SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 23/6/2022.

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1) la funzione è definita su tutto $\mathbb{R}^2$.
In $\mathbb{R}^2 - \{(0,0)\}$ è C^∞ $\Rightarrow$ differenziabile, continua e derivabile (regola del gradiente).

$$f_x = \frac{2x e^{x^2+y^2}(x^2+y^2) - [e^{x^2+y^2}-1]2x}{(x^2+y^2)^2}$$

$$= \frac{2x\left[e^{x^2+y^2}(x^2+y^2-1)+1\right]}{(x^2+y^2)^2}$$

Poiché $f(x,y) = f(y,x)$

$$\Rightarrow \quad f_y = \frac{2y\left[e^{x^2+y^2}(x^2+y^2-1)+1\right]}{(x^2+y^2)^2}$$

In $(0,0)$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\rho\to 0} \frac{e^{\rho^2}-1}{\rho^2} = 1 \quad \left(\begin{matrix}\text{LIMITE}\\ \text{NOTEVOLE}\end{matrix}\right)$$

$$\Rightarrow f \in C^0(\mathbb{R}^2).$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{1}{h}\left[\frac{e^{h^2}-1}{h^2} - 1\right]$$

$$= \lim_{h\to 0} \left[\frac{h^2 + \frac{h^4}{2} + o(h^4) - h^2}{h^3} \right] = \lim_{h\to 0} \frac{h}{2} = 0$$

Analogamente, $\frac{\partial f}{\partial y}(0,0) = 0$ (2)

DIFFERENZIABILITÀ:

$$\lim_{(h,k)\to(0,0)} \left[\left(\frac{e^{h^2+k^2} - 1}{h^2+k^2} \right) - 1 \right] \frac{1}{\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0} \frac{1}{\rho} \left[\frac{\rho^2 + \frac{\rho^4}{2} + o(\rho^4) - \rho^2}{\rho^2} \right]$$

$$= \lim_{\rho\to 0} \frac{\rho}{2} = 0$$ UNIF. RISPETTO A ϑ.

$\Rightarrow$ DIFFERENZIABILE

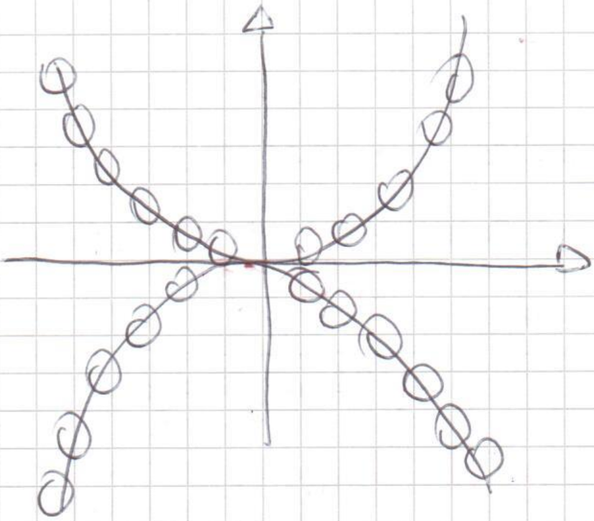
$\Rightarrow$ (regola del gradiente)

$$\frac{df(0,0)}{d\vec{v}} = \alpha f_x(0,0) + \beta f_y(0,0) = 0$$

$\forall \vec{v} = (\alpha, \beta)$

2) $f(x,y)=0 \iff y^2=x^4 \iff |y|=x^2$

$\iff y=\pm x^2$



PUNTI STAZIONARI:

