SVOLGIMENTI PROVA SCRITTA di ANALISI 2
del 24/1/2023.

1) Studiamo la convergenza assoluta, col criterio della radice:

$$\sqrt[k]{|\varphi_k(x)|} = \frac{|x-3|^k}{2^k+1} \underset{k\to+\infty}{\sim} \left(\frac{|x-3|}{2}\right)^k \xrightarrow[k\to+\infty]{} 0 < 1$$

$$\Longleftrightarrow \quad |x-3|<2 \quad \Longleftrightarrow \quad -2<x-3<2$$

$$\Longleftrightarrow \quad 1<x<5$$

$\Rightarrow$ converge assolutamente e semplicemente in $(1,5)$.

In $(-\infty,1)\cup(5,+\infty)$ non converge.

Estremi: $x=5$: $\displaystyle\sum_{k=0}^{+\infty} \frac{2^{k^2}}{(2^k+1)^k}$.

$$\frac{2^{k^2}}{(2^k+1)^k} \underset{k\to+\infty}{\sim} 1 \not\xrightarrow[k\to+\infty]{} 0$$ la serie diverge ass. e sempl.

Per $x=1$ $$\sum_{k=0}^{+\infty} \frac{(-2)^{k^2}}{(2^k+1)^k} = \sum_{k=0}^{+\infty} \frac{(-1)^{k^2}2^{k^2}}{(2^k+1)^k}$$

$$= \sum_{k=0}^{+\infty} \frac{(-1)^k 2^{k^2}}{(2^k+1)^k}$$ serie a segno alterno.

Ma $|a_k| = \dfrac{2^{k^2}}{(2^k+1)^k} \not\longrightarrow 0$, come prima. (2)

$\Rightarrow$ CONV. ASS. e SEMPLICE in $(1,5)$.

CONV. TOTALE:

$$\sup_{(1,5)} |\varphi_k(x)| = \frac{2^{k^2}}{(2^k+1)^k} \underset{k\to+\infty}{\not\longrightarrow} 0$$

$\Rightarrow$ NO CONV. TOTALE in $(1,5)$.

Consideriamo $[1+\varepsilon, 5-\varepsilon]$

$$\Rightarrow \sup_{[1+\varepsilon,5-\varepsilon]} |\varphi_k(x)| = \frac{(2-\varepsilon)^k}{(2^k+1)^k} \underset{k\to 0}{\sim} \left(\frac{2-\varepsilon}{2}\right)^k$$

$\sum \left(\frac{2-\varepsilon}{2}\right)^k$ serie geometrica convergente

$\Rightarrow$ CONV. TOTALE (e UNIFORME)

in ogni $[1+\varepsilon, 5-\varepsilon]$ o, più in generale, in ogni compatto $[\alpha,\beta] \subset (1,5)$.

2) $\forall (x,y) \neq (0,0)$

$$|f(x,y)| = \left| \frac{x\left[y^2 - y^2 + \frac{y^6}{6} + o(y^6)\right]}{\sqrt{x^2+y^2}} \right| \underset{(x,y)\to(0,0)}{\sim} \left| \frac{xy^6}{6\sqrt{x^2+y^2}} \right|$$

$$\leq \left| \frac{xy^6}{6x} \right| = \frac{y^6}{6} \underset{(x,y)\to(0,0)}{\longrightarrow} 0$$

Analogamente, in coordinate polari: (3)

$$|f(\rho,\vartheta)| \sim \frac{\rho^7 |\cos\vartheta \sin^6\vartheta|}{6\rho} \leq \frac{\rho^6}{6} \xrightarrow[\rho\to 0]{} 0$$

$\Rightarrow f$ CONTINUA in $(0,0)$

$f(x,0)=0 \Rightarrow \frac{\partial f}{\partial x}(0,0)=0$

$f(0,y)=0 \Rightarrow \frac{\partial f}{\partial y}(0,0)=0$

DIFFERENZIABILITÀ:

$$\lim_{(h,k)\to(0,0)} \left|\frac{\Delta f - df}{\sqrt{h^2+k^2}}\right| = \lim_{(h,k)\to(0,0)} \left|\frac{hk^6}{6(h^2+k^2)}\right|$$

$$= \lim_{\rho\to 0} \frac{\rho^7 |\cos\vartheta \sin^6\vartheta|}{6\rho^2} \leq \lim_{\rho\to 0} \frac{\rho^5}{6} = 0$$

f differenziabile in $(0,0) \Rightarrow$ formula del gradiente $\frac{df}{d\vec{v}}(0,0) = \alpha f_x(0,0) + \beta f_y(0,0)$

