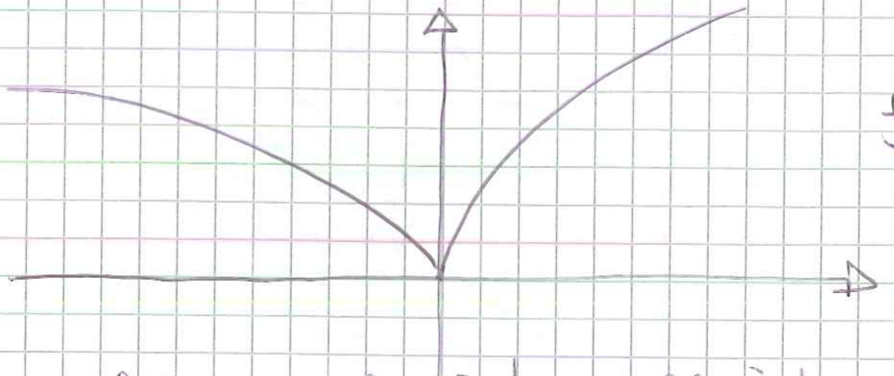
SVOLGIMENTO PROVA SCRITTA di ANALISI MAT. 2 del 17/2/2020

1)
$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases} = f(x)$$

Poiché $f_n(x) \in C^0(\mathbb{R})$, ma $f(x) \notin C^0(\mathbb{R})$

$\Rightarrow$ NON C'È CONVERGENZA UNIFORME su $\mathbb{R}$.



Cerchiamo gli intervalli di convergenza uniforme

$$|f_n(x) - f(x)| = \begin{cases} \left|x^{1/n} - 1\right| & \text{se } x > 0 \\ 0 & \text{se } x = 0 \\ \left||x|^{1/n^2} - 1\right| & \text{se } x < 0 \end{cases}$$

$$= \begin{cases} x^{\frac{1}{n}} - 1 & \text{se } x \geq 1 \\ 1 - x^{\frac{1}{n}} & \text{se } 0 < x < 1 \\ 0 & \text{se } x = 0 \\ (-x)^{1/n^2} - 1 & \text{se } x < -1 \\ 1 - (-x)^{1/n^2} & \text{se } 0 > x \geq -1 \end{cases}$$

In $[1, +\infty)$ $\quad \sup\left(x^{1/n} - 1\right) = +\infty$

$\Rightarrow$ considero $[1, \beta]$. Poichè $x^{\frac{1}{n}} - 1$ è crescente, $\sup\limits_{[1,\beta]} \left(x^{\frac{1}{n}} - 1\right) = \beta^{\frac{1}{n}} - 1 \longrightarrow 0$

$\Rightarrow$ CONV. UNIF. in ogni intervallo $[1, \beta]$.

In $(0, 1]$ $\quad \sup\left(1 - x^{\frac{1}{n}}\right) = 1$

$\Rightarrow$ considero $[\alpha, 1]$. Poichè $1 - x^{\frac{1}{n}}$ è decrescente, $\sup\limits_{[\alpha,1]} \left(1 - x^{\frac{1}{n}}\right) = \left(1 - \alpha^{\frac{1}{n}}\right) \to 0$

$\Rightarrow$ CONV. UNIF. in ogni intervallo $[\alpha, 1]$

$\Rightarrow$ CONV. UNIF. in ogni intervallo $[\alpha, \beta]$, con $0 < \alpha < \beta < +\infty$.

Analoghe considerazioni possono essere fatte per $x < 0$:

CONV. UNIF. in ogni intervallo $[\gamma, \delta]$, con $-\infty < \gamma < \delta < 0$.

2) a) $f \in C^0(\mathbb{R}^2 - \{(0,0)\})$.

Controlliamo in $(0,0)$:

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} \left| \frac{(x+y)(x^2+5y^2)}{x^2+y^2} \right|$$

$$= \lim_{\rho\to 0} \left| \frac{\rho(\cos\vartheta+\sin\vartheta)\cancel{\rho^2}(\cos^2\vartheta+5\sin^2\vartheta)}{\cancel{\rho^2}} \right|$$

$$\leq \lim_{\rho\to 0} \rho\left[(|\cos\vartheta|+|\sin\vartheta|)(1+5)\right] \leq 12 \lim_{\rho\to 0} \rho = 0$$

UNIFORMEMENTE RISPETTO A ϑ.

