SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 24/6/2024 - COMPITO A

A_1

1) $D_c = \mathbb{R}$. $\varphi_n(x) > 0$

Per $x=0$: $\varphi_n(0) = \dfrac{n}{n+1} \xrightarrow[n\to+\infty]{} 1$

Per $x \neq 0$: $e^{nx} \longrightarrow \begin{cases} +\infty & \text{se } x>0 \\ -\infty & \text{se } x<0 \end{cases}$

Pertanto $\lim\limits_{n\to+\infty} \varphi_n(x) = \begin{cases} 0 & \text{se } x>0 \\ 1 & \text{se } x=0 \\ 1 & \text{se } x<0 \end{cases} = \varphi(x)$

$D_{cp} = \mathbb{R}$.

Non può esserci convergenza uniforme in $\mathbb{R}$, perché $\varphi_n(x) \in C^0(\mathbb{R})$, ma $\varphi(x) \notin C^0(\mathbb{R})$.

In $(0,+\infty)$:

$$|\varphi_n(x) - \varphi(x)| = \varphi_n(x) = \frac{n}{e^{nx}+n}$$

$$\sup_{(0,+\infty)} |\varphi_n(x) - \varphi(x)| \geq \varphi_n\left(\frac{1}{n}\right) = \frac{n}{n+e} \longrightarrow 1 \neq 0$$

NO CONV. UNIF. in $(0,+\infty)$.

$$\sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} |\varphi_n - \varphi| = \sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} \varphi_n \leq \sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} \frac{n}{e^{nx}} = n e^{-n\alpha} \xrightarrow[n\to+\infty]{} 0$$

decrescente

In $(-\infty, 0]$: (A2)

$$|f_n(x) - f(x)| = \Big|\underbrace{\frac{n}{e^{nx}+n}}_{\le 1} - 1\Big| = 1 - \frac{n}{e^{nx}+n}$$

$$|f_n - f|' = \frac{n^2 e^{nx}}{(e^{nx}+n)^2} > 0 \qquad \forall x \in (-\infty, 0]$$

$$\Rightarrow \sup_{(-\infty,0]} |f_n - f| = 1 - \frac{n}{n+1} \xrightarrow[n\to+\infty]{} 0$$

$\Rightarrow$ CONV. UNIF. in $(-\infty, 0]$.

Poiché $\varphi_n(x) \not\to 0$ in $(-\infty, 0]$ $\Rightarrow$ NO CONV. PUNTUALE

In $(0, +\infty)$:
$$\varphi_n(x) \sim \frac{n}{e^{nx}}$$

la serie $\sum \frac{n}{e^{nx}}$ converge (ad es. per il criterio della radice: $\sqrt[n]{\varphi_n} = \frac{\sqrt[n]{n}}{e^x} \to \frac{1}{e^x} < 1 \quad \forall x \in (0, +\infty)$

$$D_{cp} = D_{ca} = (0, +\infty)$$

CONV. TOTALE:
$$\varphi_n(x) = \frac{n}{e^{nx}+n} \text{ decrescente in } (0,+\infty)$$

$$\Rightarrow \sup_{(0,+\infty)} |\varphi_n| = \lim_{x\to 0^+} \frac{n}{e^{nx}+n} = \frac{n}{n+1}.$$

$$= \varphi_n(0)$$

$\sum \varphi_n(0) = \sum \frac{n}{n+1}$ diverge $\Rightarrow$ NO CONV. TOT. in $(0,+\infty)$.

$\sup_{\substack{[\alpha,+\infty) \\ \alpha>0}} |\varphi_n| = \varphi_n(\alpha) = \frac{n}{e^{n\alpha}+n} \le \frac{n}{e^{n\alpha}}$

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e $\sum \frac{n}{e^{n\alpha}}$ (come già visto) converge $\forall \alpha > 0$

$\Rightarrow$ CONV. TOT. in ogni $[\alpha,+\infty)$, $\alpha>0$.

