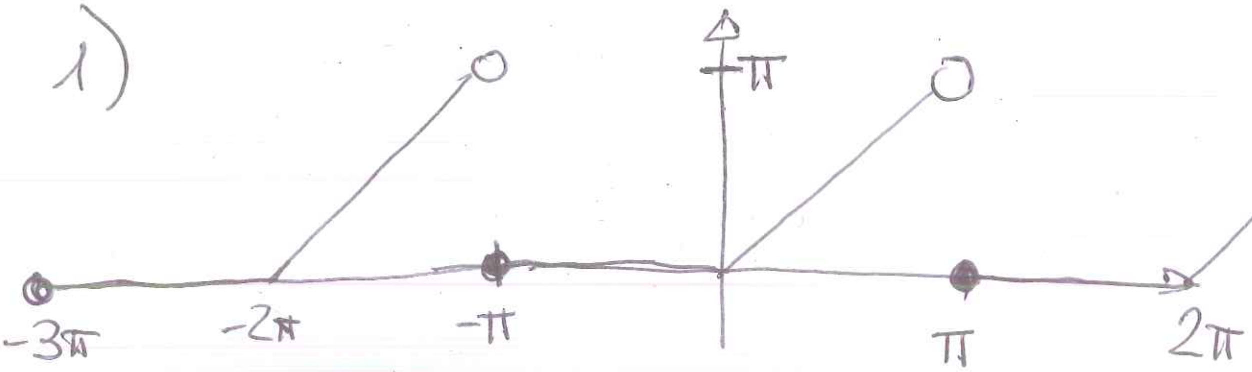
SVOLGIMENTI PROVA SCRITTA di ANALISI 2 - 24/6/21.

1)



$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} x\cos kx\,dx = \frac{1}{\pi}\left[\frac{1}{k}x\sin kx\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx\,dx\right]$$

$$= \frac{1}{\pi k}\left[\frac{1}{k}\cos kx\right]_0^{\pi} = \frac{1}{\pi k^2}\left[(-1)^k - 1\right]$$

$$= \cancel{\frac{1}{\pi}}\begin{cases} \dfrac{-2}{\pi(2m+1)^2} & \text{se } k = 2m+1 \\ 0 & \text{se } k = 2m \end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} x\sin kx\,dx = \frac{1}{\pi}\left[-\frac{x}{k}\cos kx\Big|_0^{\pi}\right.$$

$$\left. + \frac{1}{k}\int_0^{\pi}\cos kx\,dx\right] = \frac{1}{\pi k}\left[-\pi(-1)^k + \cancel{\frac{1}{k}\sin kx}\right]_0^{\pi}$$

$$= \frac{1}{k}(-1)^{k+1}$$

$$\Rightarrow f(x) \sim \frac{\pi}{4} + \sum_{m=0}^{+\infty} \frac{-2}{\pi(2m+1)^2} \cos[(2m+1)x]$$

$$+ \sum_{k=0}^{+\infty} (-1)^{k+1} \frac{1}{k} \operatorname{sen}(kx) = S(x)$$

$$S(x) = \begin{cases} f(x) & \forall x \neq (2k+1)\pi \\ \frac{\pi}{2} & \forall x = (2k+1)\pi \end{cases}$$

CONVERGENZA PUNTUALE in $\mathbb{R}$

CONVERGENZA UNIFORME

$$\forall [a,b] \subset ((2k-1)\pi, (2k+1)\pi)$$

NON SI HA CONVERGENZA TOTALE

2) f definito in tutto $\mathbb{R}^2$.

