SVOLGIMENTO PROVA SCRITTA di ANALISI 2 del 15/7/2021

1

1) $$f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$\Rightarrow$ la serie di McLaurin è

$$\frac{1}{2} - \frac{1}{2}\sum_{k=0}^{+\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!}$$

$$= \frac{1}{2} - \frac{1}{2}\left[1 + \sum_{k=1}^{+\infty} \frac{(-1)^k 4^k}{(2k)!} x^{2k}\right]$$

$$= \frac{1}{2}\sum_{k=1}^{+\infty} \frac{(-1)^{k+1} 4^k}{(2k)!} x^{2k}$$

La serie del coseno converge assolutamente e puntualmente su tutto $\mathbb{R}$. La somma della serie è la funzione stessa. Inoltre converge totalmente e quindi uniformemente in ogni compatto.

2) La curva è la cicloide

$$\begin{cases} x'(t) = R(1-\cos t) \\ y'(t) = R\sin t \end{cases}$$

$$v(t) = R\sqrt{(1-\cos t)^2 + \sin^2 t}$$

$$= R\sqrt{2-2\cos t} = \sqrt{2}\,R\sqrt{1-\cos t}$$

$$\int_\gamma \sqrt{2y}\,ds = \cancel{\int_0^\pi \sqrt{2}\sqrt{R\sin t}\,\sqrt{2}\,R}$$

$$= \sqrt{2}\int_0^\pi \sqrt{2}\,R\sqrt{1-\cos t}\cdot\sqrt{R(1-\cos t)}\,dt$$

$$= 2R\sqrt{R}\int_0^\pi |1-\cos t|\,dt = 2R^{\frac{3}{2}}\int_0^\pi (1-\cos t)\,dt$$

$$= 2R^{3/2}\left[t-\sin t\right]_0^\pi = 2R^{\frac{3}{2}}\pi$$

3)

3

$$f_x = -3y^2 - 3x^2 = -3(x^2+y^2)$$

$$f_y = 4y^3 - 6xy$$

$f_x = 0 \iff (x,y) = (0,0) \Rightarrow f_y = 0$

$\Rightarrow (0,0)$ unico punto stazionario.

Poichè $f(x,0) = -x^3 \begin{cases} >0 & \text{se } x<0 \\ <0 & \text{se } x>0 \end{cases}$

$\Rightarrow (0,0)$ punto di sella.

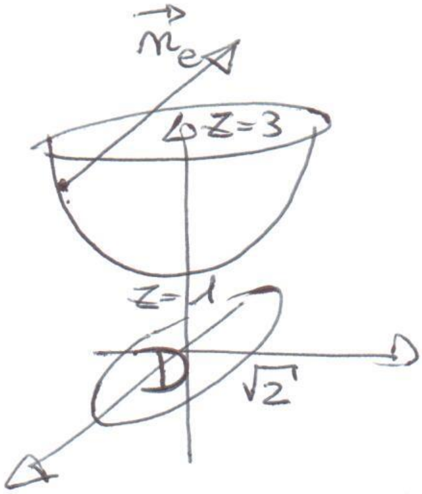
Inoltre $\lim\limits_{x \to \pm\infty} f(x,0) = \lim\limits_{x \to \pm\infty} (-x^3) = \mp\infty$

$\Rightarrow$ FUNZIONE ILLIMITATA ($\nexists$ MAX-MIN ASSOLUTI)

4) ④

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & (x-1) & (2y+1) \end{vmatrix}$$

$$= 2\vec{i} + \vec{k}.$$



Superficie: $S: \begin{cases} z = 1 + x^2 + y^2 \\ z \leq 3 \Rightarrow x^2 + y^2 \leq 2 \end{cases}$

$$J = \begin{pmatrix} 1 & 0 & 2x \\ 0 & 1 & 2y \end{pmatrix}$$

$$\Rightarrow L = -2x; \quad M = -2y; \quad N = 1$$

$$\left.(\vec{\nabla} \wedge \vec{F}) \cdot \vec{n}_e = -4x + 1 \right|_S$$

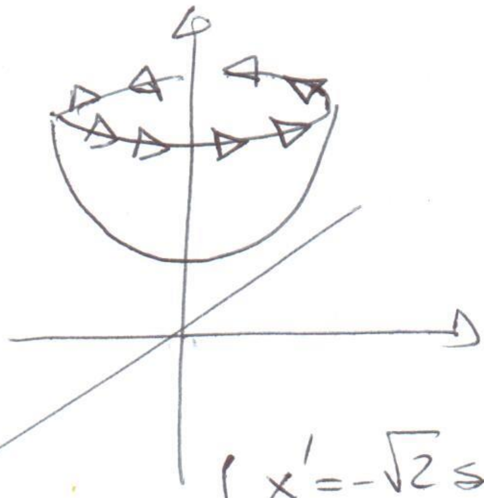
$$\Phi_S(\vec{\nabla} \wedge \vec{F}) = \iint\limits_{x^2+y^2 \leq 2} (-4x+1)\,dx\,dy$$

$$= -4 \underbrace{\iint\limits_{x^2+y^2 \leq 2} x\,dx\,dy}_{=0} + \text{area}(D) = 2\pi.$$

