frac{xy\cos(xy) - \operatorname{sen}(xy)}{xy^2}$$

<u>Lungo l'asse x:</u>

$$\left.\frac{\partial \tilde{f}}{\partial x}\right|_{(x_0,0)} = 0 \quad ; \quad \left.\frac{\partial \tilde{f}}{\partial y}\right|_{(x_0,0)} = \lim_{(x_0,y)\to(x_0,0)} \frac{f(x_0,y) - f(x_0,0)}{y}$$

$$= \lim_{(x_0,y)\to(x_0,0)} \frac{\frac{\operatorname{sen}(x_0 y)}{x_0 y} - 1}{y}$$

$$= \lim_{(x_0,y)\to(x_0,0)} \left[\frac{\left[x_0 y - \frac{(x_0 y)^3}{3!}\right]}{x_0 y} - 1\right]\frac{1}{y}$$

$$= \lim_{(x_0,y)\to(x_0,0)} \left[\cancel{1} - \frac{(x_0 y)^2}{6} \cancel{-1}\right]\frac{1}{y}$$

$$= \lim_{(x_0,y)\to(x_0,0)} \left[\frac{-x_0^2 y}{6}\right] = 0$$

Per simmetria, lungo l'asse y: (5)

$$\frac{\partial \tilde f}{\partial y}(0, y_0) = 0 \quad ; \quad \frac{\partial \tilde f}{\partial x}(0, y_0) = 0$$

Differenziabilità lungo asse x:

$$\lim_{(h,k)\to(0,0)} \left| \frac{\frac{\sin((x_0+h)k)}{(x_0+h)k} - 1}{\sqrt{h^2+k^2}} \right| =$$

$$\lim_{(h,k)\to(0,0)} \left| \frac{-(x_0+h)^{\cancel{3}\,2} k^{\cancel{3}\,2}}{\cancel{(x_0+h)k}\cdot 3!} \cdot \frac{1}{\sqrt{h^2+k^2}} \right|$$

$$= \lim_{\rho\to 0} \left| \frac{-(x_0+\rho\cos\vartheta)\,\rho^{\cancel{2}}\sin^2\vartheta}{6\cancel{\rho}} \right|$$

$$\leq \lim_{\rho\to 0} \frac{1}{6}\rho\,|x_0+\rho\cos\vartheta| \leq \lim_{\rho\to 0}\frac{1}{6}\rho(|x_0|+\rho) = 0.$$

$\tilde f$ differenziabile $\forall (x_0, 0)$ e, per simmetria, $\forall (0, y_0)$.

3) Punti stazionari: $\begin{cases} f_x = 2x+y=0 \\ f_y = x-6y=0 \end{cases}$

$\iff (x,y)=(0,0)$.

Poiché $f_{xx}=2$; $f_{xy}=f_{yx}=1$; $f_{yy}=-6$

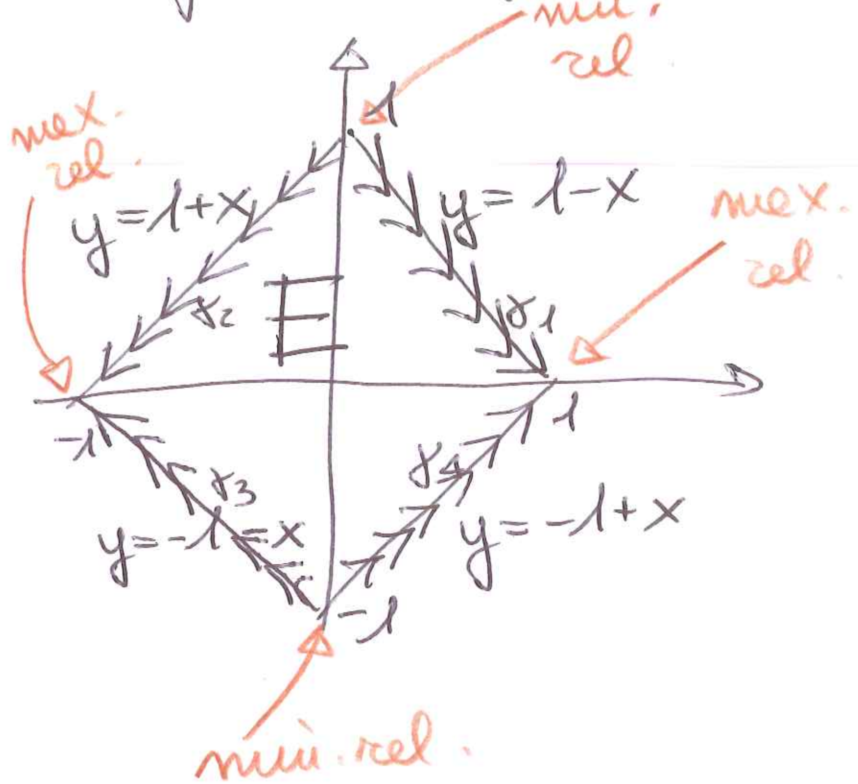
$\Rightarrow H_f(0,0)=\begin{vmatrix} 2 & 1 \\ 1 & -6 \end{vmatrix} = -14$