$$\begin{cases} f_x = -4x^3 \\ f_y = 2y \end{cases}$$

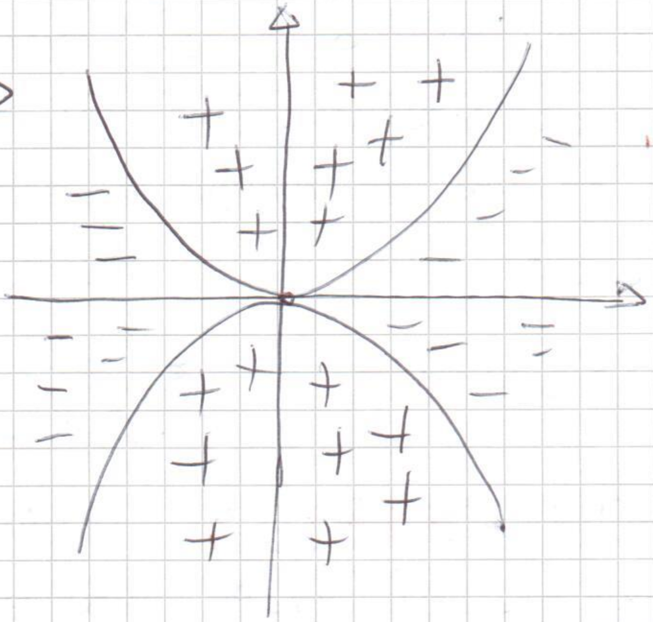
$\Rightarrow \vec{\nabla} f(x,y)=\vec{0} \iff (x,y)=(0,0)$

$f(x,y)$ SIMMETRICA RISPETTO AGLI ASSI.

BASTA STUDIARLA PER $y \geq 0$

$f(x,y) \geq 0$ se $x^2 \leq y$

$\Rightarrow$



$(0,0)$ punto di sello.

Altrimenti:

$f(0,y)=y^2 \geq 0$

$f(x,0)=-x^4 \leq 0$

In $\mathring{D}$ solo un punto di sella.

Lungo ∂D:

a) con la lagrangiana

$$\mathcal{L}(x,y,\lambda) = y^2 - x^4 - \lambda\left(x^2+y^2-1\right)$$

$$\begin{cases} \mathcal{L}_x = -4x^3 - 2\lambda x = 0 \\ \mathcal{L}_y = 2y - 2\lambda y = 2y(1-\lambda) = 0 \\ \mathcal{L}_\lambda = 1 - x^2 - y^2 = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ x = \pm 1 \\ \cancel{2x}\left(2x^2+\lambda\right) = 0 \end{cases} \cup \begin{cases} \lambda = 1 \\ 2x\left(1+2x^2\right) = 0 \\ x^2+y^2 = 1 \end{cases}$$

$$\begin{cases} y = 0 \\ x = \pm 1 \\ \lambda = -2 \end{cases} \cup \begin{cases} \lambda = 1 \\ x = 0 \\ y = \pm 1 \end{cases}$$

$f(\pm 1, 0) = -1$ PUNTI DI MIN. ASS.

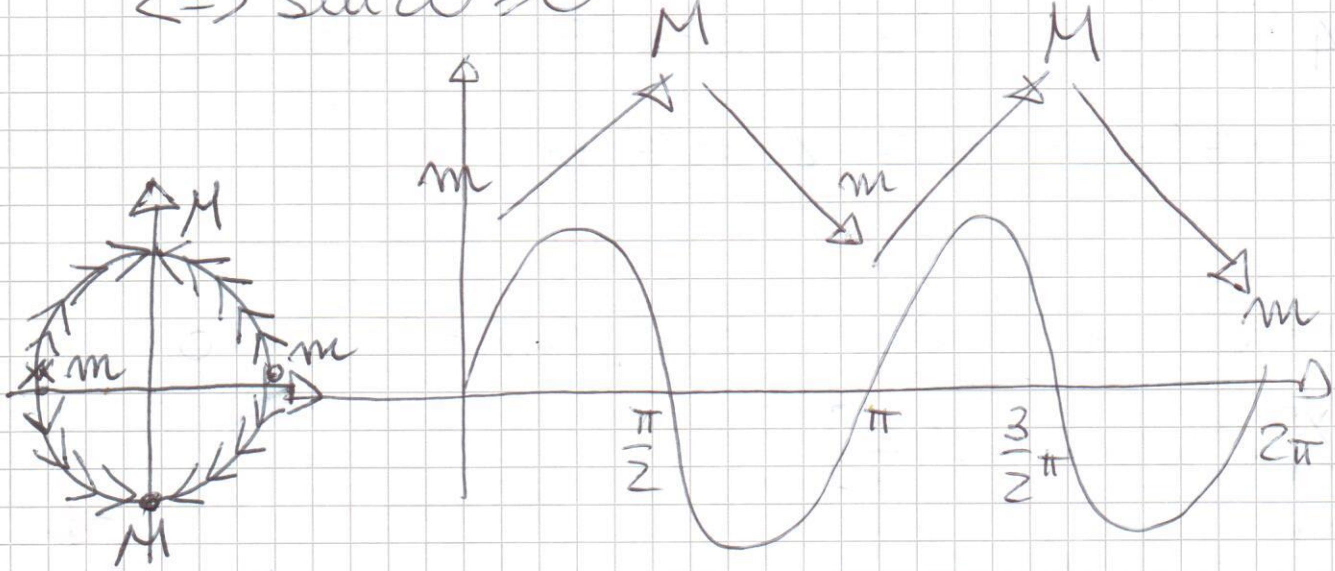
$f(0, \pm 1) = 1$ PUNTI DI MAX. ASS.

