SVOLGIMENTO PROVA SCRITTA di ANALISI 2 del 18/6/2018.

COMPITO A

1) $\begin{cases} x'(t) = 2t \\ y'(t) = t^2 - 1 \\ z'(t) = t^2 + 1 \end{cases}$ $\quad\begin{cases} x''(t) = 2 \\ y''(t) = 2t \\ z''(t) = 2t \end{cases}$

$$\ell(\gamma) = \int_0^1 \sqrt{4t^2 + (t^2-1)^2 + (t^2+1)^2}\, dt$$

$$= \int_0^1 \sqrt{2(t^2+1)^2}\, dt = \sqrt{2}\int_0^1 (t^2+1)\, dt$$

$$= \sqrt{2}\left[\frac{t^3}{3} + t\right]_0^1 = \frac{4}{3}\sqrt{2}$$

$$x_B = \frac{\int_\gamma x\, ds}{\ell(\gamma)} \ ; \quad y_B = \frac{\int_\gamma y\, ds}{\ell(\gamma)} \ ; \quad z_B = \frac{\int_\gamma z\, ds}{\ell(\gamma)}$$

$$\int_\gamma x\, ds = \sqrt{2}\int_0^1 t^2(t^2+1)\, dt = \sqrt{2}\left[\frac{t^5}{5} + \frac{t^3}{3}\right]_0^1 = \sqrt{2}\,\frac{8}{15}.$$

$$\Rightarrow x_B = \frac{2}{5}$$

$$\int_\gamma y\,ds = \sqrt{2}\int_0^1 \left(\frac{t^3}{3}-t\right)\left(t^2+1\right)dt$$

$$=\sqrt{2}\int_0^1 \left[\frac{t^5}{3}+\frac{t^3}{3}-t^3-t\right]dt =$$

$$=\sqrt{2}\left[\frac{t^6}{18}-\frac{1}{6}t^4-\frac{t^2}{2}\right]_0^1 = \sqrt{2}\left(\frac{-11}{18}\right)$$

$$\Rightarrow y_B = \frac{-11}{24}$$

$$\int_\gamma z\,ds = \sqrt{2}\int_0^1 \left(\frac{t^3}{3}+t\right)\left(t^2+1\right)dt = \sqrt{2}\int_0^1 \left(\frac{t^5}{3}+\frac{4}{3}t^3+t\right)dt$$

$$=\sqrt{2}\left[\frac{t^6}{18}+\frac{1}{3}t^4+\frac{t^2}{2}\right]_0^1 = \sqrt{2}\cdot\frac{8}{9}$$

$$\Rightarrow z_B = \frac{2}{3}$$

$$\vec{r}\,'(t)\wedge\vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & t^2-1 & t^2+1 \\ 2 & 2t & 2t \end{vmatrix}$$