$= 0$

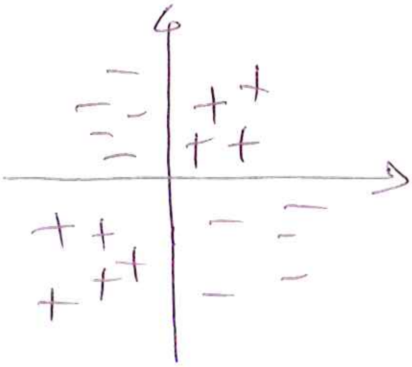
$\forall \vec{v} = (\alpha,\beta)$.

3) PUNTI STAZIONARI:

$$f_x = y \quad ; \quad f_y = x \quad \Rightarrow \vec{\nabla} f = \vec{0}$$
$$\Leftrightarrow (x,y) = (0,0)$$

$(0,0)$ punto di sello:

$[f(0,0) = 0]$



Analogamente,
con l'hessiano:

$$f_{xx} = f_{yy} = 0 \quad ; \quad f_{xy} = f_{yx} = 1$$

$$\Rightarrow |H_f(0,0)| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0.$$

$(0,0) \in \mathring{D}$.

Su ∂D:

Con la Lagrangiana:

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(x^2 + 4y^2 - 1)$$

$$\begin{cases} \mathcal{L}_x = y - 2\lambda x = 0 \\ \mathcal{L}_y = x - 8\lambda y = 0 \\ x^2 + 4y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} y = 2\lambda x \\ x - 16\lambda^2 x = 0 \\ x^2 + 16\lambda^2 x^2 - 1 = 0 \end{cases}$$

$$\begin{cases} y = 2\lambda x \\ x(1 - 16\lambda^2) = 0 \\ (1 + 16\lambda^2) x^2 = 1 \end{cases}$$

$$\begin{cases} x=0 \\ y=0 \\ \cancel{0=1} \end{cases} \cup \begin{cases} \lambda = \pm \frac{1}{4} \\ x^2 = \frac{1}{1+16\lambda^2} = \frac{1}{2} \\ y = 2\lambda x = \pm \frac{1}{2} x \end{cases}$$

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$$\begin{cases} x = \pm \frac{1}{\sqrt{2}} \\ y = \pm \frac{1}{2\sqrt{2}} \\ \lambda = \pm \frac{1}{4} \end{cases}$$

N.B.: il $\pm$ della x è indipendente dal $\pm$ di y. Abbiamo cioè 4 punti, coerentemente col fatto che $f(x,y)$ è ANTISIMMETRICA rispetto a entrambi gli assi.

$P_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$; $P_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$; $P_3 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$, $P_4 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$.

Per le simmetrie, $f(P_1) = f(P_3) = \frac{1}{4}$ MAX. ASS.

$f(P_2) = f(P_4) = -\frac{1}{4}$. MIN. ASS.

Analogamente, con la parametrizzazione:

$$\partial D: \begin{cases} x = \cos\vartheta \\ y = \frac{1}{2}\sin\vartheta \end{cases} \quad \vartheta \in [0, 2\pi]$$

$$f(\vartheta) = \frac{1}{2}\cos\vartheta \sin\vartheta = \frac{1}{4}\sin 2\vartheta$$

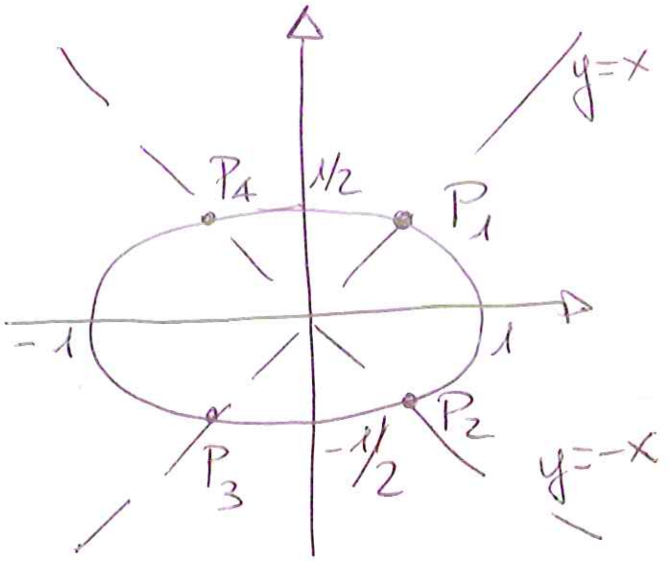
la funzione assume MAX. ASS.