$\Rightarrow f \in C^0(\mathbb{R}^2)$

b) f derivabile in $\mathbb{R}^2 - \{(0,0)\}$

$$\frac{\partial f}{\partial x} = \left\{ \left[\log(x^2+5y^2+1) + \frac{(x+y)2x}{x^2+5y^2+1}\right](x^2+y^2) - (x+y)\log(x^2+5y^2+1)\,2x \right\} \cdot \frac{1}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \left\{ \left[\log(x^2+5y^2+1) + \frac{(x+y)10y}{x^2+5y^2+1}\right](x^2+y^2) - (x+y)\log(x^2+5y^2+1)\,2y \right\} \frac{1}{(x^2+y^2)^2}$$

In $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{h\log(1+h^2)}{h^3}=\lim_{h\to 0}\frac{h^3}{h^3}=\underline{1}$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{k\log(1+5k^2)}{k^3}=\lim_{k\to 0}\frac{5h^3}{h^3}=5$$

$\Rightarrow f$ derivabile in $\mathbb{R}^2$.

c) f differenziabile in $\mathbb{R}^2-\{(0,0)\}$

In $(0,0)$: poiché

$$\frac{df}{d\vec{v}}(0,0)=\lim_{t\to 0}\frac{t(\alpha+\beta)\log\left(1+(\alpha^2+5\beta^2)t^2\right)}{t^3}$$

$$=\lim_{t\to 0}\frac{t^3(\alpha+\beta)(\alpha^2+5\beta^2)}{t^3}=(\alpha+\beta)(\alpha^2+5\beta^2)$$

e di conseguenza NON VALE la formula del gradiente, allora f NON è differenziabile in $(0,0)$.

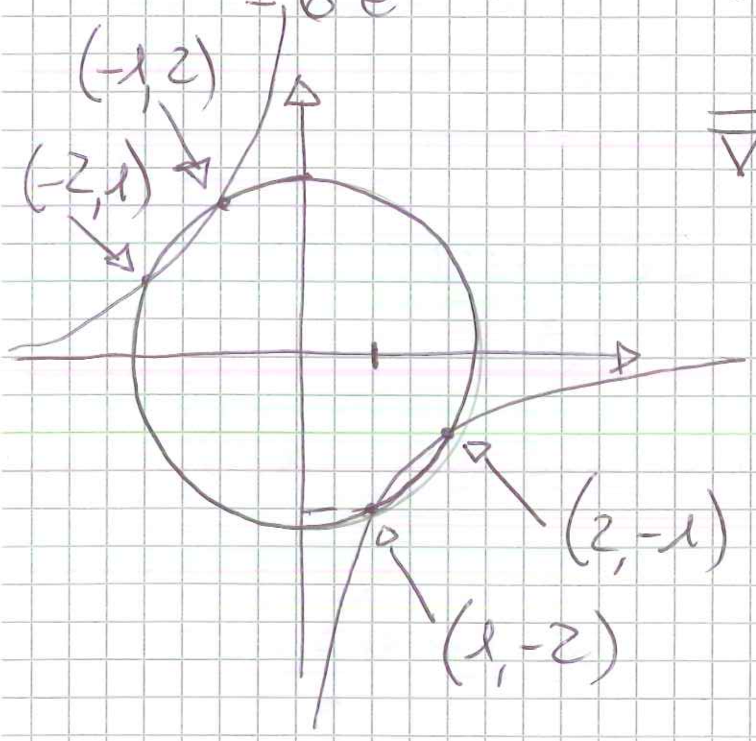
3) Punti stazionari: (5)

$$f_x = e^{3xy^2+x^2+12y-15x}\left(3y^2+3x^2-15\right)$$

$$= 3e^{3xy^2+x^2+12y-15x}\left(y^2+x^2-5\right)$$

$$f_y = e^{3xy^2+x^3+12y-15x}\left(6xy+12\right)$$

$$= 6e^{3xy^2+x^3+12y-15x}\left(xy+2\right)$$



$$\vec{\nabla} f = \vec{0} \iff \begin{cases} \underbrace{y^2+x^2=5}_{\text{CIRCONFERENZA}} \\ \underbrace{xy=-2}_{\text{IPERBOLE}} \end{cases}$$

$$\begin{cases} y = -\dfrac{2}{x} \\ \dfrac{4}{x^2}+x^2-5=0 \end{cases}$$

$$\Rightarrow x^4-5x^2+4=0$$

$$\Rightarrow (x^2-1)(x^2-4)=0$$

$\Rightarrow P_1=(1,-2)$; $P_2=(2,-1)$; $P_3=(-1,2)$;
$P_4=(-2,1)$.

$$f_{xx} = 3\left[3(y^2+x^2-5)^2+2x\right]e^{(\ )}$$

$$f_{xy} = 3\left[6(y^2+x^2-5)(xy+2)+2y\right]e^{(\ )} = f_{yx}$$

$$f_{yy} = 6\left[6(xy+2)^2+x\right]e^{(\ )}$$

$$H_f(1,-2)=\begin{vmatrix} 6e^{-26} & -12e^{-26} \\ -12e^{-26} & 6e^{-26} \end{vmatrix} = -108e^{-52} < 0$$

