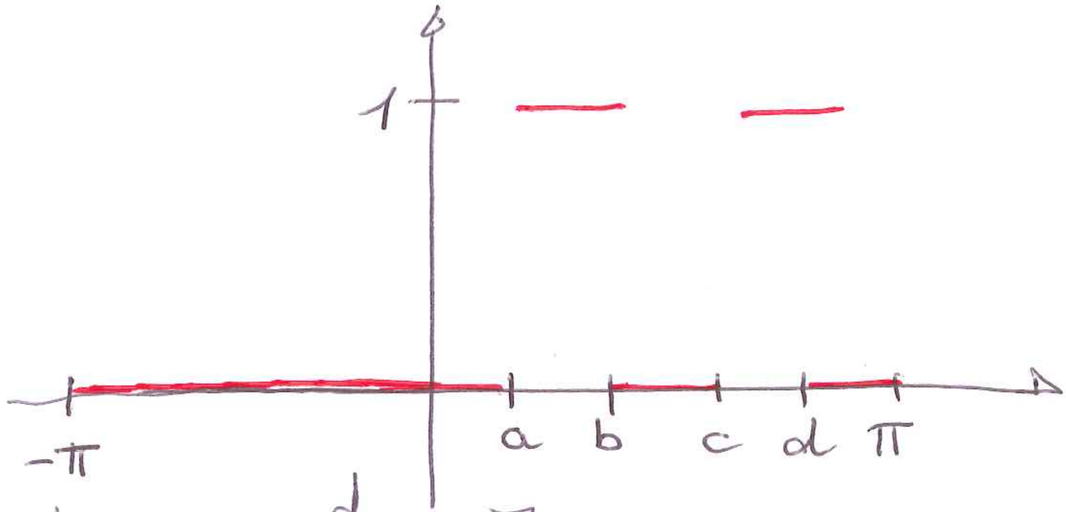
SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 13/9/2018.

1)



$$a_0 = \frac{1}{\pi}\left[\int_a^b 1\,dx + \int_c^d 1\,dx\right]$$

$$= \frac{b+d-(a+c)}{\pi}$$

$$a_k = \frac{1}{\pi}\left[\int_a^b \cos(kx)\,dx + \int_c^d \cos(kx)\,dx\right]$$

$$= \frac{1}{\pi k}\left[\operatorname{sen}(kb) + \operatorname{sen}(kd) - \sin(ka) - \sin(kc)\right]$$

$$b_k = \frac{1}{\pi}\left[\int_a^b \operatorname{sen}(kx)\,dx + \int_c^d \sin(kx)\,dx\right]$$

$$= \frac{1}{\pi k}\left[\cos(ka) + \cos(kc) - \cos(kb) - \cos(kd)\right]$$

la serie converge a $f(x)$ $\forall x \in [-\pi, \pi]$, $x \neq a, b, c, d$ e converge a $\frac{1}{2}$ in $x = a, b, c, d$.

$$S(x) = \begin{cases} f(x) & x \in [-\pi, \pi] - \{a, b, c, d\} \\ \frac{1}{2} & x \in \{a, b, c, d\} \end{cases}$$

Convergenza uniforme in ogni compatto contenuto negli intervalli di continuità.
No convergenza totale.

2) la funzione $g(x,y) = \dfrac{x}{\log(x^2+y^2)}$

è definita per $\begin{cases} x^2+y^2 > 0 \\ x^2+y^2 \neq 1 \end{cases} \Rightarrow \begin{cases} (x,y) \neq (0,0) \\ x^2+y^2 \neq 1 \end{cases}$.

Pertanto, $f(x,y)$ è definita in $\mathbb{R}^2 - \{x^2+y^2 = 1\}$.

Studiamo la continuità in $(0,0)$.

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{\rho\to 0} \left|\frac{\rho\cos\vartheta}{\log(\rho^2)}\right|$$

$$\leq \lim_{\rho\to 0} \frac{\rho}{|\log(\rho^2)|} = \frac{0^+}{+\infty} = 0 \qquad \forall \vartheta$$

$\Rightarrow$ f continua in $(0,0)$.

la funzione è inoltre continua negli altri punti del suo insieme di definizione, nonché C^∞, quindi in particolare differenziabile, perché tale è $g(x,y) = \dfrac{x}{\log(x^2+y^2)}$.