2) La $f(x,y)$ è sicuramente di classe C^∞ in $\mathbb{R}^2-\{(0,0)\}$, quindi è di classe $C^1(\mathbb{R}-\{(0,0)\})$.

In $(0,0)$: $\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{\rho\to0} \left| \frac{\cos(\rho^2)-e^{\rho^2\cos^2\vartheta}}{\rho} \right|$

$= \lim_{\rho\to0} \left| \frac{\cancel{1}-\frac{\rho^4}{2}+o(\rho^4)-\cancel{1}-\rho^2\cos^2\vartheta+o(\rho^2)}{\rho} \right|$

$= \lim_{\rho\to0} \left| \frac{-\rho^2\cos^2\vartheta+o(\rho^2)}{\rho} \right| \le \lim_{\rho\to0} \rho = 0$

f CONTINUA in $(0,0)$.

$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to0} \frac{\cos(h^2)-e^{h^2}}{h|h|} = \lim_{h\to0} \frac{-h^2}{h|h|}$ $\nexists$

$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to0} \frac{\cos(k^2)-1}{k|k|} = \lim_{k\to0} \frac{\left(-\frac{k^4}{2}\right)}{k|k|}$

$$= -\frac{1}{2}\lim_{k\to 0} \frac{k^{\cancel{2}}|k|^{\cancel{2}}}{\cancel{k}\cancel{|k|}} = -\frac{1}{2}\lim_{k\to 0} k|k| = 0.$$

A_4

Poiché $\nexists\ f_x(0,0)$, f NON è differenziabile in $(0,0)$, quindi NON è C^1 in $(0,0)$.

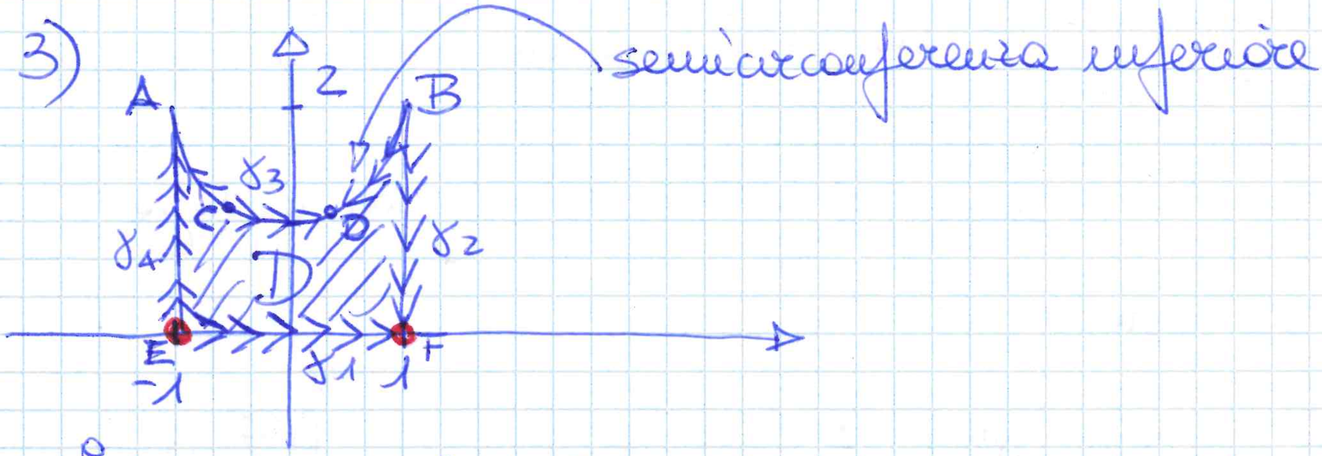
$$\frac{df}{d\vec{v}}(0,0) = \lim_{t\to 0} \frac{\cos(t^2) - e^{\alpha^2 t^2}}{t|t|}$$

$$= \lim_{t\to 0} \frac{-\alpha^2 t^2}{t|t|}$$

Se $\alpha = 0$ $\quad \frac{df}{d\vec{v}}(0,0) = 0$ (UNICA DER. DIR: infatti $f_y(0,0)=0$)

Se $\alpha \neq 0$ $\quad \nexists\ \frac{df}{d\vec{v}}(0,0)$.