$\Rightarrow (0,0)$ punto di sella.

Infatti, $f(0,0)=0$

lungo l'asse x $\quad f(x,0)=x^2 \geq 0$

lungo l'asse y $\quad f(0,y) -3y^2 \leq 0$



$\gamma_1: \begin{cases} x=t\in[0,1] \\ y=1-t \end{cases}$

$\gamma_2: \begin{cases} x=t\in[-1,0] \\ y=1+t \end{cases}$

$\gamma_3: \begin{cases} x=t\in[-1,0] \\ y=-1-t \end{cases}$

$\gamma_4: \begin{cases} x=t\in[0,1] \\ y=-1+t \end{cases}$

$$f|_{\gamma_1} = t^2 + t(1-t) - 3(1-t)^2 \qquad \text{parabola}$$

$$= \cancel{t^2} + t - \cancel{t^2} - 3 - 3t^2 + 6t = -3t^2 + 7t - 3 \qquad 7$$

$$t \in [0,1]$$

$$\left(f|_{\gamma_1}\right)' = -6t + 7 > 0 \quad \forall t < \frac{7}{6}$$

$$\Rightarrow f|_{\gamma_1} \text{ ~~decresce~~ cresce } \forall t \in [0,1]$$

$$f|_{\gamma_2} = t^2 + t(1+t) - 3(1+t)^2 = 2t^2 + t - 3 - 3t^2 - 6t$$

$$= -t^2 - 5t - 3 \qquad \text{parabola}$$

$$t \in [-1,0]$$

$$\left(f|_{\gamma_2}\right)' = -2t - 5 > 0 \quad \forall t < -\frac{5}{2}$$

$$\Rightarrow f|_{\gamma_2} \text{ decresce } \forall t \in [-1,0]$$

$$f|_{\gamma_3} = t^2 + t(-1-t) - 3(-1-t)^2 =$$

$$-t - 3 - 3t^2 - 6t = -3t^2 - 7t - 3$$

$$t \in [-1,0] \qquad \text{parabola}$$

$$\left(f|_{\gamma_3}\right)' = -6t - 7 > 0$$

$$\forall t < -\frac{7}{6}$$

$$\Rightarrow f|_{\gamma_3} \text{ decresce } \forall t \in [-1,0]$$

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$$f\big|_{\gamma_4} = t^2 + t(-1+t) - 3(-1+t)^2 = \cancel{2t}$$
$$= 2t^2 - t - 3 - 3t^2 + 6t = \cancel{} - t^2 + 5t - 3$$

parabola $t \in [0,1]$.

$$\left(f\big|_{\gamma_4}\right)' = -2t + 5 > 0 \quad \forall t < \frac{5}{2}$$
$$\Rightarrow f\big|_{\gamma_4} \text{ cresce } \forall t \in [0,1]$$

$f(-1,0) = f(1,0) = 1$ MAX. ASS.

$f(0,-1) = f(0,1) = -3$ MIN. ASS.

4) $D = \{(x,y) \in \mathbb{R}^2 \mid y > -1\}$

semipiano aperto, sempl. lin. connesso.