~~Si osservi per inciso~~

In particolare, anche se non esplicitamente richiesto, si ha che $\forall (x,y) \in I_{def}$; $(x,y) \neq (0,0)$, ③

$$f_x = \frac{\log(x^2+y^2) - \frac{2x^2}{x^2+y^2}}{\log^2(x^2+y^2)} \; ; \quad \cancel{f_y = \log}$$

$$f_y = \frac{-2xy}{\left[\log^2(x^2+y^2)\right](x^2+y^2)}$$

In $(0,0)$: $\quad f(0,y) \equiv 0 \quad \Rightarrow \frac{\partial f}{\partial y}(0,0) = 0.$

$$f_x(0,0) = \lim_{x \to 0} \frac{x}{x \log(x^2)} = \lim_{x \to 0} \frac{1}{\log(x^2)} = 0$$

$\Rightarrow f$ derivabile in I_{def}.

DIFFERENZIABILITÀ in $(0,0)$:

$$\lim_{(h,k) \to (0,0)} \left| \frac{\frac{k}{\log(h^2+k^2)}}{\sqrt{h^2+k^2}} \right| = \lim_{\rho \to 0} \left| \frac{\rho \cos\theta}{\rho \log(\rho^2)} \right|$$

$$\leq \lim_{\rho \to 0} \frac{1}{|\log(\rho^2)|} = 0$$

$\Rightarrow$ DIFFERENZIABILE in $(0,0)$

$\Rightarrow$ " in I_{def}.

$$\Rightarrow \quad \forall (x,y) \in I_{df} ; \forall \vec{r} = (\alpha, \beta)$$

$$\frac{df}{d\vec{r}}(x,y) = \vec{r} \cdot \vec{\nabla} f = \alpha f_x(x,y) + \beta f_y(x,y)$$

In particolare, in $(0,0)$

$$\frac{df}{d\vec{r}}(0,0) = 0.$$

3) All'interno di Q:

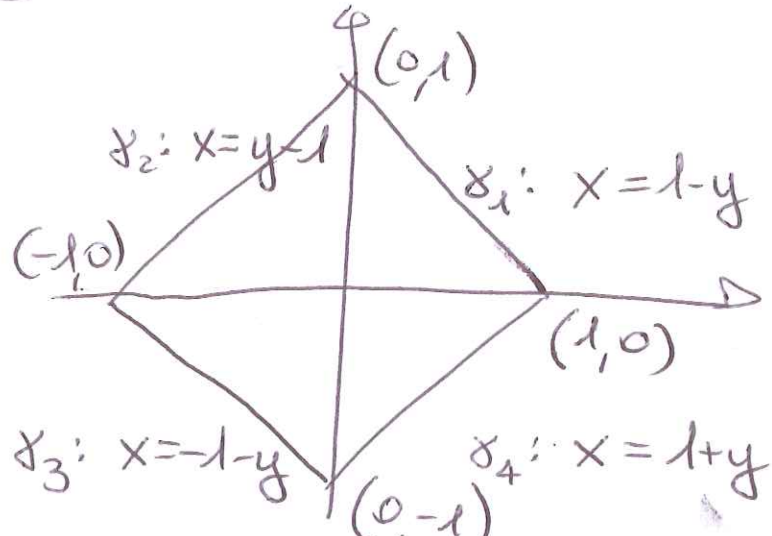
$$f_x = y \; ; \quad f_y = x - \frac{3}{2} y$$

$\Rightarrow$ unico punto stazionario $(0,0)$.

$$H_f(0,0) = \begin{vmatrix} 0 & 1 \\ 1 & -\frac{3}{2} \end{vmatrix} = -1 < 0$$

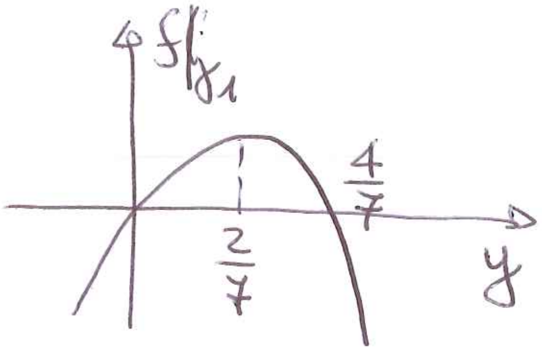
$\Rightarrow$ punto di sella.