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} \left| \frac{\left[xy + \frac{(xy)^3}{3}\right] - xy}{(x^2+y^2)} \right| =$$

$$= \lim_{(x,y)\to(0,0)} \left| \frac{1}{3} \left[\frac{x^3 y^3}{(x^2+y^2)} \right] \right|$$

$$= \frac{1}{3} \lim_{\rho\to 0} \left[\rho^6 \frac{|\sin^3\vartheta \cos^3\vartheta|}{\rho^2} \right] \leq \frac{1}{3} \lim_{\rho\to 0} \rho^4 = 0$$

f CONTINUA in $(0,0)$.

$$f(x,0) = f(0,y) = 0 \quad \Rightarrow f_x(0,0) = f_y(0,0) = 0$$

DIFFERENZIABILITÀ:

$$\lim_{(h,k)\to(0,0)} \left| \frac{\Delta f - df}{\sqrt{h^2+k^2}} \right| = \lim_{(h,k)\to(0,0)} \left| \frac{1}{3} \left[\frac{h^3 k^3}{(h^2+k^2)\sqrt{h^2+k^2}} \right] \right|$$

$$= \frac{1}{3} \lim_{\rho\to 0} \frac{\rho^6 |\cos^3\vartheta \sin^3\vartheta|}{\rho^3} \leq \frac{1}{3} \lim \rho^3 = 0$$

$\Rightarrow f$ DIFF. BILE in $(0,0)$

$\Rightarrow$ Formula del gradiente

$$\frac{df}{d\vec{v}}(0,0) = \alpha f_x(0,0) + \beta f_y(0,0)$$

$$= 0 \qquad \forall \vec{v}$$

3) $$f_x = \left[2x + (x^2+y^2)(-2x)\right] e^{-x^2-y^2}$$

$$= 2x\left[1-x^2-y^2\right] e^{-x^2-y^2}$$

Analogamente:

$$f_y = 2y\left[1-x^2-y^2\right] e^{-x^2-y^2}$$

$$f_x = 0 \Rightarrow \begin{cases} x=0 \\ 2y(1-y^2)e^{-y^2}=0 \end{cases} \cup \begin{cases} x^2+y^2=1 \\ f_y=0 \end{cases}$$

$$\Rightarrow (0,0) \cup \{x^2+y^2=1\} \cup \begin{cases} x=0 \\ y=\pm 1 \end{cases}$$ (già presenti nella circonferenza)

CIRCONFERENZA

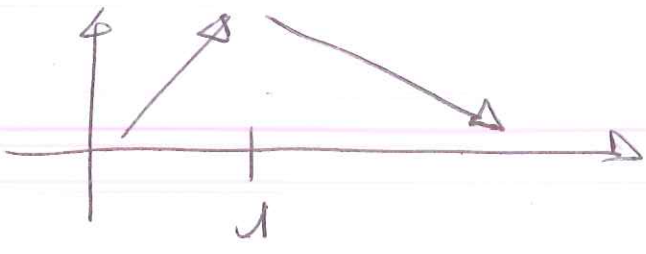
$$f(x,y) \geq -1$$

$$f(0,0) = -1$$ MIN. ASSOLUTO.

Operiamo la sostituzione $t = x^2+y^2 \geq 0$

$$f(t) = t e^{-t} - 1$$

$$f'(t) = (1-t)e^{-t} > 0 \iff t < 1$$



$$\lim_{t\to+\infty} f(t) = -1$$

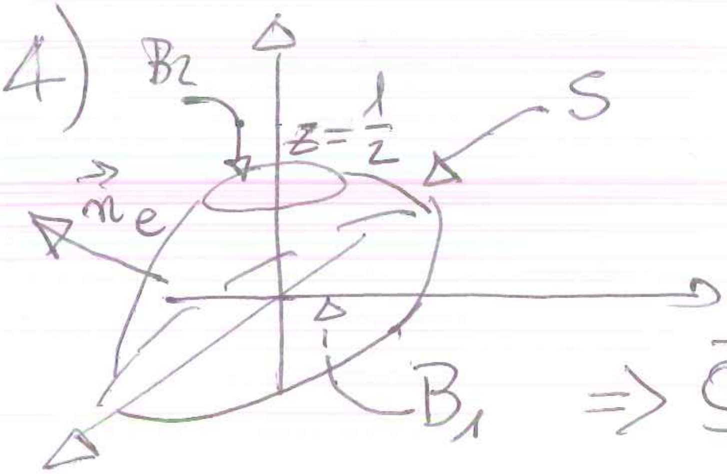
$$f(1) = \frac{1}{e} - 1 \quad \text{MAX. ASSOLUTO}$$

Ma $t = 1 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow$ TUTTI I PUNTI DELLA CIRCONFERENZA $\{x^2+y^2=1\}$ SONO PUNTI DI MAX. ASSOLUTO.