3)

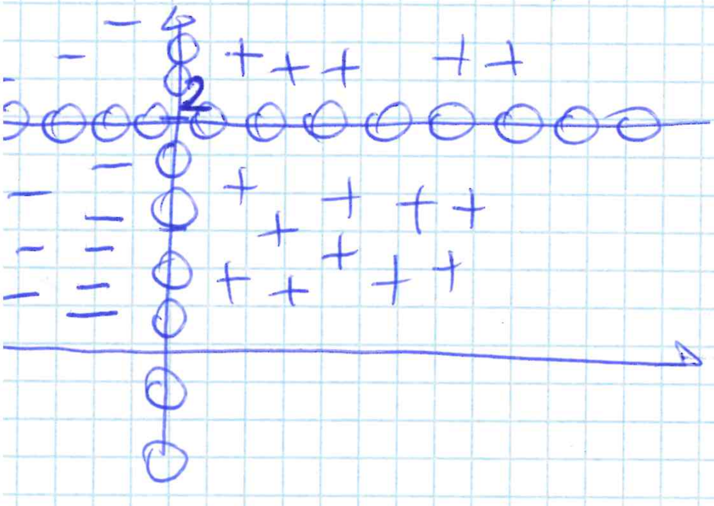


In $\mathring{D}$: $\begin{cases} f_x = (y-2)^2 = 0 \\ f_y = 2x(y-2) = 0 \end{cases} \Rightarrow \begin{cases} y = 2 \\ x \in \mathbb{R} \end{cases}$

Pertanto gli unici punti stazionari si trovano sulla frontiera di D.

: se non richiesto, si osservi che il segno di f è

A5



i i punti stazionari liberi $(x, 2)$ sono di
EL. per $x<0$; di MIN. REL. per $x>0$;
LLA $x=0$.
amo pertanto aspettarci che, anche sulla ∂D,
$(1,2)$ sia di MIN. REL. e $A\equiv(-1,2)$ sia di
EL.
e, si osservi che D è SIMMETRICO rispetto
sse y, mentre f è ANTISIMMETRICA:

$$(-x, y) = -x(y-2)^2 = -f(x,y).$$

li in punti simmetrici rispetto all'asse y
ortamento di f è opposto.

$\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.

$\begin{cases} x\in[-1,1] \\ y=0 \end{cases}$; $\gamma_2: \begin{cases} x=1 \\ y\in[0,2] \end{cases}$; ~~$\gamma_3:$~~

$$f\big|_{\gamma_3} = f(x, 2-\sqrt{1-x^2}) = x\left(-\sqrt{1-x^2}\right)^2 = x(1-x^2)$$
$$= x - x^3$$

A6

$$\left(f\big|_{\gamma_3}\right)' = 1-3x^2 \geq 0 \iff -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$$

Se $x = \pm\frac{1}{\sqrt{3}} \Rightarrow y = 2-\sqrt{\frac{2}{3}} > 0$

$f\big|_{\gamma_4} = -(y-2)^2$ (parabola con vertice in $y=2$ e rivolta verso il basso)

$A \equiv (-1,2)$ di MAX. REL $\Rightarrow B \equiv (1,2)$ di MIN. REL.
(per antisimmetria)

$f(-1,2)=0$ $\qquad f(1,2)=0$

$C \equiv \left(-\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right)$ di MIN. REL.

$\Rightarrow D = \left(\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right)$ di MAX. REL.

$$f\left(\pm\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right) = \pm\frac{1}{\sqrt{3}}\cdot\frac{2}{3}$$

$E = (-1,0)$ di MIN. REL. $\Rightarrow F \equiv (1,0)$ di MAX. REL.