Sulla frontiera:



$$f\big|_{\gamma_1} = (1-y)y - \frac{3}{4}y^2 = y - \frac{7}{4}y^2$$

Parabola rivolta verso il basso.



Cresce per $y \in \left[0, \frac{2}{7}\right)$;

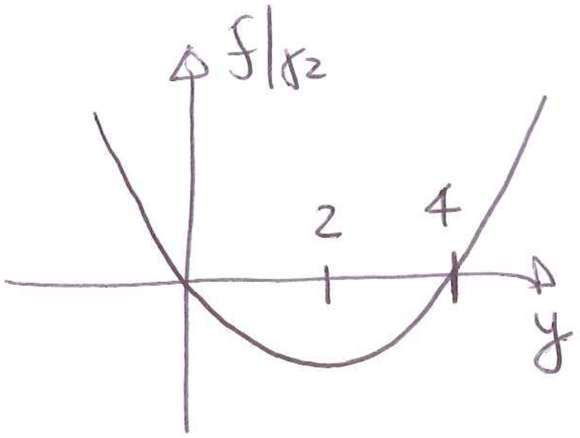
decresce per $y \in \left(\frac{2}{7}, 1\right]$.

$P_1 = \left(\frac{5}{7}, \frac{2}{7}\right)$ PUNTO DI MASSIMO RELATIVO:

$$f(P_1) = \frac{10}{49} - \frac{3}{4}\left(\frac{2}{7}\right)^2 = \frac{1}{7}$$

$$f\big|_{\gamma_2} = y(y-1) - \frac{3}{4}y^2 = \frac{1}{4}y^2 - y$$

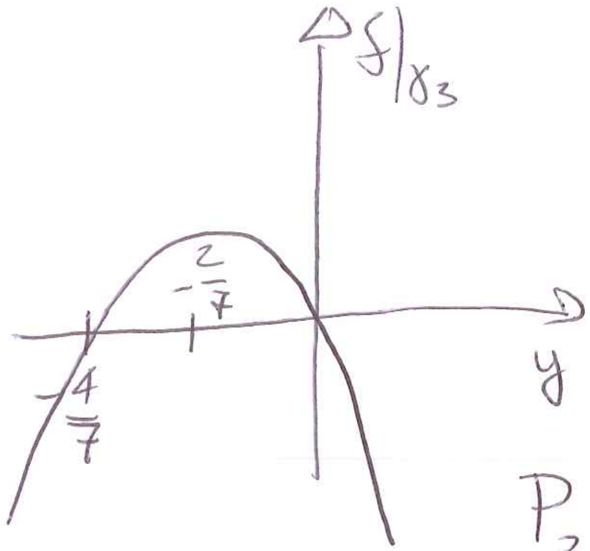
Parabola rivolta verso l'alto.



$f\big|_{\gamma_2}$ decrescente in $[0,1]$

$$f\big|_{\gamma_3} = (-1-y)y - \frac{3}{4}y^2 = -\frac{7}{4}y^2 - y .$$

~~come per~~ $\cancel{f\big|_{\gamma_2}}$ ~~:~~ $\cancel{f\big|_{\gamma_3}}$ ~~decrescente in~~ $\cancel{[-1, 0]}$.



Parabola rivolta verso il basso.

$f|_{\gamma_3}$ cresce in $\left[-1,-\frac{2}{7}\right)$ e decresce in $\left(-\frac{2}{7},0\right]$

$P_3=\left(-\frac{5}{7},-\frac{2}{7}\right)$ PUNTO DI MASSIMO RELATIVO. $f(P_3)=f(P_1)=\frac{1}{7}$