ω ~~NON~~ è CHIUSA :

$$X_y = \frac{1}{1+y} = Y_x \Rightarrow \omega \text{ esatta in } D.$$

$$V(x,y) = \int Y(x,y)\,dy =$$

$$\int \frac{(1+x)}{(1+y)}\,dy = (1+x)\log(1+y) + \varphi(x)$$

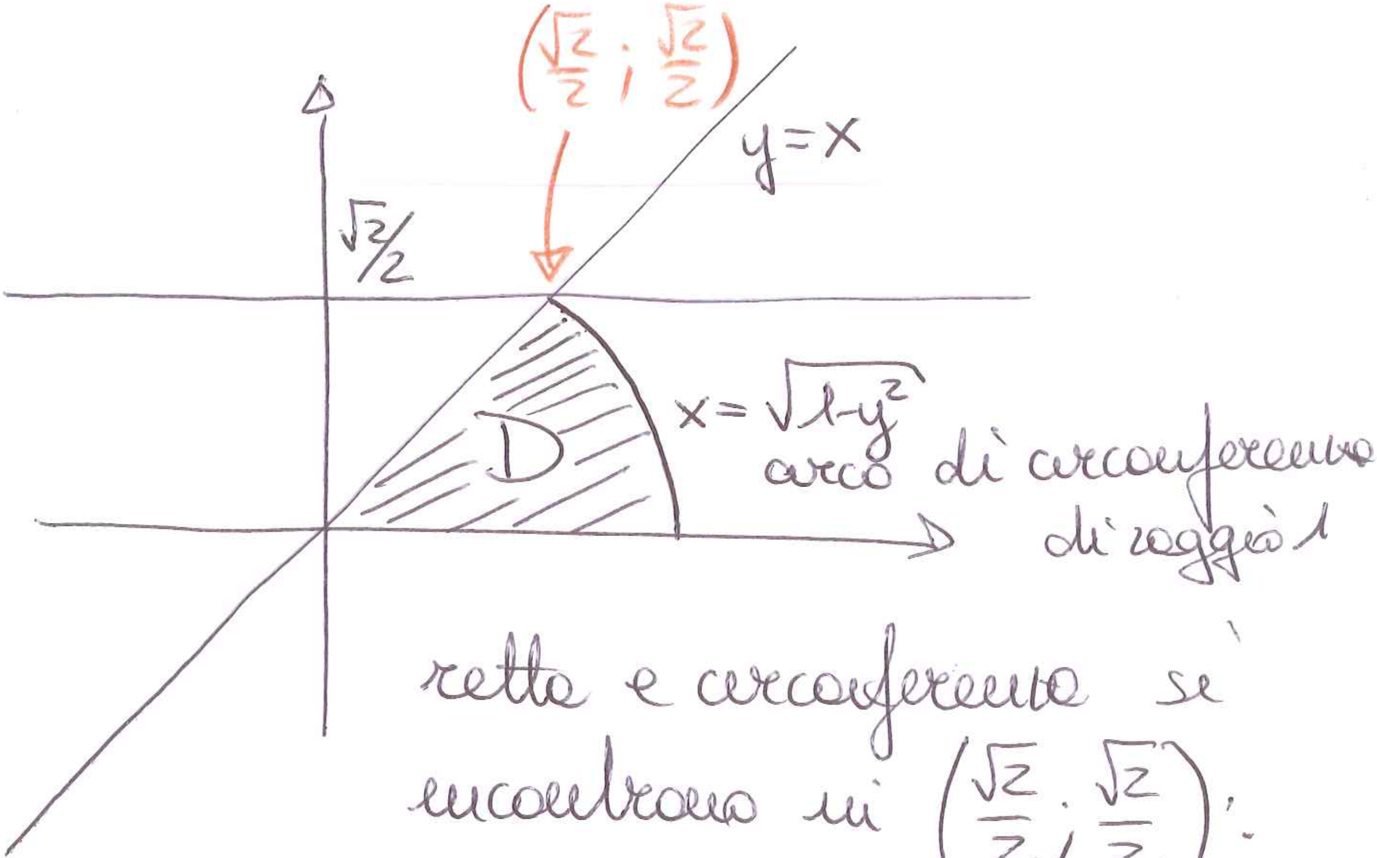
$$V_x = \log(1+y) + \varphi'(x) = X = \log(1+y)$$

$$\Rightarrow \varphi'(x) = 0 \quad \Rightarrow \varphi(x) = C.$$

$\Rightarrow$ tutte le primitive sono della forma

$$\underline{V(x,y) = (1+x)\log(1+y) + C};$$

5)



retta e circonferenza si incontrano in $\left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$.