$$= -4t\,\vec{i} + 2(1-t^2)\,\vec{j} + 2(t^2+1)\,\vec{k}$$

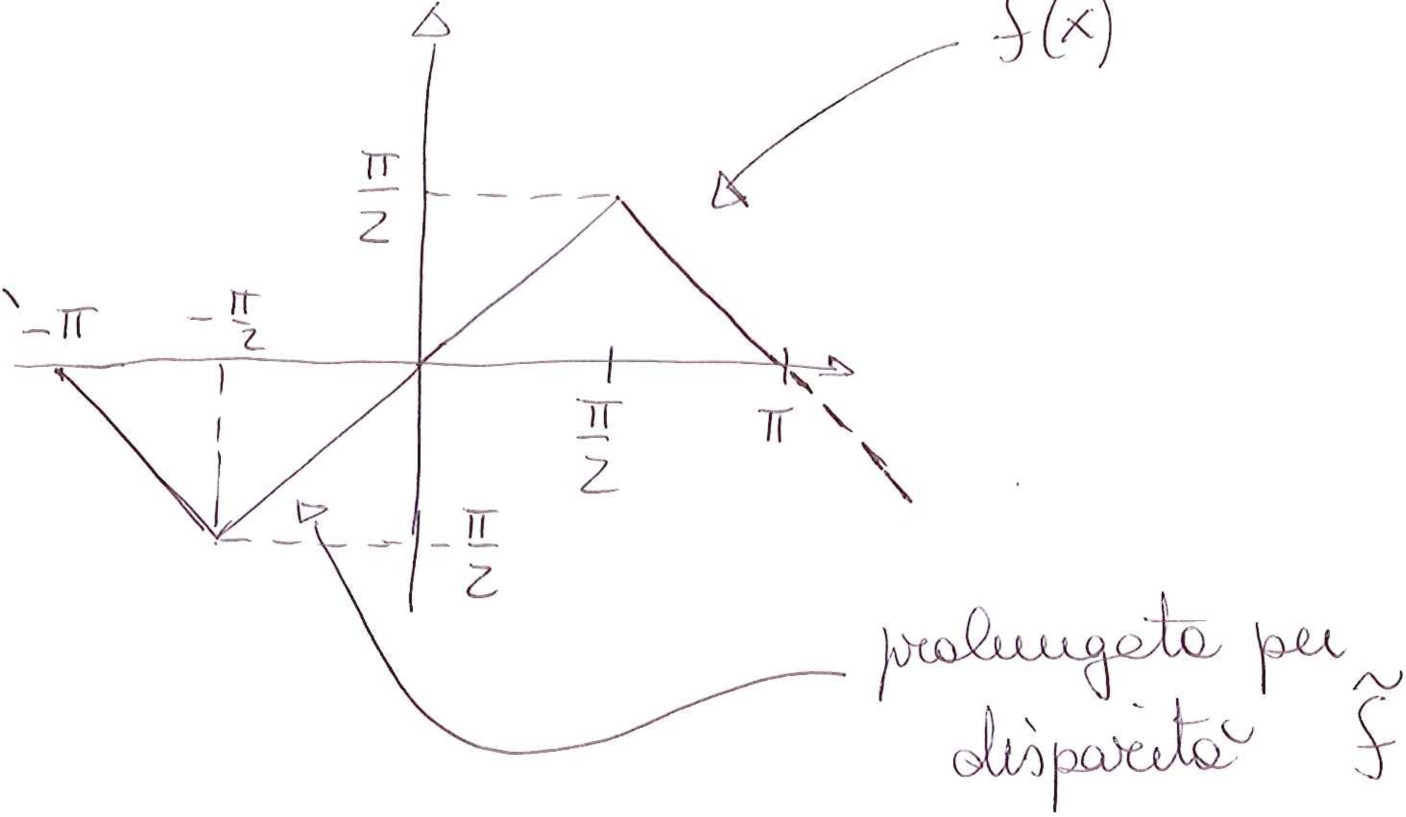
$$\Rightarrow \quad \left\| \vec{r}\,'(t) \wedge \vec{r}\,''(t) \right\| = \sqrt{16t^2 + 4(t^2-1)^2 + 4(t^2+1)^2}$$

$$= \cancel{4}\,2 \sqrt{4t^2 + (t^2-1)^2 + (t^2+1)^2} = \cancel{4}\,2\,v(t)$$

$$\Rightarrow k(t) = \frac{\cancel{4}\,2\,v(t)}{v^3(t)} = \frac{\cancel{4}\,2}{v^2(t)} = \frac{\cancel{4}\,2}{2(t^2+1)^2} = \frac{\cancel{2}\,1}{(t^2+1)^2}$$

2)

$$f(x) = \begin{cases} \dfrac{\pi}{2} - \left(x - \dfrac{\pi}{2}\right) = \pi - x & \text{se } \dfrac{\pi}{2} \le x \le \pi \\[2ex] \dfrac{\pi}{2} + \left(x - \dfrac{\pi}{2}\right) = x & \text{se } 0 \le x < \dfrac{\pi}{2} \end{cases}$$



la prolungata per disparità è CONTINUA e REGOLARE A TRATTI $\forall x \in \mathbb{R}$

