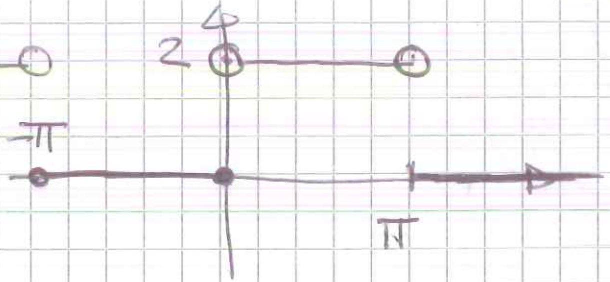
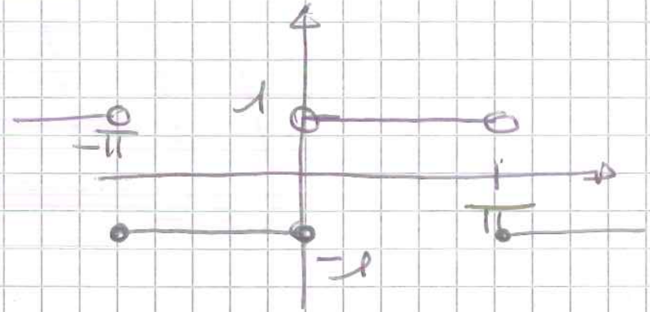
SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 21/6/2023

1) la funzione $f(x)$ può essere riscritta come $f(x) = 1 + g(x)$, dove

$$g(x) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in (0, \pi) \end{cases}$$





$g(x)$ è dispari $\Rightarrow a_k = 0$, $k \geq 1$

Inoltre, poiché $\frac{a_0}{2} = \mu(f) \Rightarrow a_0 = 2$

Calcoliamo i b_k di $g(x)$

$$b_k = \frac{2}{\pi} \int_0^\pi \sin(kx)\,dx = \frac{-2}{\pi k} \cos(kx) \Big|_0^\pi$$

$$= \frac{-2}{\pi k}\left[(-1)^k - 1\right] = \begin{cases} 0 & \text{se } k = 2m \\ +\frac{4}{\pi(2m+1)} & \text{se } k = 2m+1 \end{cases}$$

$$\Rightarrow f(x) \sim 1 + \sum_{m=0}^{+\infty} \frac{4}{\pi(2m+1)} \sin\left[(2m+1)x\right]$$

$$S(x)=\begin{cases} f(x) & \text{se } x \neq k\pi \\ 1 & \text{se } x = k\pi \end{cases}$$

CONV. PUNTUALE in $\mathbb{R}$

CONV. UNIFORME in ogni $[\alpha, \beta]$ in cui f sia continua

NO CONV. TOTALE

Ponendo $x=1$ e calcolando $S(1) = 2 = \displaystyle\sum_{n=0}^{+\infty} \frac{4}{(2n+1)\pi} \operatorname{sen}(2n+1)$.

2)

$$|f(x,y)| \leq |xy| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow f$ CONTINUA IN $(0,0)$

$$f(x,0) = f(0,y) = 0 \Rightarrow f_x(0,0) = f_y(0,0) = 0.$$

DIFFERENZIABILITA':

$$\lim_{(h,k)\to(0,0)} \left| \frac{\Delta f - df}{\sqrt{h^2+k^2}} \right| = \lim_{\rho \to 0} \left| \frac{\rho^2 \cos\vartheta \sin\vartheta \cos\left(\frac{1}{\rho^2}\right)}{\rho} \right|$$

$$\leq \lim_{\rho\to 0} \rho = 0$$

$\Rightarrow$ DIFF. BILE in $(0,0)$ $\Rightarrow$ FORMULA del GRADIENTE

$$\frac{df}{d\vec{v}}(0,0) = \alpha f_x(0,0) + \beta f_y(0,0) = 0$$

$$\forall \vec{v} = (\alpha, \beta)$$

$f \in C^\infty(\mathbb{R}^2 - \{(0,0)\})$, pertanto $f \in C^1(\mathbb{R}^2 - \{(0,0)\})$.

Devo solo verificare che le derivate parziali siano continue ANCHE in $(0,0)$.