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$$0 \le y = \sqrt{1-y^2} \Rightarrow y^2 = 1-y^2 \Rightarrow y^2 = \frac{1}{2}$$

$$\Rightarrow y = \frac{\sqrt{2}}{2} \quad \left(\begin{array}{l}\text{sono} \\ \text{compatibili} \\ \text{solo se } y \ge 0\end{array}\right).$$

$$f(x,y) = |x+y| = \begin{cases} x+y & \text{se } y \ge -x \\ -(x+y) & \text{se } y < -x \end{cases}$$

Ma ogni punto di D è tale che $y \ge -x$

$$\Rightarrow \iint_D f(x,y)\,dx\,dy = \int_0^{\sqrt{2}/2} dy \int_y^{\sqrt{1-y^2}} (x+y)\,dx$$

$$= \int_0^{\sqrt{2}/2} dy \left(\frac{x^2}{2} + yx\right)\Big|_y^{\sqrt{1-y^2}} = \int_0^{\frac{\sqrt{2}}{2}} dy \left[\frac{1-y^2}{2} + y\sqrt{1-y^2} - \frac{y^2}{2} - y^2\right]$$

$$= \int_0^{\frac{\sqrt{2}}{2}} \left[\frac{1}{2} - 2y^2 + y\sqrt{1-y^2}\right] dy$$

$$= \frac{1}{2}y - \frac{2}{3}y^3 - \frac{1}{3}(1-y^2)^{\frac{3}{2}} \Big|_0^{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{4} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \frac{1}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{1}{3}$$

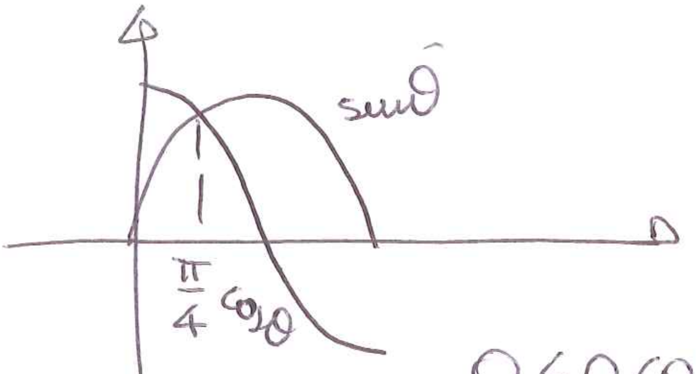
$$= \frac{\sqrt{2}}{4} - \frac{\cancel{4}\sqrt{2}}{\cancel{24}6} - \frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{1}{3}$$

$$= \frac{6-4-2+4\sqrt{2}}{12\sqrt{2}} = \frac{4\sqrt{2}}{12\sqrt{2}} = \frac{1}{3}.$$

Altrimenti, in coordinate polari: dalla figura si osserva che $\rho \in [0,1]$, $\vartheta \in \left[0, \frac{\pi}{4}\right]$.

Infatti $0 \leq y \leq x \leq \sqrt{1-y^2}$

$$\Rightarrow \underbrace{0 \leq \rho \sin\vartheta}_{\vartheta \in [0,\pi]} \underbrace{\leq \rho\cos\vartheta}_{0 \leq \sin\vartheta \leq \cos\vartheta} \leq \sqrt{1-\rho^2\sin^2\vartheta}$$



$$\Rightarrow \vartheta \in \left[0, \frac{\pi}{4}\right]$$

Inoltre

$$0 \leq \rho\cos\vartheta \leq \sqrt{1-\rho^2\sin^2\vartheta}$$

$$\Rightarrow \rho^2\cos^2\vartheta \leq 1 - \rho^2\sin^2\vartheta$$

$$\Rightarrow \rho^2 \leq 1 \Rightarrow \rho \in [0,1].$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{4}} d\theta \int_0^1 \rho\left(\rho\cos\theta + \rho\,\mathrm{sen}\,\theta\right) d\rho$$

$$= \int_0^{\frac{\pi}{4}} (\cos\theta + \mathrm{sen}\,\theta)\,d\theta \cdot \int_0^1 \rho^2\,d\rho$$

$$= \Big[\mathrm{sen}\,\theta - \cos\theta\Big]_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3}\right]_0^1$$

$$= \left(\cancel{\frac{\sqrt{2}}{2}} - \cancel{\frac{\sqrt{2}}{2}} + 1\right) \cdot \frac{1}{3} = \frac{1}{3}.$$