$\Rightarrow$ CONV. PUNTUALE, UNIFORME e TOTALE in $\mathbb{R}$

$$S(x) = \tilde{f}(x) \qquad \forall x \in \mathbb{R}$$

$$a_k = 0 \qquad \forall k$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(kx)\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(kx)\,dx\right] =$$

$$= \frac{2}{\pi}\left[-\frac{1}{k}x\cos(kx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{k}\int_0^{\frac{\pi}{2}}\cos(kx)\,dx - \frac{1}{k}(\pi - x)\cos(kx)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{k}\int_{\frac{\pi}{2}}^{\pi}\cos(kx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\cancel{-\frac{\pi}{2k}\cos\left(k\frac{\pi}{2}\right)} + \frac{1}{k^2}\sin(kx)\Big|_0^{\frac{\pi}{2}} + \cancel{\frac{\pi}{2k}\cos\left(k\frac{\pi}{2}\right)} - \frac{1}{k^2}\sin(kx)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{4}{\pi k^2} \operatorname{sen}\left(k\frac{\pi}{2}\right)$$

$$\Rightarrow \tilde{f}(x) \sim \sum_{k=1}^{\infty} \frac{4}{\pi k^2} \operatorname{sen}\left(k\frac{\pi}{2}\right) \operatorname{sen}(kx)$$

Si osservi che

$$\operatorname{sen}\left(k\frac{\pi}{2}\right) = \begin{cases} 0 & \text{se } k = 2m \\ (-1)^m & \text{se } k = 2m+1 \end{cases}$$

$$\Rightarrow \tilde{f}(x) \sim \sum_{m=0}^{\infty} \frac{(-1)^m 4}{\pi (2m+1)^2} \operatorname{sen}[(2m+1)x]$$

3)

⑥

$\alpha > 0: \quad I_{def} = \mathbb{R}^2$

$\alpha < 0: \quad I_{def} = \mathbb{R}^2 - \{y = -x\}$

$\alpha > 0: \quad f(x,y) \in C^\infty(\mathbb{R}^2)$ $\Rightarrow$ CONTINUE, DERIVABILI, DIFFERENZIABILI

$\alpha < 0: \quad f(x,y) \in C^\infty(I_{def})$

$$f(x,y) = \begin{cases} (x+y)^\alpha & \text{se } y > -x \\ [-(x+y)]^\alpha & \text{se } y < -x \end{cases}$$

$\forall (x,y) \in \mathbb{R}^2 - \{y = -x\}, \quad \forall \alpha \neq 0,$

$$f_x(x,y) = f_y(x,y) = \begin{cases} \alpha (x+y)^{\alpha - 1} & \text{se } y > -x \\ -\alpha [-(x+y)]^{\alpha - 1} & \text{se } y < -x \end{cases}$$

Rimangono da studiare (<u>SOLO PER $\alpha>0$</u>)

i punti $(x,-x)$. In tali punti $f(x,-x)=0$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(x_0,-x_0)=\lim_{h\to 0}\frac{f(x_0+h,-x_0)}{h}$$ (7)

$$=\lim_{h\to 0}\frac{|\cancel{x_0}+h-\cancel{x_0}|^{\alpha}}{h}=\begin{cases}\lim_{h\to 0^+} h^{\alpha-1}\\ \lim_{h\to 0^-} -(-h)^{\alpha-1}\end{cases}$$

Tale limite esiste solo per $\alpha>1$:

$$\frac{\partial f}{\partial x}(x_0,-x_0)=\lim_{h\to 0^+} h^{\alpha-1}=\lim_{h\to 0^-}-(-h)^{\alpha-1}=0.$$

Analogamente per $\frac{\partial f}{\partial y}(x_0,-x_0)$.

Dunque

$$\frac{\partial f}{\partial x}(x_0,-x_0)=\frac{\partial f}{\partial y}(x_0,-x_0)=0 \qquad \forall \alpha>1$$