3

$\forall (x,y) \neq (0,0)$

$$f_x(x,y) = y\left[\cos\left(\frac{1}{x^2+y^2}\right) + \frac{2x^2}{(x^2+y^2)^2}\sin\left(\frac{1}{x^2+y^2}\right)\right]$$

Affinché f_x sia continua in $(0,0)$, deve aversi

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0.$$

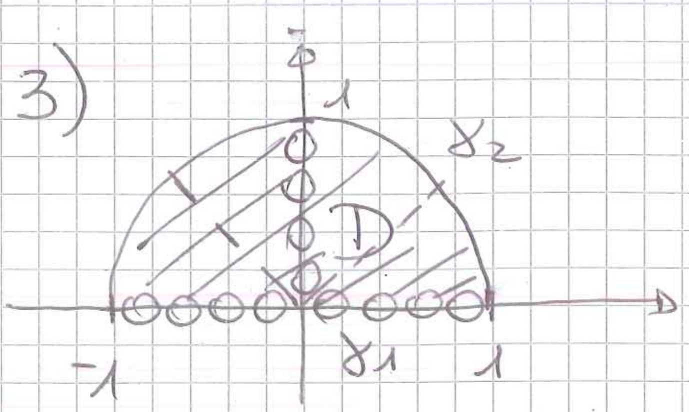
Ma, ad esempio, lungo $y=x$

$$\lim_{x\to 0} f_x(x,x) = \lim_{x\to 0} x\left[\cos\left(\frac{1}{2x^2}\right) + \frac{2x^2}{(2x^2)^2}\sin\left(\frac{1}{2x^2}\right)\right]$$

$$= \underbrace{\lim_{x\to 0} x\cos\left(\frac{1}{2x^2}\right)}_{=0} + \underbrace{\lim_{x\to 0} \frac{1}{2x}\sin\left(\frac{1}{2x^2}\right)}_{\nexists}$$

Pertanto f_x (e analogamente f_y) non è continua in $(0,0)$ e quindi $f \notin C^1(\mathbb{R}^2)$.

3)



Si ~~osservi~~ osservi che $f(x,y) = \ln(1+x^2y^2) \geq 0$ e che $f(x,y)=0$ sull'asse x $(y=0)$ e sull'asse y $(x=0)$

Pertanto tutti i punti degli assi sono di MIN. ASS. LIBERO.

(4)

Infatti: $f_x = \dfrac{2xy^2}{1+x^2+y^2}$; $f_y = \dfrac{2yx^2}{1+x^2+y^2}$

$$\vec{\nabla} f = \vec{0} \iff x=0 \vee y=0.$$

Lungo la frontiera: $\partial D = \gamma_1 \cup \gamma_2$

$\gamma_1 : \begin{cases} x \in [-1,1] \\ y = 0 \end{cases}$; $\gamma_2 : \begin{cases} x \in [-1,1] \\ y = \sqrt{1-x^2} \end{cases}$ (a) oppure $\begin{cases} x = \cos\vartheta \\ y = \sin\vartheta \end{cases}$ (b) $\vartheta \in [0,\pi]$

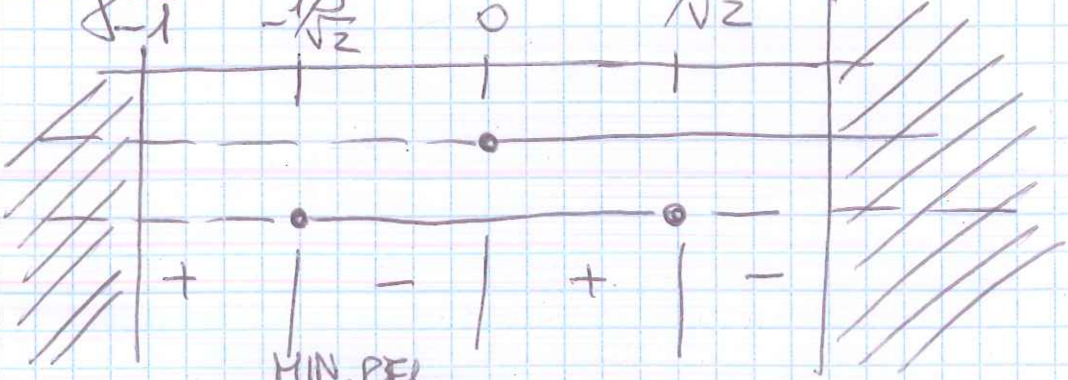
$f\big|_{\gamma_1} = f(x,0) = 0$ COME GIÀ VISTO, PUNTI DI MIN. ASSOLUTO.