b) $f|_{\partial D} = \sin^2\vartheta - \cos^4\vartheta \quad ; \quad \vartheta \in [0, 2\pi]$

$$\left(f|_{\partial D}\right)_\vartheta = 2\sin\vartheta\cos\vartheta + 4\cos^3\vartheta\sin\vartheta$$

$$= \sin 2\vartheta \left(1 + 2\cos^2\vartheta\right) \geq 0$$

$\iff \sin 2\vartheta \geq 0$



~~f~~ In $\frac{\pi}{2}$ e $\frac{3}{2}\pi$ MAX. REL. e ASS.

In $0, \pi, \cancel{2\pi}$ MIN. REL. e ASS.

3) ~~Spezziamo la~~ La serie è definita per $x \neq 0$.

Spezziamo la serie:

$$\sum_{n=0}^{\infty} (n+1) x^n + \sum_{n=0}^{+\infty} (n+1) \left(\frac{1}{2x}\right)^n$$

Ponendo nella seconda serie $t = \frac{1}{2x}$,

abbiamo che le due serie convergono

respettivamente per $|x| < 1$; $|t| < 1$

Negli estremi abbiamo $\sum |\varphi_n| = \sum (n+1)$

$(n+1) \not\to 0 \Rightarrow$ NON CONVERGENZA.

La serie converge per

$$\begin{cases} -1 < x < 1 \\ x \neq 0 \\ \left|\frac{1}{2x}\right| < 1 \end{cases} \Rightarrow \begin{cases} (-1,0) \cup (0,1) \\ |2x| > 1 \end{cases}$$

$$\begin{cases} (-1,0) \cup (0,1) \\ x < -\frac{1}{2} \vee x > \frac{1}{2} \end{cases}$$

da cui CONV. PUNTUALE e ASSOLUTA

in $\left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$

CONV. UNIF. e TOTALE in ogni intervallo

$[\alpha, \beta] \subset \left(-1, -\frac{1}{2}\right)$ oppure $[\gamma, \delta] \subset \left(\frac{1}{2}, 1\right)$

Calcoliamo la somma:

$$\sum_{n=0}^{\infty}(n+1)x^n=\sum_{n=0}^{+\infty}\left(x^{n+1}\right)'=\sum_{k=1}^{+\infty}\left(x^k\right)'$$

$$\sum_{n=0}^{+\infty}(n+1)t^n=\sum_{n=0}^{+\infty}\left(t^{n+1}\right)'=\sum_{k=1}^{+\infty}\left(t^k\right)'$$

$$\forall x \in \left(-1,-\frac{1}{2}\right)\cup\left(+\frac{1}{2},1\right)$$

$\exists\ [\alpha,\beta]\subset\left(-1,-\frac{1}{2}\right)$, oppure $[\gamma,\delta]\subset\left(\frac{1}{2},1\right)$

in cui la serie converge totalmente

$$\Rightarrow \sum_{k=1}^{+\infty}\left(x^k\right)'=\left(\sum_{k=1}^{+\infty}x^k\right)'=\left(\frac{x}{1-x}\right)'$$