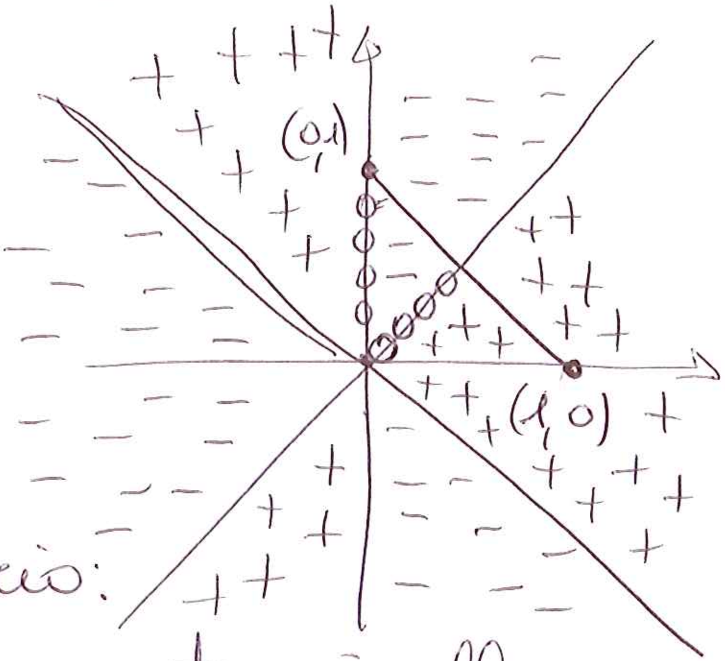
$$\nexists\ \frac{\partial f}{\partial x}(x_0,-x_0);\ \frac{\partial f}{\partial y}(x_0,-x_0) \qquad \forall \alpha\le 1$$

4)

$$f(x,y) = 2x\left(x^2-y^2\right)$$

$f(x,y)=0$ per $x=0$; $y=\pm x$

Estremi liberi:

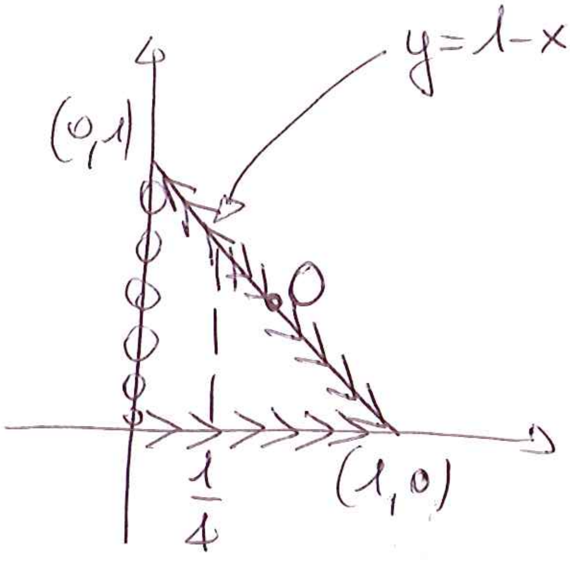


$$\begin{cases} f_x = 6x^2-2y^2 \\ f_y = -2xy \end{cases}$$

Unico punto stazionario:

(0,0), che però appartiene alla frontiera di T.

Sulla frontiera:



$f(x,0)=2x^3$ crescente

$f(0,y)=0$ costante

$$f(x,1-x) = 2x^3-2x(1-x)^2$$

$$= 2\left[2x^2-x\right] \qquad f'(x)=2(4x-1)$$

$\Rightarrow$ f decresce per $x\in\left[0,\frac{1}{4}\right)$ e cresce per $x\in\left(\frac{1}{4},1\right]$

$$y = 1 - x \Big|_{x=\frac{1}{4}} = \frac{3}{4}$$

$(0,1)$ punto di MAX. REL. ; $f(0,1) = 0$

$\left(\frac{1}{4}, \frac{3}{4}\right)$ punto di MIN. REL. ; $f\left(\frac{1}{4}, \frac{3}{4}\right) = -\frac{1}{4}$

$(1,0)$ punto di MAX. REL. ; $f(1,0) = 2$

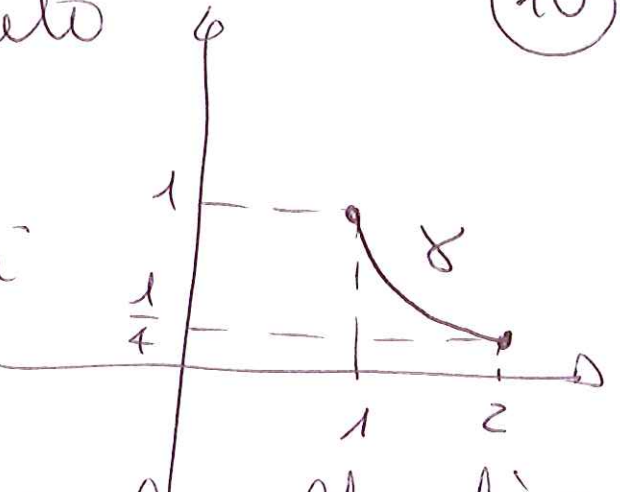
$(0,0)$ punto di MIN. REL. ; $f(0,0) = 0$

