SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 24/10/2019

1)
$$\sum f_n(x) = \sum \frac{1}{1+\frac{1}{n}} \left(\frac{x}{e^2}\right)^n = \sum \frac{n}{n+1} \left(\frac{x}{e^2}\right)^n$$

Poniamo $t = \frac{x}{e^2}$. La serie $\sum \frac{n}{n+1} t^n$ converge assolutamente e ~~semplicemente~~ puntualmente in $(-1, 1)$

Per $t = \pm 1$ $\quad \left| \frac{n}{n+1}(-1)^n \right| = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1 \neq 0$

quindi la serie NON converge negli estremi.

$\Rightarrow$ la serie di potenza converge assolutamente e puntualmente

per $x \in (-e^2; e^2) = I$

Converge totalmente e uniformemente in ogni intervallo $[a, b] \subset I$.

2) $$|f(x,y)| \underset{(x,y)\to(0,0)}{\sim} \left|\frac{x^n}{x^2+y^2}\right| = \left|\frac{\rho^n \cos^n\vartheta}{\rho^2}\right|$$ 2

$$= \rho^{n-2}|\cos^n\vartheta| \le \rho^{n-2} \longrightarrow 0$$

$$\iff \underline{n>2}$$

$$f(0,y)=0 \implies \frac{\partial f}{\partial y}(0,0)=0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\left[\frac{h^n}{\operatorname{sen}(h^2)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{h^{n-1}}{h^2} = \lim_{h\to 0} h^{n-3} = \begin{cases} 0 & \text{se } n>3 \\ 1 & \text{se } n=3 \\ \nexists & \text{se } n<3 \end{cases}$$

DERIVATE DIREZIONALI (per $\alpha \neq 0$):

$$\lim_{t\to 0}\left[\frac{(\alpha t)^n}{\operatorname{sen}(t^2)}\right]\frac{1}{t} = \lim_{t\to 0} t^{n-3}\alpha^n \begin{cases} 0 & \text{se } n>3 \\ \alpha^3 & \text{se } n=3 \\ \nexists & \text{se } n<3 \end{cases}$$

$\implies f$ NON DIFFERENZIABILE per $n=3$.

Verifica:

Per $n=3$:

3

$$\lim_{(h,k)\to(0,0)} \left[\frac{h^3}{\operatorname{sen}(h^2+k^2)} - h \right] \frac{1}{\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0} \left[\frac{\rho^3\cos^3\vartheta - \rho\cos\vartheta\,\operatorname{sen}(\rho^2)}{\operatorname{sen}(\rho^2)\rho} \right]$$

$$= \lim_{\rho\to 0} \frac{\cancel{\rho^3}}{\cancel{\rho^3}} \left[\cos^3\vartheta - \cos\vartheta \left(1 - \frac{\rho^4}{6} + o(\rho^4) \right) \right]$$

$$= \lim_{\rho\to 0} \left[\cos^3\vartheta - \cos\vartheta + \cos\vartheta \frac{\rho^4}{6} \right]$$

$$= \cos^3\vartheta - \cos\vartheta .$$

$\Rightarrow$ NON DIFFERENZIABILE.

Per $n>3$:

$$\lim_{(h,k)\to(0,0)} \left| \left[\frac{h^n}{\operatorname{sen}(h^2+k^2)} \right] \frac{1}{\sqrt{h^2+k^2}} \right|$$

$$= \lim_{\rho\to 0} \left| \left[\frac{\rho^n \cos^n\vartheta}{\operatorname{sen}(\rho^2)\rho} \right] \right| = \lim_{\rho\to 0} \left| \rho^{n-3} \cos^n\vartheta \right|$$

$$\le \lim_{\rho\to 0} \rho^{n-3} = 0 \qquad \forall n>3$$

3) $\begin{cases} f_x = -2x\,e^{y^2-x^2} = 0 \iff x = 0 \\ f_y = 2y\,e^{y^2-x^2} = 0 \iff y = 0 \end{cases}$ 4

$\Rightarrow$ unico punto stazionario: $(0,0)$.

$f_{xx} = -2e^{y^2-x^2}\left[1-2x^2\right]$

$f_{xy} = f_{yx} = -4xy\,e^{y^2-x^2}$

$f_{yy} = 2e^{y^2-x^2}\left[1+2y^2\right]$

$H_f(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix}$ $\Rightarrow$ punto di sella.