per $2\vartheta = \frac{\pi}{2} + 2k\pi \Rightarrow \vartheta = \frac{\pi}{4} + k\pi$

$$\Rightarrow \vartheta_1 = \frac{\pi}{4} \; ; \; \vartheta_2 = \frac{5}{4}\pi$$

e assume MIN. ASS. per $2\vartheta = -\frac{\pi}{2} + 2k\pi$

$$\Rightarrow \vartheta = -\frac{\pi}{4} + k\pi \Rightarrow \vartheta_3 = \frac{3\pi}{4} \; ; \; \vartheta_4 = \frac{7}{4}\pi$$



$(x(\vartheta_1), y(\vartheta_1)) = \cancel{\left(\frac{\sqrt{2}}{2}\right)}$
$= \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = P_1$

$(x(\vartheta_2), y(\vartheta_2))$
$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = P_3$

$$(x(\vartheta_3), y(\vartheta_3)) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = P_4$$

$$(x(\vartheta_4), y(\vartheta_4)) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = P_2$$

4) $\vec{F} = \vec{F}_1 + \vec{F}_2$

con $\vec{F}_1 = (x^2 + \cos x \sin y \, ; \, 2zy + \sin x \cos y \, ; \, y^2)$

che è conservativo:

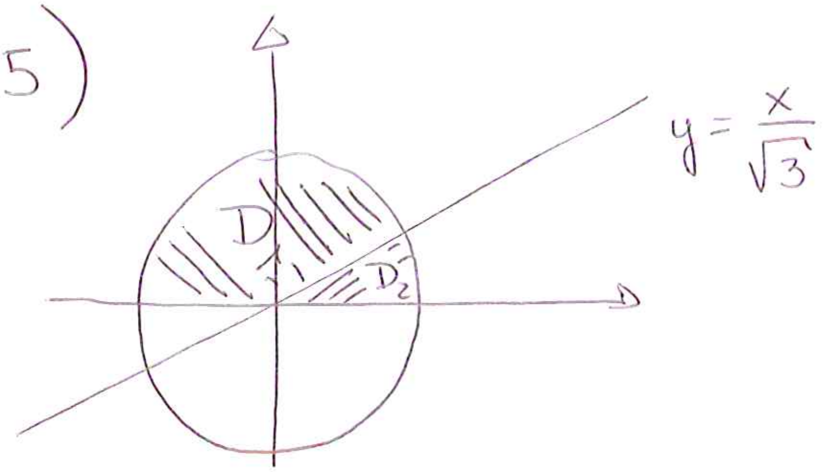
$$(X_1)_y = \cos x \cos y = (Y_1)_x$$
$$(X_1)_z = 0 = (Z_1)_x \qquad \Rightarrow \oint \vec{F}_1 d\vec{s} = 0$$
$$(Y_1)_z = 2y = (Z_1)_y$$

$$\Rightarrow \oint \vec{F} = \oint \vec{F}_2 \qquad \text{con } \vec{F}_2 = (0, 0, 2y) \qquad 7$$

$$= \int_0^{2\pi} \left[2 \sin t \, (-\sin t) \right] dt = -2 \int_0^{2\pi} \sin^2 t \, dt$$

$$= \int_0^{2\pi} \left[\cos 2t - 1 \right] dt = \left[\frac{1}{2} \sin 2t - t \right]_0^{2\pi} = -2\pi .$$

5)



$$|x - \sqrt{3} y| = \begin{cases} x - \sqrt{3} y & \text{se } y \le \frac{x}{\sqrt{3}} \\ \sqrt{3} y - x & \text{se } y \ge \frac{x}{\sqrt{3}} \end{cases}$$

$$D = D_1 \cup D_2$$

$$\text{con } D_1 = \left\{ x^2 + y^2 \le 1 \; ; \; y \ge 0 \; ; \; y \ge \frac{x}{\sqrt{3}} \right\}$$

$$= \left\{ 0 \le \rho \le 1 \; ; \; \vartheta \in [0, \pi] \; ; \; \vartheta \in \left[\frac{\pi}{6}, \frac{7}{6} \pi \right] \right\}$$