$f(\pm 1, 0) = \pm 4$ $\qquad$ (si osservi che $0 < \frac{2}{3\sqrt{3}} < 2$)

$\Rightarrow$ MAX. ASS. in $(1,0)$: $f(1,0) = 4$

MIN. ASS. in $(-1,0)$: $f(-1,0) = -4$.

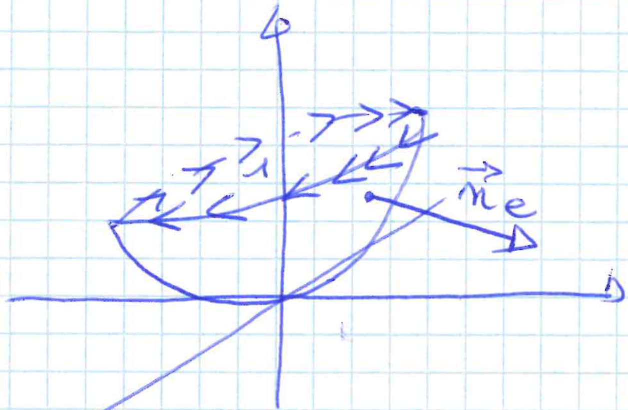
4) A 7

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x-y & y+z & x^2 z \end{vmatrix} = -\hat{\imath} - 2xz\,\hat{\jmath} + \hat{k}$$

Se calcoliamo direttamente il flusso, allora

$$S: \begin{cases} z = x^2+y^2 \\ x^2+y^2 \leq 1 \end{cases} ; \quad J = \begin{pmatrix} 1 & 0 & 2x \\ 0 & 1 & 2y \end{pmatrix}$$

$$(L,M,N) = \pm(-2x, -2y, 1)$$



Serve $N<0$

$$\Rightarrow (L,M,N) = (2x, 2y, -1)$$

$$(\vec{\nabla} \wedge \vec{F}) \cdot \vec{n}_e = -2x - 4xyz - 1$$

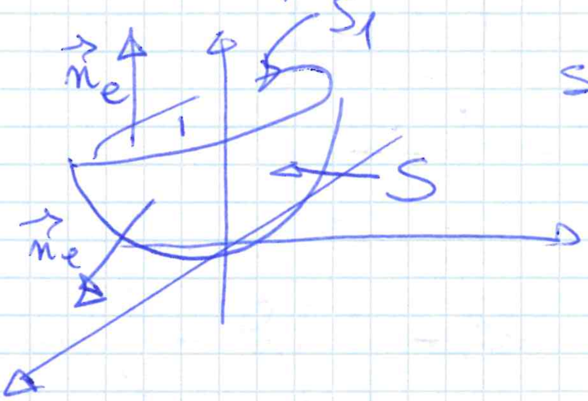
$$\Phi_S(\vec{\nabla} \wedge \vec{F}) = -\iint_{D(x,y)} (+2x + 4xy(x^2+y^2) + 1)\,dx\,dy$$

$$= -\int_0^1 \rho\, d\rho \int_0^{2\pi} [2\rho\cos\vartheta + 4\rho^4 \cos\vartheta \sin\vartheta + 1]\, d\vartheta$$

$$= -\int_0^1 d\rho \left[\rho\left(\cancel{2\rho\sin\vartheta} + \cancel{2\rho^4 \frac{\sin^2\vartheta}{2}} + \vartheta\right)_0^{2\pi}\right]$$

$$= -2\pi \int_0^1 \rho\, d\rho = -2\pi \frac{\rho^2}{2}\Big|_0^1 = -\pi$$

In alternativa, considerando la superficie CHIUSA $\Sigma = S \cup S_1$,



sappiamo che

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$$0 = \Phi_\Sigma = \Phi_{S \cup S_1}$$
$$= \Phi_S + \Phi_{S_1}$$