Sulla frontiera: $\begin{cases} x = \cos\vartheta \\ y = \sin\vartheta \end{cases}$

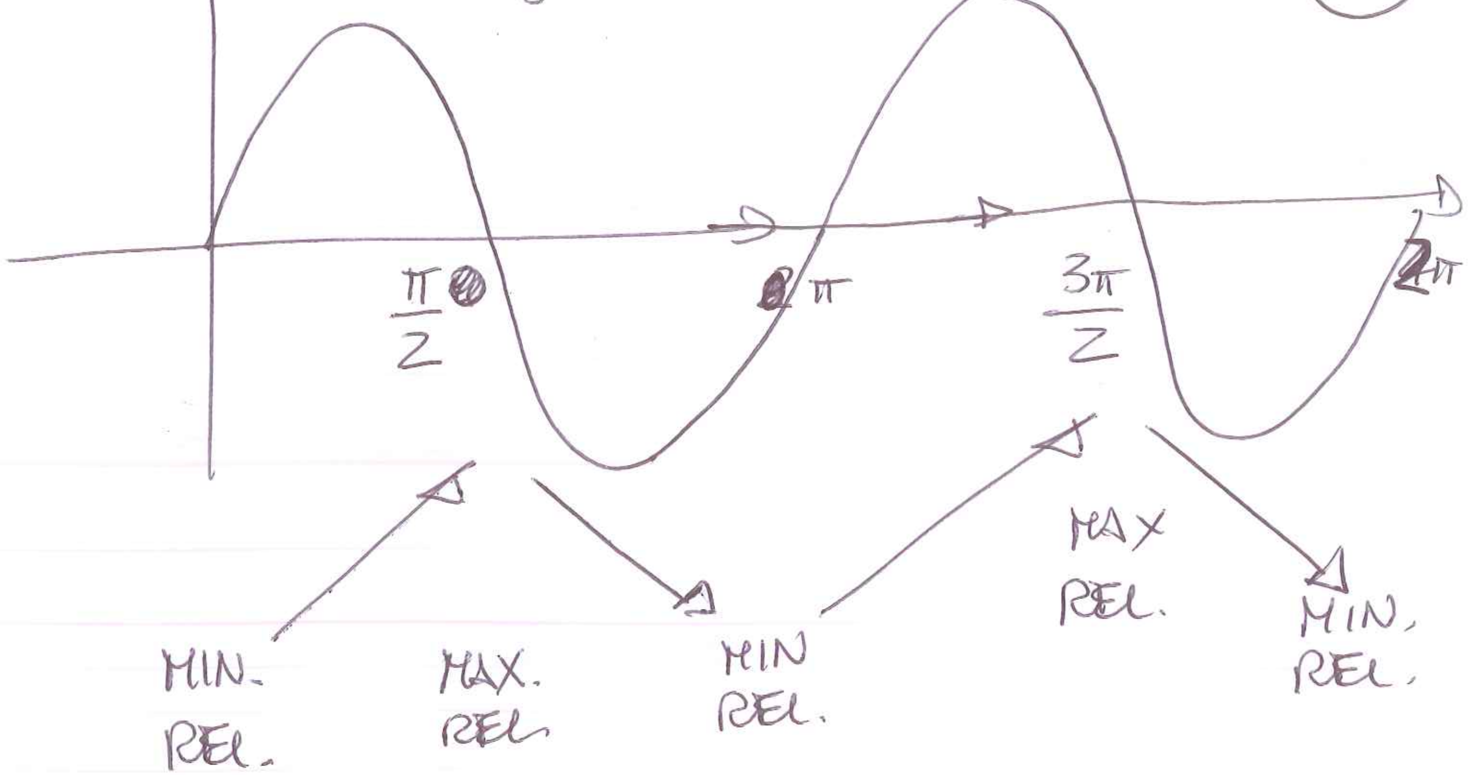
$$f|_\gamma = f(\cos\vartheta, \sin\vartheta) = e^{\sin^2\vartheta - \cos^2\vartheta} = e^{-\cos 2\vartheta}$$