Quindi f è limitata: $-1 \leq f(x,y) \leq -1 + \frac{1}{e}$

4)



$$\operatorname{div} \vec{F} = y + 1 - y = 1$$

$$\Rightarrow \Phi_{\partial\Omega}(\vec{F}) = \iiint_{\Omega} 1 \, dx\,dy\,dz$$

$= \operatorname{vol} \Omega.$

Integrazione per strati: $\quad \forall z \in [0, \frac{1}{2}] \quad x^2 + y^2 \leq 1 - z^2$

$$= \int_0^{1/2} \text{Area } A(z)\, dz = \int^{1/2} \pi (1 - z^2)\, dz$$

$$= \pi \left[z - \frac{z^3}{3} \right]_0^{\frac{1}{2}} = \pi \left[\frac{1}{2} - \frac{1}{24} \right] = \frac{11}{24} \pi.$$

Può essere didatticamente utile riscrivere Ω in coordinate sferiche.

$$\Omega = \left\{ 0 \le \rho \le 1 \; ; \; 0 \le \rho \underbrace{\cos\varphi}_{\ge 0} \le \frac{1}{2} \right\} \qquad \varphi \in \left[0, \frac{\pi}{2}\right]$$

$$= \left\{ 0 \le \rho \le 1 \; ; \; 0 \le \rho \le \frac{1}{2\cos\varphi} \right\}$$

Queste due condizioni devono essere rispettate entrambe. Si ha

$$\begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le 1 \\ \text{se } 1 \le \frac{1}{2\cos\varphi} \end{cases} \cup \begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le \frac{1}{2\cos\varphi} \\ \text{se } 1 \ge \frac{1}{2\cos\varphi} \end{cases}$$

$$\begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le 1 \\ \cos\varphi \le \frac{1}{2} \end{cases} \cup \begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le \frac{1}{2\cos\varphi} \\ \cos\varphi \ge \frac{1}{2} \end{cases}$$

$$\begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le 1 \\ \varphi \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \end{cases} \cup \begin{cases} \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \le \rho \le \frac{1}{2\cos\varphi} \\ \varphi \in \left[0, \frac{\pi}{3}\right] \end{cases}$$

$$\Rightarrow \Omega = \left\{ \varphi \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right];\ \rho \in [0,1] \right\} \cup \qquad 7$$

$$\left\{ \varphi \in \left[0, \frac{\pi}{3}\right];\quad 0 \le \rho \le \frac{1}{2\cos\varphi} \right\}$$

$$\Rightarrow \operatorname{vol} \Omega = 2\pi \left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^1 \rho^2 \sin\varphi \, d\rho + \int_0^{\frac{\pi}{3}} d\varphi \int_0^{\frac{1}{2\cos\varphi}} \rho^2 \sin\varphi \, d\rho \right]$$

$$= \left[\cancel{\sin\varphi}\right]^{2\pi} \left[-\left[\cos\varphi\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{\rho^3}{3}\right]_0^1 \right.$$

$$\left. + \int_0^{\frac{\pi}{3}} d\varphi \sin\varphi \left[\frac{\rho^3}{3}\right]_0^{\frac{1}{2\cos\varphi}} \right] =$$

$$= 2\pi \left[\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \left[\int_0^{\frac{\pi}{3}} d\varphi \sin\varphi \left[\frac{1}{8\cos^3\varphi} \right] \right] \right]$$

$$2\pi \left[\frac{1}{6} + \frac{1}{\cancel{24}_{12}} \left[\frac{-\cancel{2}1}{\cos^2\varphi} \right]_0^{\frac{\pi}{3}} \right]$$

$$= 2\pi \left[\frac{1}{6} - \frac{1}{12} \left[\frac{1}{4} - 1 \right] \right] = \cancel{\frac{1}{48}}\pi \left[\frac{1}{3} + \frac{1}{8} \right] = \frac{11}{24}\pi.$$