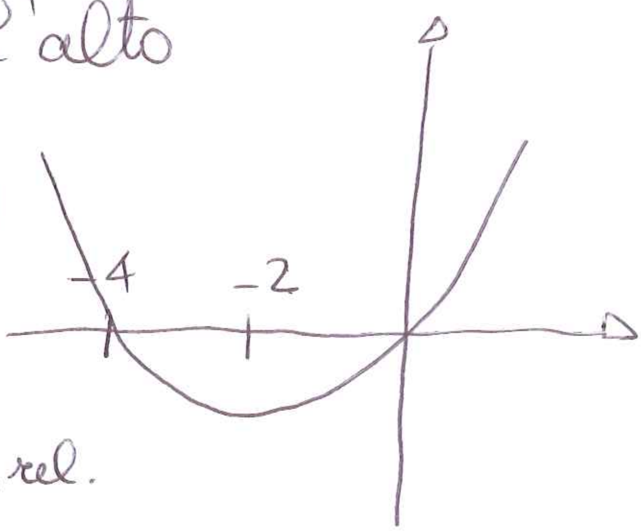
N.B.: f è dotata di simmetria radiale:

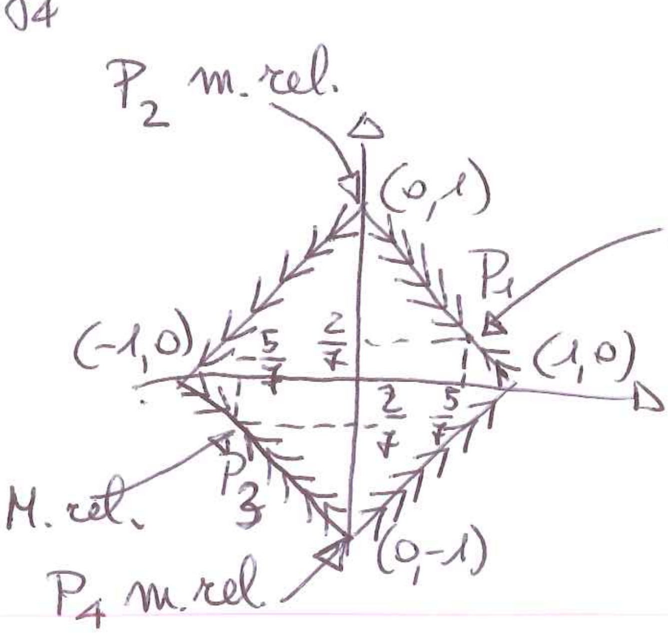
$$f(-x,-y)=(-x)\cdot(-y)-\frac{3}{4}(-y)^2=xy-\frac{3}{4}y^2$$
$$=f(x,y).$$

$$f|_{\gamma_4}=(1+y)y-\frac{3}{4}y^2=\frac{1}{4}y^2+y$$

Parabola rivolta verso l'alto

$f|_{\gamma_4}$ crescente in $[-1,0]$





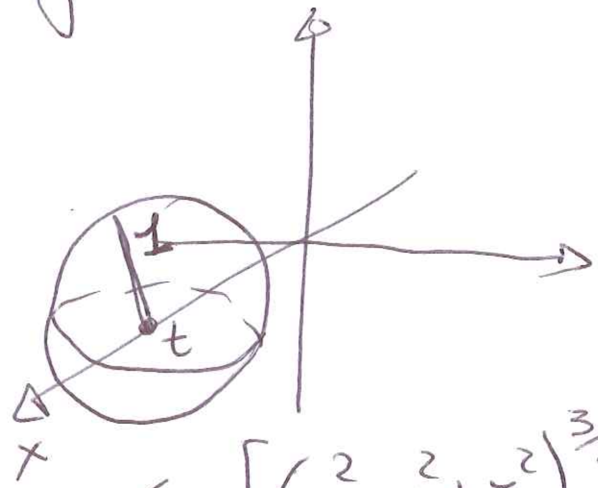
Dal disegno si può apprezzare il fatto che $P_2 = (0,1)$ e $P_4 = (0,-1)$ sono punti di MIN. REL. $f(P_2) = f(P_4) = -\frac{3}{4}$

P_1, P_3 punti di MAX. ASS.

P_2, P_4 " " MIN. ASS.

4) $S_t = \{(x-t)^2 + y^2 + z^2 = 1\}$

Sfera centrata in $(t,0,0)$ e di raggio 1.