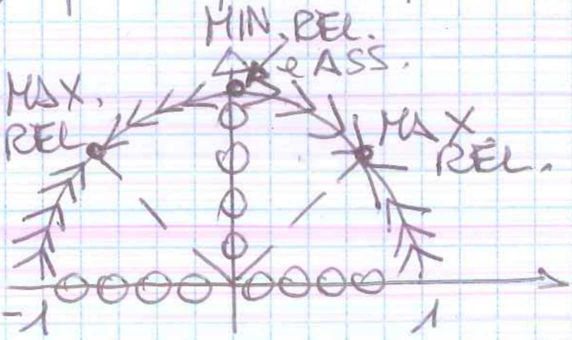
Su γ_2, con la parametrizzazione (a):

$$f\big|_{\gamma_2} = f(x, \sqrt{1-x^2}) = \ln(1+x^2(1-x^2)) = \ln(1+x^2-x^4)$$

$$\left(f\big|_{\gamma_2}\right)' = \frac{1}{1+x^2-x^4}(2x-4x^3) = \frac{2x(1-2x^2)}{1+x^2-x^4}$$

Segno di f':





$$f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \ln\left(1+\frac{1}{4}\right) = \ln\left(\frac{5}{4}\right).$$

MAX. ASS.

Usando la parametrizzazione (b):

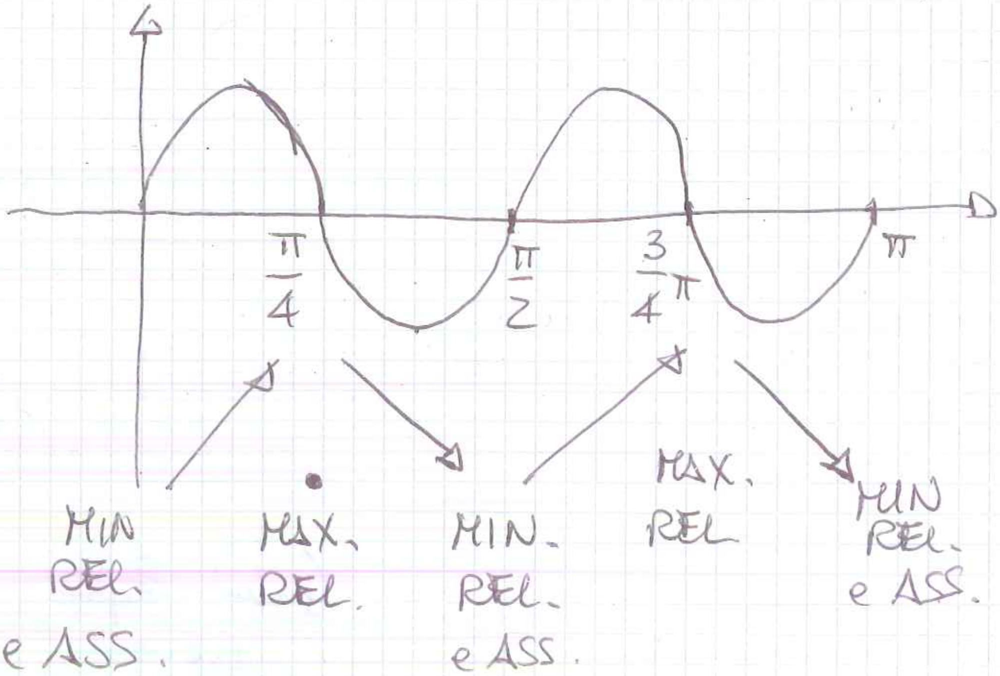
$$f(\vartheta) = \ln\left(1 + \cos^2\vartheta \sin^2\vartheta\right)$$