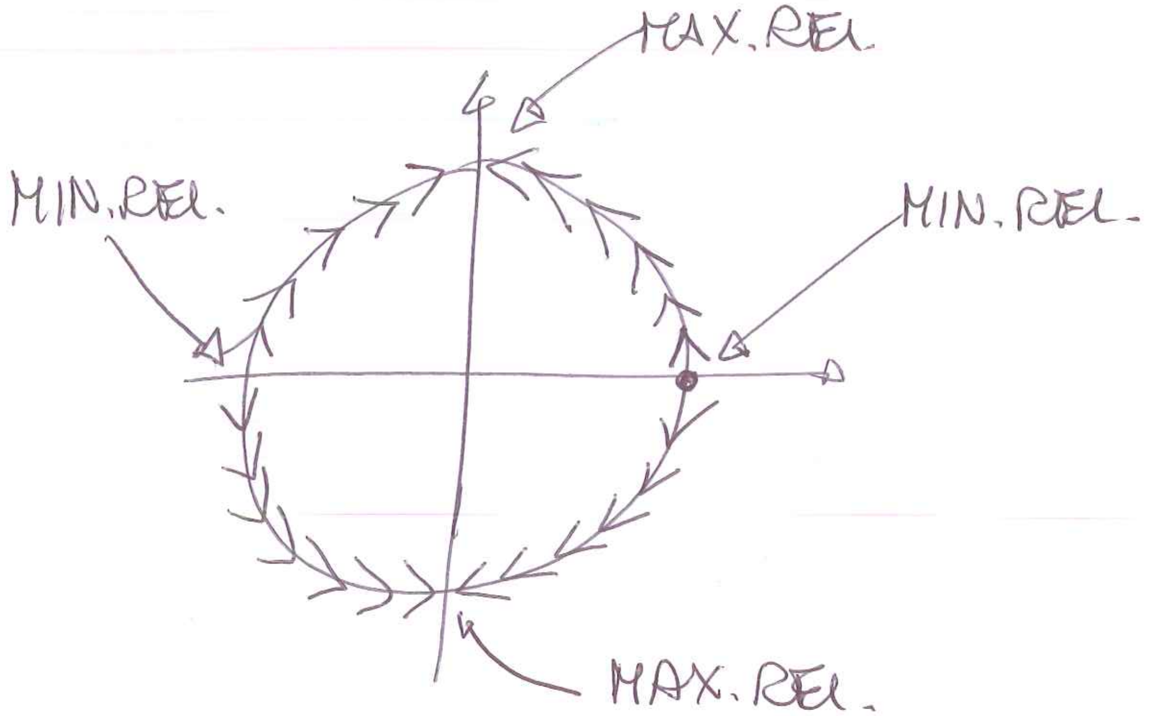
$$\vartheta \in [0, 2\pi]$$

$$(f|_\gamma)_\vartheta = 2\sin 2\vartheta\, e^{-\cos 2\vartheta} \geq 0$$

$$\iff \sin 2\vartheta \geq 0$$

$f(\theta) = \sin 2\theta$





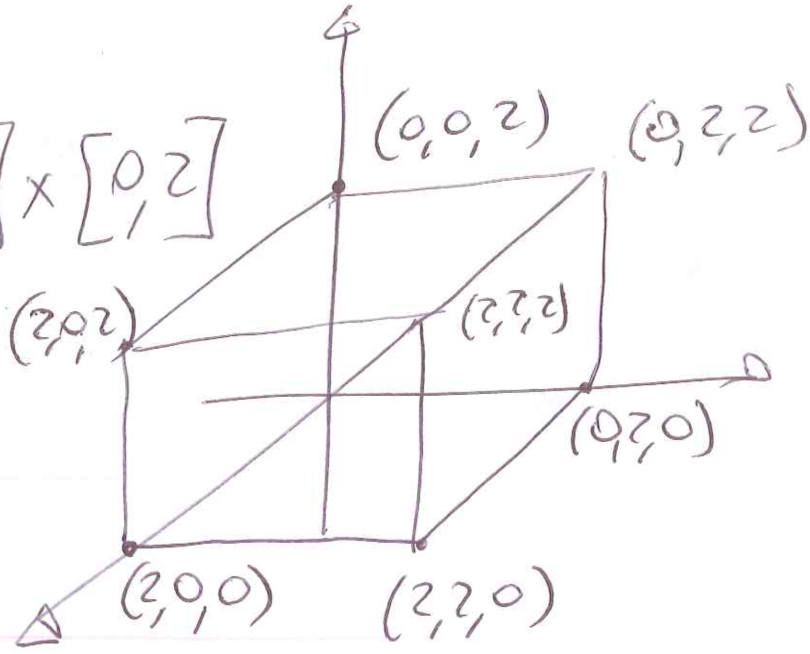
$f(0,1) = f(0,-1) = e$ MAX. ASS.