Se avessimo calcolato direttamente (8)
il flusso, avremmo ottenuto

$$\Phi_{\partial\Omega}(\vec{F}) = \Phi_{B_1}(\vec{F}) + \Phi_{B_2}(\vec{F}) + \Phi_{S}(\vec{F})$$

In B_1: $\vec{n}_e = (0,0,1)$

$$\vec{F}\cdot\vec{n}_e = \left(x+y-y\frac{1}{2}\right) = x + \frac{y}{2}$$

In B_2: $\vec{n}_e = (0,0,-1)$

$$\vec{F}\cdot\vec{n}_e = -(x+y)$$

In S: $\begin{cases} x = \sin\varphi\cos\vartheta \\ y = \sin\varphi\sin\vartheta \\ z = \cos\varphi \end{cases}$ $\quad \varphi\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ $\quad \vartheta\in[0,2\pi]$

$$J = \begin{pmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta & -\sin\varphi \\ -\sin\varphi\sin\vartheta & \sin\varphi\cos\vartheta & 0 \end{pmatrix}$$

$L = \sin^2\varphi\cos\vartheta$; $M = \sin^2\varphi\sin\vartheta$

$N = \sin\varphi\cos\varphi \geq 0$ per $\varphi\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$

$$\Rightarrow \vec{n}_e = \left(\sin^2\varphi \cos\vartheta \,;\, \sin^2\varphi \sin\vartheta \,;\, \sin\varphi\cos\varphi \right)$$

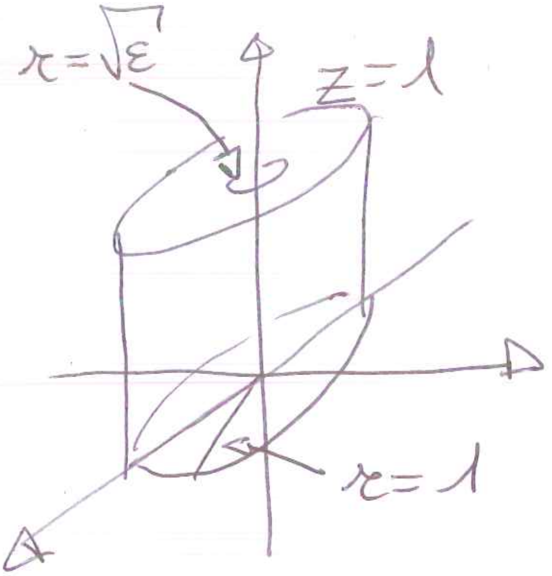
$\Rightarrow$ Su S

$$\vec{F}\cdot\vec{n}_e = \sin^2\varphi\cos\vartheta\sin\vartheta\left(\sin^2\varphi\cos\vartheta\right) + \sin\varphi\sin\vartheta\left(\sin^2\varphi\sin\vartheta\right) + \left[\left(\sin\varphi\cos\vartheta + \sin\varphi\sin\vartheta - \sin\varphi\cos\varphi\sin\vartheta\right)\cdot\sin\varphi\cos\varphi\right]$$

L'integrale diventa troppo complicato.

5) Coordinate cilindriche

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = t \end{cases}$$



$J = \rho$

$$\Rightarrow \left\{ \sqrt{\varepsilon} \le \rho \le 1 \,;\, 0 \le \vartheta \le 2\pi \,;\, 0 \le t \le 1 \right\}$$

$$\Rightarrow \int_0^{2\pi} d\vartheta \int_0^1 t\,dt \int_{\sqrt{\varepsilon}}^1 \rho\left[\frac{\rho-(1+\rho)\ln(1+\rho)}{\rho^3(1+\rho)}\right]d\rho$$

Dominio: parallelepipedo.