$$= \ln\left(1 + \frac{1}{4}\sin^2 2\vartheta\right)$$

$$f_\vartheta = \frac{1}{1+\frac{1}{4}\sin^2 2\vartheta}\left[\frac{1}{4} 2 \sin 2\vartheta \cdot 2\cos 2\vartheta\right]$$

$$= \frac{1}{1+\frac{1}{4}\sin^2 2\vartheta}\left[\sin 2\vartheta \cos 2\vartheta\right] = \frac{2}{4+\sin^2 2\vartheta} \sin 4\vartheta$$

$f_\vartheta \geq 0 \iff \operatorname{sen} 4\vartheta \geq 0 \quad \vartheta \in [0, \pi]$



come nel caso a.

4) $\vec{F}(x,y,z) = \vec{F}_1(x,y) + \vec{F}_2(y,z)$ definito in $\mathbb{R}^3$

~~Si verifica facilmente che~~

con $\vec{F}_1(x,y) = \left(\dfrac{2xy}{1+x^2}, \ln(1+x^2)\right)$

$\vec{F}_2(y,z) = \left(-z \operatorname{sen} zy, -y \operatorname{sen} zy\right)$

$\gamma(0) = (1, 0, 1) \; ; \; \gamma(1) = \left(2, \frac{\pi}{4}, e\right)$

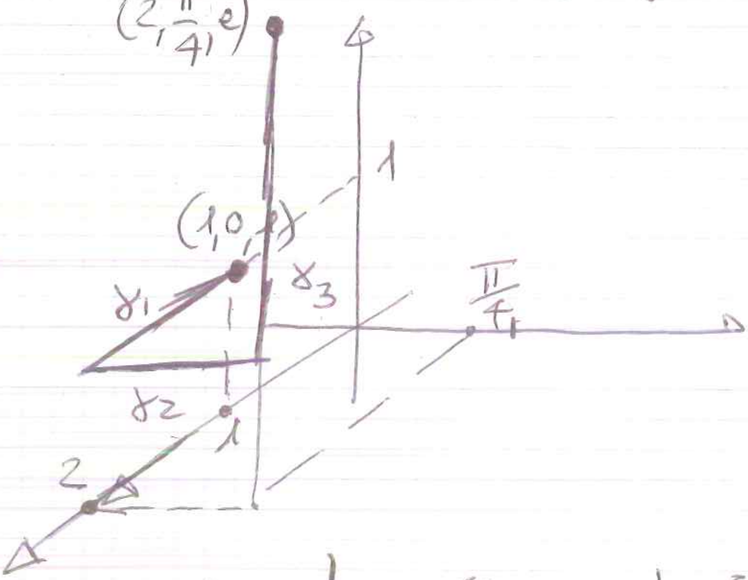
Si vede facilmente che $\vec{F}_1$ ed $\vec{F}_2$ (e quindi $\vec{F}$) sono irrotazionali (e quindi conservative) e tali che

$$U(x,y,z) = U_1(x,y) + U_2(y,z)$$
$$= y \ln(1+x^2) + \cos(zy) + c$$

Quindi $\displaystyle\int_\gamma \omega = U\left(2,\frac{\pi}{4},e\right) - U(1,0,1)$

$$= \frac{\pi}{4}\ln 5 + \cos\left(\frac{\pi}{4}e\right) - 1$$

Oppure possiamo scegliere una curva più semplice congiungente i due punti



$$\gamma_1 : \begin{cases} x = t & \cancel{dx} & t \in [1,2] \\ z = 1 & dz = 0 \\ y = 0 & dy = 0 \end{cases}$$

$$\gamma_2 : \begin{cases} y = t & t \in [0, \frac{\pi}{4}] \\ x = 2 & dx = 0 \\ z = 1 & dz = 0 \end{cases}$$

$$\gamma_3 : \begin{cases} z = t & t \in [1, e] \\ x = 2 & dx = 0 \\ y = \frac{\pi}{4} & dy = 0 \end{cases}$$