$$\Rightarrow \underbrace{\Phi_S = -\Phi_{S_1}}$$

Ma $S_1 : \begin{cases} x^2+y^2 \le 1 \\ z = 1 \end{cases} \Rightarrow \cancel{\begin{cases} z = 1 \\ x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \quad \begin{matrix} \rho \in [0,1] \\ \vartheta \in [0,2\pi]. \end{matrix}}$

e $\vec{n}_e = (0,0,1)$

$$\Rightarrow (\vec{\nabla} \wedge \vec{F}) \wedge \vec{n}_e \Big|_{S_1} = (-1, -2\rho\cos\vartheta, 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow \Phi_S = -\Phi_{S_1} = -\iint_{D(x,y)} 1 \, dx\,dy = -\text{Area}(S_1) = -\pi.$$

Applicando il Teorema del Rotore, invece,

$$\Phi_S(\vec{\nabla} \wedge \vec{F}) = \oint_{\partial S^+} \vec{F} \cdot \vec{ds} =$$

$BS: \begin{cases} z = 1 \\ x = \cos\vartheta \\ y = \sin\vartheta \end{cases} \quad \vartheta \in [0, 2\pi]$ $\qquad A_9$

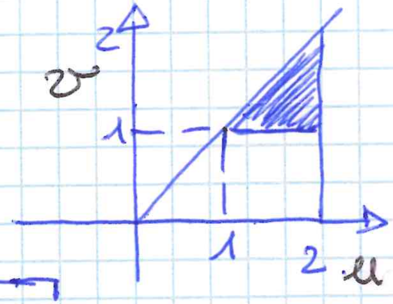
BS^+ è CONTROVERSO rispetto al verso di crescita di ϑ

$$\Rightarrow \Phi_S(\vec{\nabla} \wedge \vec{F}) = -\int_0^{2\pi} \Big[(\cos\vartheta - \sin\vartheta)(-\sin\vartheta) + (1 + \sin\vartheta)\cos\vartheta\Big] d\vartheta$$

$$= -\int_0^{2\pi} \Big[\cancel{2\sin\vartheta\cos\vartheta} + \sin^2\vartheta + \cos\vartheta\Big] d\vartheta$$

$$= -\Big[\cancel{\frac{-\cos 2\vartheta}{2}} + \frac{1}{2}\Big[\vartheta - \cancel{\frac{1}{2}\cos 2\vartheta}\Big] \cancel{+ \sin\vartheta}\Big]_0^{2\pi} = -\pi.$$

5) $$J = \begin{pmatrix} 1 & 1 & v \\ 2v & -2v & u \end{pmatrix}$$



$$W(u,v) = \sqrt{(u + 2v^2)^2 + (2v^2 - u)^2 + (4v)^2}$$

$$f(x(u,v),y(u,v),z(u,v)) = \frac{uv}{\sqrt{\frac{(2u)^2}{2}+2(2v^2)^2+8(2v^2)}}$$

A$_{10}$

$$= \frac{uv}{\sqrt{2u^2+8v^4+16v^2}}$$

$$\Rightarrow \int f\,dS = \int_1^2 du \int_1^u dv \frac{uv}{\cancel{\sqrt{2u^2+8v^4+16v^2}}} \cdot \cancel{\sqrt{2u^2+8v^4+16v^2}}$$

$$= \int_1^2 du \cdot u\left[\frac{v^2}{2}\right]_1^u = \frac{1}{2}\int_1^2 \left[u^3-u\right] du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}-\frac{u^2}{2}\right]_1^2 = \frac{1}{2}\left[\left(\frac{16}{4}-\frac{4}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[2+\frac{1}{4}\right] = \frac{9}{8}$$

6) In forma normale (ponendo $x>0$)

$$y' - \frac{1}{2\sqrt{x}}\,y = 0$$

$$\Rightarrow y(x) = -e^{\int_1^x \frac{1}{2\sqrt{t}}dt} = -e^{\left[\sqrt{t}\right]_1^x} = -e^{\sqrt{x}-1}.$$