Attraverso il Teorema di Stokes: 5

Per la scelta di $\vec{n}_e$, la faccia esterna volge lo sguardo verso l'alto

$\Rightarrow$ BS va percorso in verso ANTI-ORARIO



$$BS: \begin{cases} x = \sqrt{2}\cos\vartheta \\ y = \sqrt{2}\sin\vartheta \\ z = 3 \end{cases}$$

$$\begin{cases} x' = -\sqrt{2}\sin\vartheta \\ y' = \sqrt{2}\cos\vartheta \\ z' = 0 \end{cases}$$

$$\Phi_S(\vec{\nabla} \wedge \vec{F}) = \int_{BS^+} \vec{F} \cdot \hat{T}\, ds$$

$$= \int_0^{2\pi} \left[(\sqrt{2}\cos\vartheta - 1)\sqrt{2}\cos\vartheta \right] d\vartheta$$

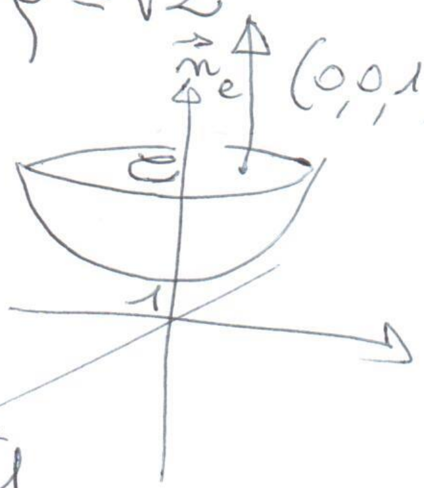
$$= 2\int_0^{2\pi} \cos^2\vartheta\, d\vartheta - \cancel{\sqrt{2}\int_0^{2\pi} \cos\vartheta\, d\vartheta} = 0$$

$$= \int_0^{2\pi} [\cos 2\vartheta + 1]\, d\vartheta = 2\pi.$$

È ancor più immediato applicare ~~a~~ a ritroso il Teorema di Stokes: la circuitazione lungo ~~BS~~ BS è uguale al flusso di $\vec{F}$ del rotore attraverso QUALSIASI superficie avente BS come bordo. Possiamo quindi prendere il cerchio sul piano $z=3$ che ha come frontiera BS:

5b

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \\ z = 3 \end{cases} \qquad 0 \le \rho \le \sqrt{2}$$



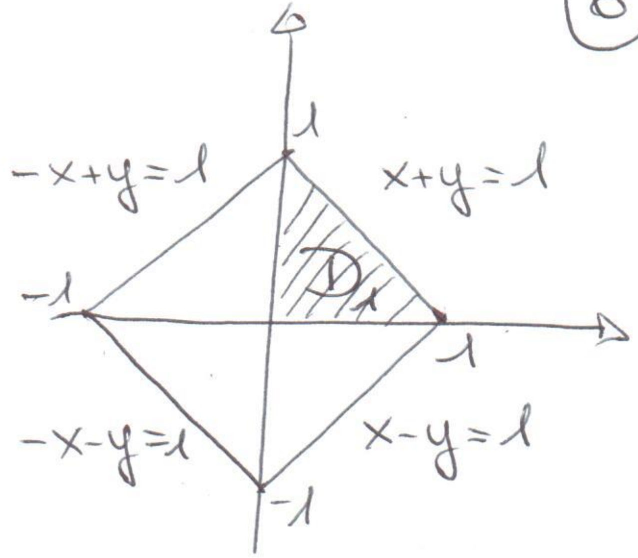
$$(\vec{\nabla} \wedge \vec{F}) \cdot \vec{n}_e =$$

$$(2,0,1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow \Phi_\Sigma(\vec{\nabla} \wedge \vec{F}) = \iint_\Sigma 1 \, dx\, dy =$$

$$\text{area}(\Sigma) = \pi(\sqrt{2})^2 = 2\pi.$$

5)



Poiché $f(x,y) = f(-x,y) = f(x,-y)$, data la simmetria del dominio,

$$\iint_D f(x,y) = 4 \iint_{D_1} f(x,y)\,dx\,dy$$

$$= 4 \iint_{D_1} (x^2+y^4)\,dx\,dy - 4 \text{ area } D_1$$

Ma $D_1 = \{ 0 \le x \le 1 \,;\, 0 \le y \le 1-x \}$

$$= 4 \int_0^1 dx \int_0^{1-x} (x^2+y^4)\,dy - 2$$

$$= 4 \int_0^1 dx \left[x^2 y + \frac{y^5}{5} \right]_0^{1-x} - 2$$

$$= 4\int_0^1 \left[x^2(1-x) + \frac{(1-x)^5}{5}\right] dx - 2$$

$$= 4\left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^6}{30}\right]_0^1 - 2$$