I punti sull'asse delle y, $y \in (0,1)$ sono indifferentemente di MAX. e MIN. REL.

$\left(\frac{1}{4}, \frac{3}{4}\right)$ punto di MIN. ASSOLUTO

$(1,0)$ " " MAX. "

5) Il campo è definito nei 4 quadranti privi degli assi, quindi disgiunti. Ci mettiamo nel primo quadrante, per la scelta di γ.



Si può verificare che il campo è irrotazionale (e quindi conservativo) nel quadrante.

Però qui si determina "a occhio" un potenziale:

$$\vec{F} = \vec{F}_1(x,y) + \vec{F}_2(x,y)$$

con
$$\vec{F}_1(x,y) = \frac{2x}{x^2+y^2}\vec{\imath} + \frac{2y}{x^2+y^2}\vec{k}$$

$$\Rightarrow V_1(x,y) = \ln(x^2+y^2) + C_1$$

$$\vec{F}_2(x,y) = \frac{1}{x^2}\vec{\imath} - \frac{1}{y^2}\vec{k} \Rightarrow V_2(x,y) = -\frac{1}{x} + \frac{1}{y} + C_2$$

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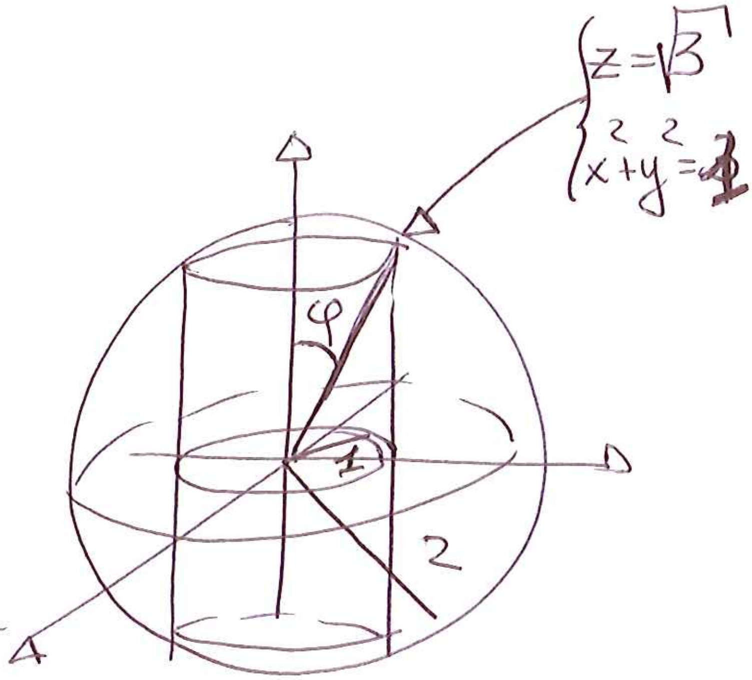
$$\Rightarrow V(x,y) = \ln(x^2+y^2) - \frac{1}{x} + \frac{1}{y} + C.$$

$$\Rightarrow \int_\gamma \vec{F}\cdot d\vec{P} = V\left(2,\frac{1}{4}\right) - V(1,1)$$

$$= \ln\left(\frac{65}{16}\right) - \frac{1}{2} + 4 - \ln(2) + 1 - 1$$

$$= \ln\left(\frac{65}{32}\right) + \frac{7}{2}$$

6) Il dominio è la parte di sfera esterna al cilindro.



Data la simmetria della figura, basterà calcolare il volume del solido in un ottante e moltiplicare per 8.