$f(1,0) = f(-1,0) = \frac{1}{e}$ MIN. ASS.

4)

(6)

$$Q = [0,2] \times [0,2] \times [0,2]$$



$$\operatorname{div} \vec{F} = 2x + 2y + 2z$$

$$\Phi_{\partial Q}(\vec{F}) = \iiint_Q 2(x+y+z)\, dx\,dy\,dz$$

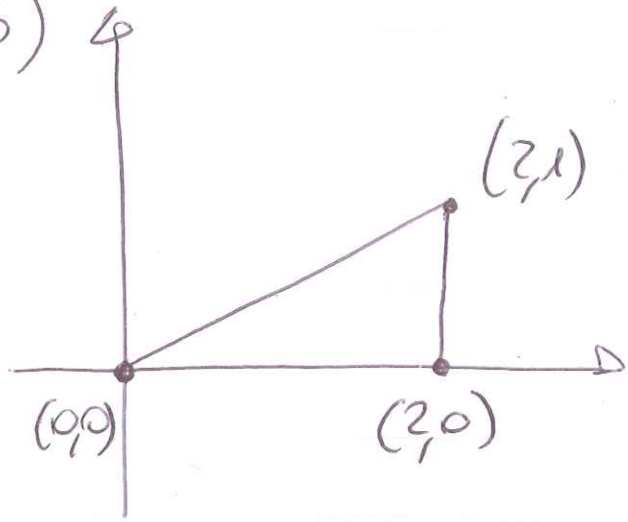
$$\iiint_Q x\, dx\,dy\,dz = \iiint_Q y\, dx\,dy\,dz$$

$$= \iiint_Q z\, dx\,dy\,dz = \dot{x}_B \operatorname{Vol}(Q)$$

$$= 1 \cdot 8 = 8$$

$$\Rightarrow \Phi_{\partial D}(\vec{F}) = 2 \cdot 8 \cdot 3 = 48$$

5)



$$T = \left\{ 0 \le x \le 2;\ 0 \le y \le \frac{x}{2} \right\}$$

$$\iint_T x^2 e^{xy}\,dx\,dy = \int_0^2 x^2\,dx \int_0^{\frac{x}{2}} e^{xy}\,dy =$$

$$= \int_0^2 x^2\,dx \left[\frac{1}{x} e^{xy} \right]_0^{\frac{x}{2}} = \int_0^2 x \left[e^{\frac{x^2}{2}} - 1 \right] dx$$

$$= \left[e^{x^2/2} - \frac{x^2}{2} \right]_0^2 = e^2 - 2 - 1 = e^2 - 3.$$

6)

$$\gamma'(t) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{-2t}{1-t^2} \right)$$

$$\| \gamma'(t) \| = \sqrt{\frac{1}{4} + \frac{3}{4} + \frac{4t^2}{(1-t^2)^2}} =$$

$$= \sqrt{1 + \frac{4t^2}{(1-t^2)^2}} = \frac{\sqrt{(1-t^2)^2 + 4t^2}}{(1-t^2)}$$

$$= \frac{\sqrt{(1+t^2)^2}}{(1-t^2)} = \frac{(1+t^2)}{(1-t^2)}$$

$$\Rightarrow \ell(\gamma) = \int_a^b \frac{(1+t^2)}{(1-t^2)}\,dt = -\int_a^b \frac{t^2+1}{t^2-1}\,dt$$

$$= -\int_a^b \left[\frac{t^2-1}{t^2-1} + \frac{2}{t^2-1}\right] dt$$

$$= -\int_a^b \left[1 - \frac{1}{t+1} + \frac{1}{t-1}\right] dt$$

$$= -\left(b-a + \ln\left(\left|\frac{t-1}{t+1}\right|\right)\Big|_a^b\right)$$

$$= a-b - \ln\left(\left|\frac{b-1}{b+1}\right| \cdot \left|\frac{a+1}{a-1}\right|\right)$$

$$\cancel{= a-b + \ln\left(\frac{(b+1)\cdot(a+1)}{(1+b)\cdot(1-a)}\right)}$$

$$\cancel{= a-b + \ln\left(\frac{1-b^2}{1-a^2}\right) + \ln}$$

$$= a-b + \ln\left(\frac{(b+1)(1-a)}{(1-b)(1+a)}\right)$$