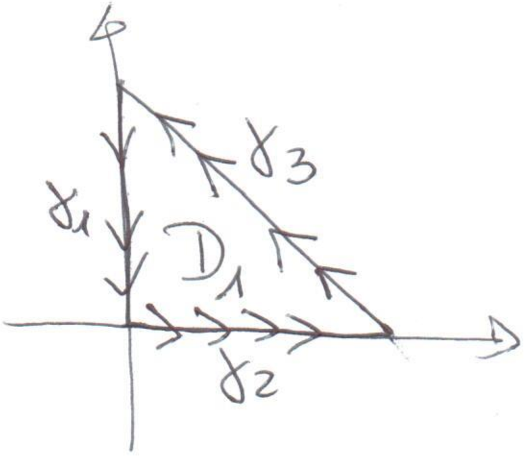
$$= 4\left[\frac{1}{3} - \frac{1}{4} + \frac{1}{30}\right] - 2 = \cancel{4}\left(\frac{20-15+2}{\cancel{60}\,15}\right) - 2$$

$$= \frac{7}{15} - 2 = \frac{-23}{15}.$$

In alternativa, dato per assunto che

$4\iint_{D_1} -1 \, dx\,dy = -2$, calcoliamo

$\iint_{D_1} (x^2+y^4)\, dx\,dy$ con le formule di Gauss-Green:



$\cancel{\iint_{D_1} (x^2+y^4)\, dx\,dy}$ =

ad esempio

$$\iint_{D_1} Q_x \, dx\,dy = \int_{\partial D_1^+} Q \, dy$$ 8

Se $Q_x(x,y) = x^2 + y^4 \Rightarrow Q(x,y) = \frac{x^3}{3} + y^4 x$

(ad esempio)

$$\Rightarrow \int_{+\partial D} \left(\frac{x^3}{3} + y^4 x \right) dy$$

Ma su γ_2 $\quad y = 0 \; ; \; dy = 0$

su γ_1 $\quad x = 0$

γ_3 : $\begin{cases} y \in [0,1] \\ x = 1 - y \end{cases}$

$$\Rightarrow \int_{+\partial D} = \int_0^1 \left[\frac{(1-y)^3}{3} + y^4 (1-y) \right] dy$$

$$= -\frac{(1-y)^4}{12} + \frac{y^5}{5} - \frac{y^6}{6} \Big|_0^1 = \frac{1}{12} + \frac{1}{5} - \frac{1}{6}$$

$$= \frac{5 + 12 - 10}{60} = \frac{7}{60}$$

$$\Rightarrow \iint_D (x^2+y^4-1)\,dx\,dy = 4\cdot\frac{7}{60}-2 = \frac{-23}{15}.$$

6) OMOGENEA ASSOCIATA:

$$\alpha^2-2\alpha+1=0 \qquad \Rightarrow \alpha_{1,2}=1$$

$$\Rightarrow y_0(x)=C_1e^x+C_2xe^x$$

$$y_p(x)=C_1(x)\,e^x+C_2(x)\,xe^x$$

$$W(y_1,y_2)=\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}=(x+1-x)e^{2x}=e^{2x}$$

$$C_1(x)=-\int\frac{\frac{e^x}{x}xe^x}{e^{2x}}dx=-\int 1\,dx=-x$$

$$C_2(x)=\int\frac{\frac{e^x}{x}e^x}{e^{2x}}dx=\int\frac{1}{x}dx=\ln(|x|)$$

~~Ma è equ~~ Ma il problema di Cauchy è definito in $(0,+\infty)$

$$\Rightarrow \quad C_2(x)=\ln(x)$$

$$y_{N.O.}(x) = C_1 e^x + \underbrace{C_2 x e^x - x e^x}_{\text{si potrebbero sommare.}} + x \ln x \cdot e^x$$

Però lasciamo così

$$y'_{N.O.}(x) = C_1 e^x + C_2 (x+1) e^x - (x+1) e^x + \ln x \cdot e^x + e^x + x \ln x \, e^x$$

$$\Rightarrow \begin{cases} y(1) = C_1 e + C_2 e - e = -e \\ y'(1) = C_1 e + 2C_2 e \cancel{- 2e + e = -e} \end{cases}$$

$$\Rightarrow \begin{cases} C_1 + C_2 = 0 \\ C_1 + 2C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$$

$$\Rightarrow y(x) = -x e^x + x \ln x \cdot e^x$$

$$= x e^x (\ln x - 1)$$