Integrando a variabili separabili

$$\Rightarrow I_\varepsilon = 2\pi \left[\frac{t^2}{2}\right]_0^1 \int_{\sqrt{\varepsilon}}^1 \underbrace{\left[\frac{1}{\rho(1+\rho)} - \frac{\ln(1+\rho)}{\rho^2}\right]}_{\text{per parti}} d\rho$$

$$= \pi \left[\frac{1}{\rho}\ln(1+\rho)\Big|_{\sqrt{\varepsilon}}^1 + \int_{\sqrt{\varepsilon}}^1 \cancel{\left[\frac{1}{\rho(1+\rho)} - \frac{1}{\rho(1+\rho)}\right]} d\rho \right]$$

$$= \pi \left[\ln 2 - \frac{\ln(1+\sqrt{\varepsilon})}{\sqrt{\varepsilon}} \right]$$

Pertanto, poichè $\lim_{t\to 0} \frac{\ln(1+t)}{t} = 1$,

allora

$$\lim_{\varepsilon \to 0} I_\varepsilon = \pi\left[\ln 2 - 1\right] < 0$$

Si osservi, per inciso, che l'integrale si sarebbe potuto risolvere subito osservando che

(11)

$$\frac{1}{\rho(1+\rho)} - \frac{\ln(1+\rho)}{\rho^2} = \left[\frac{\ln(1+\rho)}{\rho}\right]'$$

$$\Rightarrow \int_{\sqrt{\varepsilon}}^{1} \left[\frac{\ln(1+\rho)}{\rho}\right]' d\rho = \frac{\ln(1+\rho)}{\rho}\Bigg|_{\sqrt{\varepsilon}}^{1}$$

Il fatto che $\lim_{\varepsilon \to 0} I_\varepsilon < 0$ è coerente col segno dell'integrando, poiché, ~~per~~ per $\rho \to 0$, il numeratore dell'integrando è negativo:

$$\rho - (1+\rho)\ln(1+\rho) = \rho - (1+\rho)\left[\rho - \frac{\rho^2}{2} + o(\rho^2)\right]$$

$$= \cancel{\rho} - \cancel{\rho} - \rho^2 + \frac{\rho^2}{2} + o(\rho^2) = \frac{-\rho^2}{2} + o(\rho^2).$$

6) Definito per $x \neq 0$; $y \neq 0$

Visto il problema di Cauchy, considereremo

$x \in (-\infty, 0)$; $y \in (-\infty, 0)$

$$A(x) = \frac{1}{x(1+x^2)} \in C^\infty(-\infty, 0)$$

$$B(y) = \frac{1+y^2}{y} = \frac{1}{y} + y \in C^\infty(-\infty, 0)$$

$\exists!$ soluzione locale.

Soluzioni singolari: $1+y^2=0$

$\Rightarrow \nexists$ sol. sing.

Separazione variabili:

$$\int \frac{y}{1+y^2}\,dy = \int \frac{1}{x(1+x^2)}\,dx$$

$$\frac{1}{2}\ln(1+y^2) = \int \left[\frac{A}{x} + \frac{Bx+C}{1+x^2}\right] dx$$

$$\frac{A + Ax^2 + Bx^2 + Cx}{x(1+x^2)} = \frac{1}{x(1+x^2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

$$\Rightarrow \frac{1}{2}\ln(1+y^2)=\int\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx$$

$$\frac{1}{2}\ln(1+y^2)=\ln|x|-\frac{1}{2}\ln(1+x^2)+K$$

$$\ln(1+y^2)=2\ln|x|-\ln(1+x^2)+\underbrace{2K}_{=\ln C}$$

$(C>0)$

$$(1+y^2)=\frac{Cx^2}{1+x^2}=C\left[1-\frac{1}{1+x^2}\right]$$

C.I: $y(-1)=-1$

$$\Rightarrow 2=\frac{C}{2} \Rightarrow C=4$$

$$\Rightarrow 1+y^2=\frac{4x^2}{1+x^2}$$

$$\Rightarrow y^2=\frac{4x^2}{1+x^2}-1$$

N.B.: $y<0$

$$\Rightarrow \quad \underline{y=-\sqrt{\frac{4x^2}{1+x^2}-1}<0}$$

la sol. è definita per $\dfrac{4x^2-1-x^2}{1+x^2}>0$

$$\Rightarrow \quad \frac{3x^2-1}{1+x^2}>0 \quad \Rightarrow \quad \underline{x<-\frac{1}{\sqrt{3}}} \;\vee\; \cancel{x>\frac{1}{\sqrt{3}}}$$

(ci interessano le $x<0$)

cioè per $x\in\left(-\infty,-\frac{1}{\sqrt{3}}\right)$.