In coordinate sferiche:

$$\begin{cases} \rho^2 \leq 4 \\ \rho^2 \sin^2\varphi \geq 1 \end{cases} \Rightarrow \begin{cases} 0 \leq \rho \leq 2 \\ \rho \geq \dfrac{1}{\sin\varphi} \end{cases}$$

Abbiamo quindi

$$\frac{1}{\sin\varphi} \leq \rho \leq 2 \quad ; \quad \underbrace{\varphi \in \left[0, \frac{\pi}{2}\right] ; \ \vartheta \in \left[0, \frac{\pi}{2}\right]}_{1^{\circ} \text{ ottante}}$$

Condizione di compatibilità:

$$2 \geq \frac{1}{\sin\varphi} \Rightarrow \sin\varphi \geq \frac{1}{2} \Rightarrow \varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$$

$$\Rightarrow \operatorname{Vol} T = 8 \int_0^{\frac{\pi}{2}} d\vartheta \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi \int_{\frac{1}{\sin\varphi}}^{2} \rho^2 \sin\varphi \, d\rho$$

$$= 4\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\varphi \left[\frac{\rho^3}{3}\right]_{\frac{1}{\sin\varphi}}^{2} = \frac{4}{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\varphi \left[\frac{-1}{\sin^3\varphi} + 8\right] d\varphi$$

$$= \frac{4}{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8\sin\varphi - \frac{1}{\sin^2\varphi}\right] d\varphi \qquad \cancel{112\sqrt{3}\pi}$$

$$= \frac{4}{3}\pi \left[-8\cos\varphi + \cot g\varphi\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{4}{3}\pi \left[4\sqrt{3} - \sqrt{3}\right]$$

$$= 4\sqrt{3}\pi$$

In coordinate cilindriche:

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = t \end{cases} \qquad \vartheta \in \left[0, \frac{\pi}{2}\right] \quad (1^\circ \text{ quadrante})$$

$$\begin{cases} \rho^2 + t^2 \leq 4 \\ \rho^2 \geq 1 \end{cases} \Rightarrow 1 \leq \rho \leq \sqrt{4-t^2}$$

con $0 \leq t \leq 2$

Condizione di compatibilità:

$$1 \leq \sqrt{4-t^2} \Rightarrow 1 \leq 4 - t^2$$

$$\Rightarrow 0 \leq t \leq \sqrt{3}$$

$$\Rightarrow \begin{cases} \vartheta \in \left[0, \frac{\pi}{2}\right] \\ t \in \left[0, \sqrt{3}\right] \\ 1 \leq \rho \leq \sqrt{4-t^2} \end{cases}$$

$$\Rightarrow \operatorname{Vol} T = 8 \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\sqrt{3}} dt \int_1^{\sqrt{4-t^2}} \rho \, d\rho$$

$$= 4\pi \int_0^{\sqrt{3}} dt \left[\frac{\rho^2}{2}\right]_1^{\sqrt{4-t^2}} = 2\pi \int_0^{\sqrt{3}} \left[4 - t^2 - 1\right] dt$$

$$= 2\pi \left[3t - \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\pi \left[3\sqrt{3} - \sqrt{3}\right]$$

$$= 4\pi\sqrt{3}$$

In coordinate cartesiane, integrando per fili: 1° quadrante

$$\begin{cases} x^2+y^2 \geq 1 \\ 0 \leq z \leq \sqrt{4-x^2+y^2} \end{cases}$$

$$\Rightarrow \begin{cases} 1 \leq x^2+y^2 \leq 4 \leftarrow D(x,y) \\ 0 \leq z \leq \sqrt{4-x^2-y^2} \end{cases}$$

$$\mathrm{Vol} = 8 \int_{D(x,y)} dx\,dy \left[\sqrt{4-x^2-y^2}\right] \quad \left(\begin{matrix}\text{coordinate}\\ \text{polari}\end{matrix}\right)$$

$$= 8 \int_0^{\frac{\pi}{2}} d\vartheta \int_1^2 \rho\, d\rho \sqrt{4-\rho^2}$$

$$= -4\pi \left[\frac{1}{3}(4-\rho^2)^{\frac{3}{2}}\right]_1^2 = 4\pi\sqrt{3}.$$