$$= \left\{ 0 \le \rho \le 1 \; ; \; \vartheta \in \left[\frac{\pi}{6}, \pi \right] \right\}$$

$$D_2 = \left\{ x^2 + y^2 \le 1 \; ; \; y \ge 0 \; ; \; y \le \frac{x}{\sqrt{3}} \right\}$$

$$= \left\{ 0 \le \rho \le 1 \; ; \; \vartheta \in [0, \pi] \; ; \; \vartheta \in \left[-\frac{5}{6} \pi, \frac{\pi}{6} \right] \right\}$$

$$= \left\{ 0 \le \rho \le 1 \; ; \; \vartheta \in \left[0, \frac{\pi}{6} \right] \right\}$$

come si può ricavare <u>molto più velocemente</u> osservando il disegno del dominio.

$$\iint_D |x-\sqrt{3}y|\,dx\,dy = \iint_{D_1} (\sqrt{3}y - x)\,dx\,dy + \iint_{D_2} (x-\sqrt{3}y)\,dx\,dy$$

$$= \int_{\frac{\pi}{6}}^{\pi} d\vartheta \int_0^1 \rho^2(\sqrt{3}\sin\vartheta - \cos\vartheta)\,d\rho + \int_0^{\frac{\pi}{6}} d\vartheta \int_0^1 \rho^2(\cos\vartheta - \sqrt{3}\sin\vartheta)\,d\rho$$

$$= \left\{\left[-\sqrt{3}\cos\vartheta - \sin\vartheta\right]_{\frac{\pi}{6}}^{\pi} + \left[\sin\vartheta + \sqrt{3}\cos\vartheta\right]_0^{\frac{\pi}{6}}\right\}\left[\frac{\rho^3}{3}\right]_0^1$$

$$= \frac{1}{3}\left[\sqrt{3} + \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{3}{2} - \sqrt{3}\right] = \frac{4}{3}.$$

6) $z = y'$

$$\Rightarrow \begin{cases} z'' + z = x^2 + 1 \\ z(0) = z'(0) = 0 \end{cases}$$

omo: $\alpha^2 + 1 = 0 \Rightarrow \alpha_{1,2} = \pm i$

$$\Rightarrow z_0(x) = C_1\cos x + C_2 \sin x$$

$$z_p(x) = Ax^2 + Bx + C$$

$$z_p' = 2Ax + B \quad ; \quad z_p'' = 2A$$

$$2A + Ax^2 + Bx + C = x^2 + 1$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 0 \\ C = 1 - 2A \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \\ C = -1 \end{cases}$$

$$\Rightarrow z_{NO}(x) = C_1 \cos x + C_2 \sin x + x^2 - 1$$

$$z(0) = C_1 - 1 = 0$$

$$z'_{NO}(x) = -C_1 \sin x + C_2 \cos x + 2x$$

$$z'(0) = C_2 = 0$$

$$\Rightarrow z(x) = \cos x + x^2 - 1$$

$$\Rightarrow y(x) = \sin x + \frac{x^3}{3} - x + C_3$$

$$y(0) = C_3 = 0$$

$$\Rightarrow y(x) = \sin x + \frac{x^3}{3} - x.$$

Analogamente, studiando direttamente la EDO di terzo ordine

$$\Rightarrow \text{OMO: } \alpha^3 + \alpha = 0 \Rightarrow \alpha_1 = 0 \; ; \; \alpha_2 = i \; ; \; \alpha_3 = -i$$

$$= \alpha(\alpha^2 + 1)$$

$$\Rightarrow y_0(x) = C_1 \cos x + C_2 \sin x + C_3$$

Poichè $\alpha=0$ è radice del polinomio caratteristico $\Rightarrow y_p(x)=x(Ax^2+Bx+C)$

$$\Rightarrow y_p' = 3Ax^2+2Bx+C$$

$$y_p'' = 6Ax+2B$$

$$y_p''' = 6A$$

$$\Rightarrow 6A+3Ax^2+2Bx+C = x^2+1$$

$$\Rightarrow \begin{cases} 3A=1 \\ 2B=0 \\ C=1-6A \end{cases} \Rightarrow \begin{cases} A=\frac{1}{3} \\ B=0 \\ C=-1 \end{cases}$$

$$\Rightarrow y_{NO}(x) = C_1\cos x + C_2 \sin x + C_3 + \frac{1}{3}x^3 - x$$

Imponendo le 3 condizioni iniziali, si arriva alla stessa soluzione ottenuta col metodo dell'abbassamento dell'ordine.