$$\int_\gamma = \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3}$$

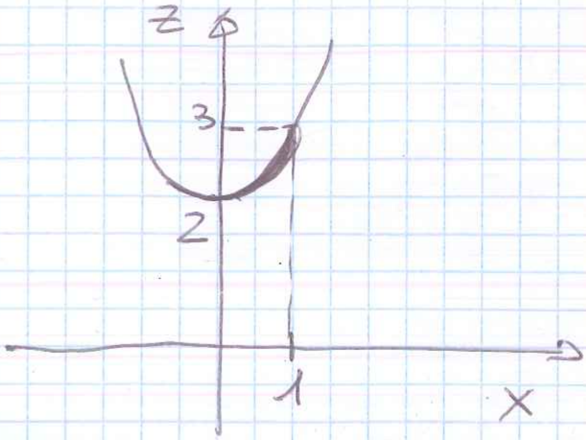
$$= \int_1^2 0 + \int_0^{\pi/4} \left[-\sin t \, dt \; (+\ln 5)\right]$$

$$+ \int_1^e -\frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right)dt$$

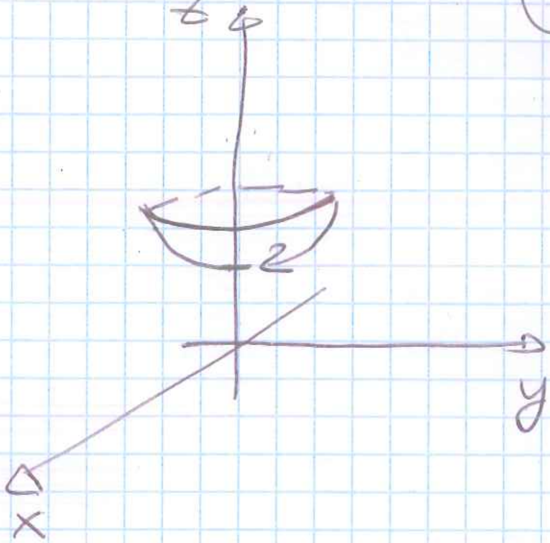
$$= \Big[\cos t\Big]_0^{\pi/4} \left(+\frac{\pi}{4}\ln 5\right) + \left[\cos\left(\frac{\pi}{4}t\right)\right]_1^e = \frac{\sqrt{2}}{2} - 1 + \cos\left(\frac{\pi}{4}e\right) - \frac{\sqrt{2}}{2}$$

$$= \cos\left(\frac{\pi}{4}e\right) - 1.$$

5)



$\Rightarrow$



$$S: \begin{cases} x = t\cos\vartheta \\ y = t\sin\vartheta \\ z = t^2+2 \end{cases} \qquad \begin{array}{l} t \in [0,1] \\ \vartheta \in [0, \frac{\pi}{2}] \end{array}$$

$$J = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 2t \\ -t\sin\vartheta & t\cos\vartheta & 0 \end{pmatrix}$$

$$\vec{n}_e = \pm(L, M, N) = \pm\left(-2t^2\cos\vartheta, -2t^2\sin\vartheta, t\right)$$

$$W(t,\vartheta) = \sqrt{4t^4 + t^2}$$

$$\Rightarrow \text{Area } S = \int_0^{\pi/2} d\vartheta \int_0^1 |t|\sqrt{4t^2+1}\, dt$$

$$= \frac{\pi}{2}\int_0^1 t\sqrt{4t^2+1}\, dt = \frac{1}{12}\left(4t^2+1\right)^{3/2}\Big|_0^1 \frac{\pi}{2} =$$

$$= \frac{\pi}{24}\left[5^{3/2} - 1\right]$$

6) EDO lineare omogenea, definita per $x>0$.

$$y(x) = -1\, e^{\int_1^x \ln t\, dt} =$$

$$- e^{[t\ln t - t]_1^x} = - e^{x\ln x - x + 1}$$

$$= -e \cdot e^{x\ln x - x} = - \frac{x^x}{e^{x-1}}$$

Poiché $\ln x \in C^0(0,+\infty) \Rightarrow \exists!$ sol. $y \in C^1(0,+\infty)$

(soluzione globale).