PUNTO DI SELLA

$$H_f(-1,2)=\begin{vmatrix} -6e^{+28} & 12e^{+28} \\ 12e^{+28} & -6e^{+28} \end{vmatrix} = -108e^{56} < 0$$

PUNTO DI SELLA

>0

$$H_f(2,-1)=\begin{vmatrix} (12e^{-32}) & -6e^{-32} \\ -6e^{-32} & 12e^{-32} \end{vmatrix} = 108e^{-64} > 0$$

PUNTO DI MINIMO REL.

<0

$$H_f(-2,1)=\begin{vmatrix} (-12e^{40}) & 6e^{40} \\ 6e^{40} & -12e^{40} \end{vmatrix} = 108e^{80} > 0$$

PUNTO DI MASSIMO REL.

Ovviamente, $f(x,y)>0 \quad \forall (x,y)\in\mathbb{R}^2$

Poichè $f(x,-x) = e^{3x^3+x^2-27x} \xrightarrow[x\to-\infty]{} 0$

$\Rightarrow$ NON ESISTE MINIMO ASS.

Poichè $f(x,-x) = e^{3x^3+x^2-27x} \xrightarrow[x\to+\infty]{} +\infty$

$\Rightarrow$ NON ESISTE MASSIMO ASS.

4) $D = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

SEMIPIANO SUPERIORE. SEMPL. LIN. CONNESSO

$X_y = \frac{1}{2\sqrt{y}} - 2x = Y_x \implies \omega$ CHIUSA ed ESATTA

Si vede facilmente che

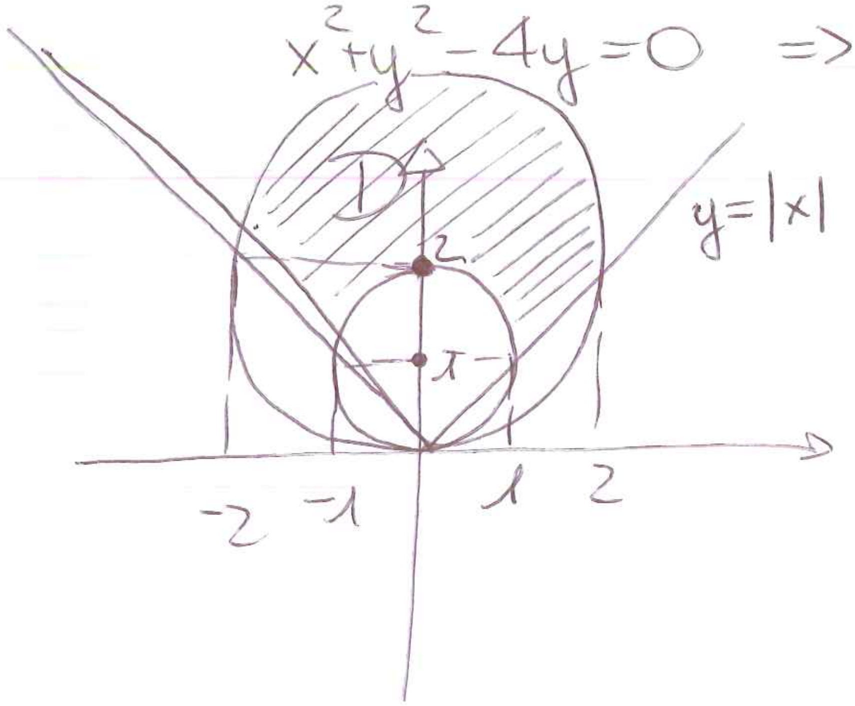
$V(x,y) = x\sqrt{y} - x^2 y + C$

$V(1,4) = 2 - 4 + C = 4 \iff C = 6$

$\implies \underline{V(x,y) = x\sqrt{y} - x^2 y + 6}$

5) $x^2 + y^2 - 2y = 0 \implies x^2 + (y-1)^2 = 1$

$x^2 + y^2 - 4y = 0 \implies x^2 (y-2)^2 = 4$

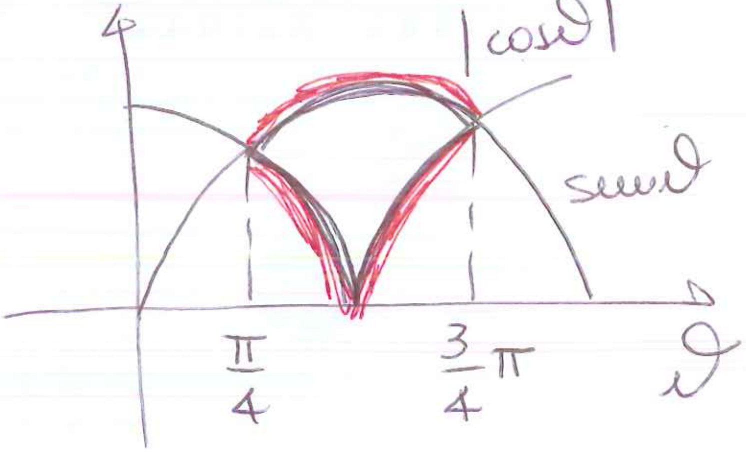


In coordinate polari: (8)