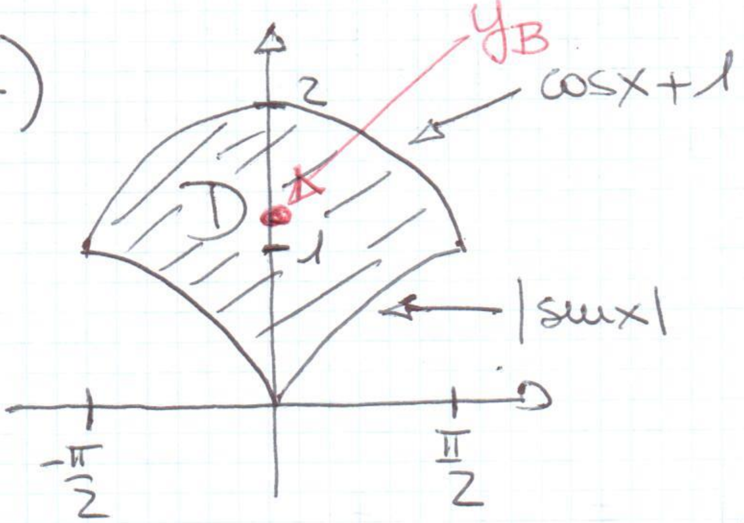
$$=\left[\frac{1-x+x}{(1-x)^2}\right]=\frac{1}{(1-x)^2}$$

$$\cancel{\sum_{k=1}^{+\infty}\left(\frac{1}{2x}\right)^k=}\ \sum_{k=1}^{+\infty}\left(t^k\right)'=\frac{1}{(1-t)^2}$$

$$\Rightarrow \sum_{n=0}^{+\infty}(n+1)\left[x^n+\frac{1}{(2x)^n}\right]=\frac{1}{(1-x)^2}+\frac{1}{\left(1-\frac{1}{2x}\right)^2}$$

$$=\frac{1}{(1-x)^2}+\frac{4x^2}{(2x-1)^2}$$

4)



Per simmetria $x_B = 0$

$$y_B = \frac{\iint_D y\,dx\,dy}{\text{Area } D} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \int_{|\sin x|}^{\cos x+1} y\,dy}{\int_{-\frac{\pi}{2}}^{\pi/2} dx \int_{|\sin x|}^{\cos x+1} dy}$$

$$= \frac{\cancel{2}\int_0^{\frac{\pi}{2}} dx \int_{\sin x}^{\cos x+1} y\,dy}{\cancel{2}\int_0^{\pi/2} dx \int_{\sin x}^{\cos x+1} dy} = \frac{\int_0^{\pi/2} dx \left[\frac{y^2}{2}\right]_{\sin x}^{\cos x+1}}{\int_0^{\pi/2} dx \left[\cos x + 1 - \sin x\right]}$$

$$= \frac{1}{2}\,\frac{\int_0^{\pi/2}\left[(\cos x+1)^2 - \sin^2 x\right]dx}{\left[\sin x + x + \cos x\right]_0^{\pi/2}}$$

$$= \frac{1}{2}\,\frac{\int_0^{\pi/2}\left[\cos^2 x - \sin^2 x + 2\cos x + 1\right]dx}{\cancel{1} + \frac{\pi}{2} - \cancel{1}}$$

$$= \frac{1}{\pi} \int_0^{\pi/2} \left[\cos 2x + 2\cos x + 1\right] dx$$

$$= \frac{1}{\pi} \left[\frac{1}{2} \operatorname{sen} 2x + 2 \operatorname{sen} x + x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \left[2 + \frac{\pi}{2} \right] = \frac{1}{2} + \frac{2}{\pi} > 1$$

Per il volume applichiamo il Teorema di Guldino

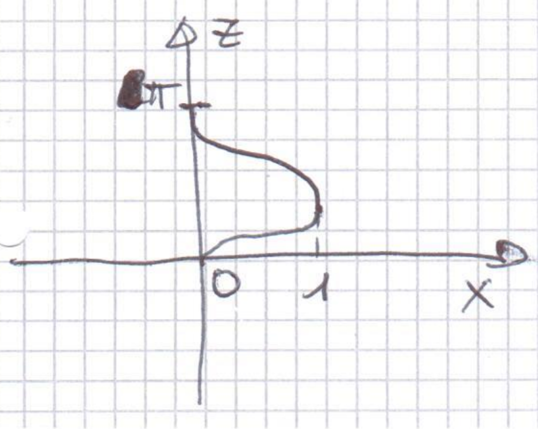
$$\cancel{Vol = 2\pi y_B \cdot Area\, D = 2\pi \left(\frac{1}{2} + \frac{2}{\pi}\right)\frac{\pi}{2}}$$