Applichiamo il Teorema della Divergenza:

$$\operatorname{div}\vec{F} = \frac{\left[\begin{array}{l}(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2} \\ + (x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2} \\ + (x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}\end{array}\right]}{(x^2+y^2+z^2)^3}$$

$$= \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$ 8

$$\Rightarrow \Phi_{S_t}(\vec{F}) = \iiint_{E_t} 0 \, dx\,dy\,dz = 0$$

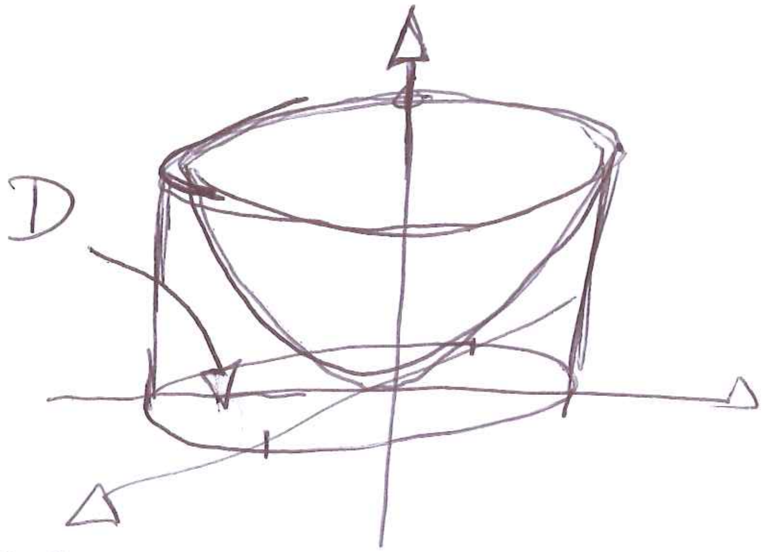
E_t ← dominio sferico ~~di~~ con frontiera S_t

$$\Rightarrow \lim_{t \to 1^+} \Phi_{S_t}(\vec{F}) = \lim_{t \to 1^+} \iint_{S_t} \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \lim_{t \to 1^+} 0 = 0.$$

5)

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, 4x^2+y^2 \le 1 \right\}$$



Integriamo per fili:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = -\iint_D \left[4x^2+y^2-1\right] \cdot \left[(1+x)y^2 + x^2\right] dx\,dy$$

Coordinate ellittiche:

$$\begin{cases} x = \frac{1}{2}\rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

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$$= -\int_0^1 \frac{1}{2}\rho\, d\rho \int_0^{2\pi} (\rho^2-1)\left[\left(1+\frac{1}{2}\rho\cos\vartheta\right)\rho^2\sin^2\vartheta + \frac{1}{4}\rho^2\cos^2\vartheta\right] d\vartheta$$

$$= -\int_0^1 \frac{1}{2}\rho^3(\rho^2-1)\, d\rho \int_0^{2\pi} \left[\left(1+\frac{1}{2}\rho\cos\vartheta\right)\sin^2\vartheta + \frac{1}{4}\cos^2\vartheta\right] d\vartheta$$

$$= -\int_0^1 \frac{1}{2}\rho^3(\rho^2-1)\, \cancel{d\rho} \left[\int_0^{2\pi}\left[\sin^2\vartheta + \frac{1}{2}\rho\sin^2\vartheta\cos\vartheta + \frac{1}{4}\cos^2\vartheta\right] d\vartheta\right] d\rho$$

$$= -\int_0^1 \frac{1}{2}\rho^3(\rho^2-1)\left[\int_0^{2\pi}\left[\frac{1-\cos 2\vartheta}{2} + \frac{1+\cos 2\vartheta}{8}\right] d\vartheta + \cancel{\frac{1}{2}\rho\frac{\sin^3\vartheta}{3}\Big|_0^{2\pi}}\right] d\rho$$

$$= -\frac{1}{2}\int_0^1 \rho^3(\rho^2-1)\left[\int_0^{2\pi}\left(\frac{5}{8} - \frac{3}{8}\cos 2\vartheta\right) d\vartheta\right] d\rho$$

$$= -\frac{1}{2}\left[\frac{\rho^6}{6} - \frac{\rho^4}{4}\right]_0^1 \left[\frac{5}{8}\vartheta - \frac{3}{16}\sin 2\vartheta\right]_0^{2\pi}$$

$$= +\frac{1}{24}\left[\frac{5}{\cancel{8}_4}\cancel{2}\pi\right] = \cancel{\frac{-5}{96}} + \frac{5}{96}\pi .$$