$$\begin{cases} 2\rho \sin\vartheta \le \rho^2 \le 4\rho \sin\vartheta \\ \cancel{\rho}\,|\cos\vartheta| \le \cancel{\rho} \sin\vartheta \end{cases}$$

$$\begin{cases} 2\sin\vartheta \le \rho \le 4\sin\vartheta \\ |\cos\vartheta| \le \sin\vartheta \end{cases}$$

← CONDIZIONE DI COMPATIBILITA':
$\sin\vartheta \ge 0$
$\Rightarrow \vartheta \in [0, \pi]$



$$\Rightarrow \begin{cases} 0 \le \vartheta \le \pi \\ \boxed{\frac{\pi}{4} \le \vartheta \le \frac{3}{4}\pi} \\ 2\sin\vartheta \le \rho \le 4\sin\vartheta \end{cases}$$

$$\iint_D \frac{1}{(x^2+y^2)^{1/2}}\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\vartheta \int_{2\sin\vartheta}^{4\sin\vartheta} \frac{1}{\rho}\,\rho\,d\rho$$

$$= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left[4\sin\vartheta - 2\sin\vartheta\right] d\vartheta = -2\cos\vartheta \Big|_{\frac{\pi}{4}}^{\frac{3}{4}\pi}$$

$$= -2\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right] = 2\sqrt{2}.$$

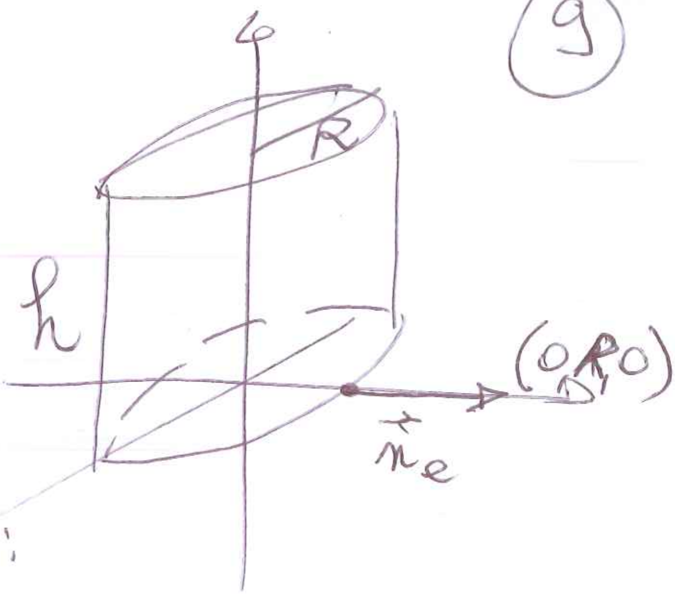
S è la superficie

TERALE del

lindro retto

lore di raggio R

ltezza h.



ordinate cilindriche:

$$=R\cos\vartheta \qquad \vartheta\in[0,2\pi]$$

$$=R\sin\vartheta$$

$$=t \qquad t\in[0,h]$$

$$= \left\| \begin{matrix} -R\sin\vartheta & R\cos\vartheta & 0 \\ 0 & 0 & 1 \end{matrix} \right\|$$

$$\vec{n}_e = \pm\left(R\cos\vartheta, R\sin\vartheta, 0\right)$$

$$= \sqrt{L^2+M^2+N^2} = \sqrt{R^2} = R$$

$$\iint_S x^2y^2\,dS = \int_0^h dz \int_0^{2\pi} d\vartheta\, R \cdot R^4 \cos^2\vartheta \sin^2\vartheta$$

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$$= R^5 h \int_0^{2\pi} \frac{1}{4} \sin^2 2\vartheta\, d\vartheta$$

$$= \frac{R^5 h}{8} \int_0^{2\pi} \left[1 - \cos 4\vartheta\right] d\vartheta = \frac{R^5 h}{8} \left[\vartheta - \frac{1}{4}\sin 4\vartheta\right]_0^{2\pi}$$

$$= \frac{R^5 h \pi}{4}.$$

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