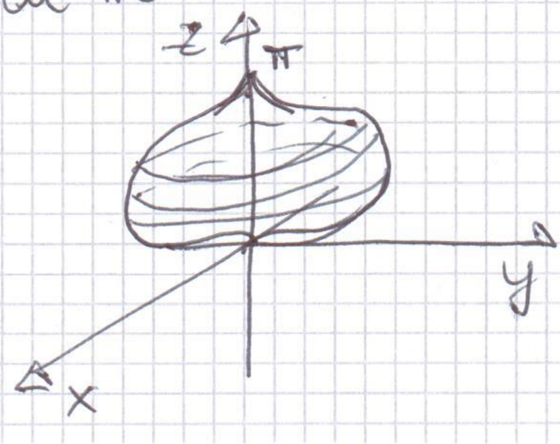
$$N.B.: \ Area\, D = 2 \int_0^{\frac{\pi}{2}} dx \int_{\operatorname{sen} x}^{\cos x + 1} dy = 2\left(\frac{\pi}{2}\right) = \pi$$

$$Vol = 2\pi y_B \, Area\, D = 2\pi \left(\frac{1}{2} + \frac{2}{\pi} \right) \pi$$

$$= (\pi + 4)\pi.$$

5) La superficie è chiusa, quindi frontiera di un dominio limitato di $\mathbb{R}^3$.





Poiché $\vec{\nabla}\cdot\vec{F} = \cos x + 1 - \cos x = 1$

$$\Rightarrow \Phi_S(\vec{F}) = \iiint_D 1\,dx\,dy\,dz = \text{vol } D.$$

$$= \pi\int_0^{\pi} \sin^4 t\,dt = \pi\int_0^{\pi} (\sin^2 t)^2\,dt =$$

$$= \pi\int_0^{\pi}\left[\frac{1-\cos 2t}{2}\right]^2 dt = \frac{\pi}{4}\int_0^{\pi}\left[1-2\cos 2t+\cos^2 2t\right]dt$$

$$= \frac{\pi}{4}\left[\pi - \cancel{\sin 2t\Big|_0^{\pi}} + \int_0^{\pi}\left[\frac{1+\cos 4t}{2}\right]dt\right.$$

$$= \frac{\pi}{4}\left[\pi + \frac{1}{2}\pi + \cancel{\frac{1}{8}\sin 4t\Big|_0^{\pi}}\right] = \frac{3}{8}\pi^2$$

6) Omogenea associata:

$$y''-2y'+2y=0 \quad \Rightarrow \alpha^2-2\alpha+2=0$$

$$\alpha_{1,2} = 1\pm\sqrt{-1} = 1\pm i$$

$$\Rightarrow y_0(x) = e^x\left[C_1\cos x + C_2\sin x\right]$$

Poiché $\alpha = i$ non è radice del polinomio caratteristico $\Rightarrow y_p(x) = A\cos x + B\sin x$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x$$

$$\Rightarrow y_p'' - 2y_p' + 2y_p = -A\cos x - 2B\cos x + 2A\cos x$$
$$-B\sin x + 2A\sin x + 2B\sin x$$

$$= (A-2B)\cos x + (B+2A)\sin x = 5\cos x$$

$$\Longleftrightarrow \begin{cases} A - 2B = 5 \\ B = -2A \end{cases} \Rightarrow \begin{cases} 5A = 5 \\ B = -2A \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases}$$

$$\Rightarrow y(x) = e^x\left[C_1\cos x + C_2\sin x\right]$$
$$+ \cos x - 2\sin x$$

$$y(0) = C_1 + 1 = 1 \quad \Rightarrow C_1 = 0$$

$$y(\pi) = \cancel{e^\pi\left[-C_1\right]} - 1 = -1 \qquad \underline{\underline{\text{ok}}}\ \forall C_2 \in \mathbb{R}$$

$\Rightarrow \infty$ soluzioni:

$$y(x) = C_2 e^x \sin x + \cos x - 2\sin x,$$

$$C_2 \in \mathbb{R}$$