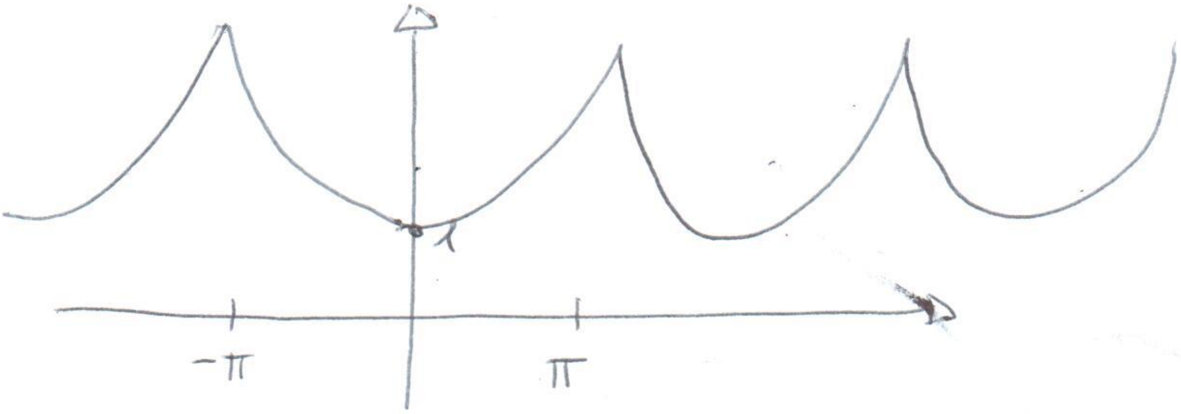
SVOLGIMENTI PROVA SCRITTA
di ANALISI 2 del 25/6/2020

1)



f PARI ; $f \in C^0(\mathbb{R})$ e regolare a tratti

$\Rightarrow$ convergenza puntuale, uniforme e totale su $\mathbb{R}$.

$$S(x) = f(x) \qquad \forall x \in \mathbb{R}.$$

$b_k = 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi \cosh(x)\,dx = \frac{2}{\pi}\sinh(x)\Big|_0^\pi$$

$$= \frac{2}{\pi}\sinh(\pi) = \frac{e^\pi - e^{-\pi}}{\pi}.$$

$$a_k = \frac{2}{\pi}\int_0^\pi \cosh(x)\cos(kx)\,dx$$

Ma $\displaystyle\int_0^\pi \cosh(x)\cos(kx)\,dx =$ (per parti) ②

$$\sinh(x)\cos(kx)\Big|_0^\pi + k\int_0^\pi \sinh(x)\sin(kx)\,dx$$

$$= \sinh(\pi)\cdot(-1)^k + k\left[\cancel{\cosh(x)\sin(kx)\Big|_0^\pi} - k\int_0^\pi \cosh(x)\cos(kx)\,dx\right]$$

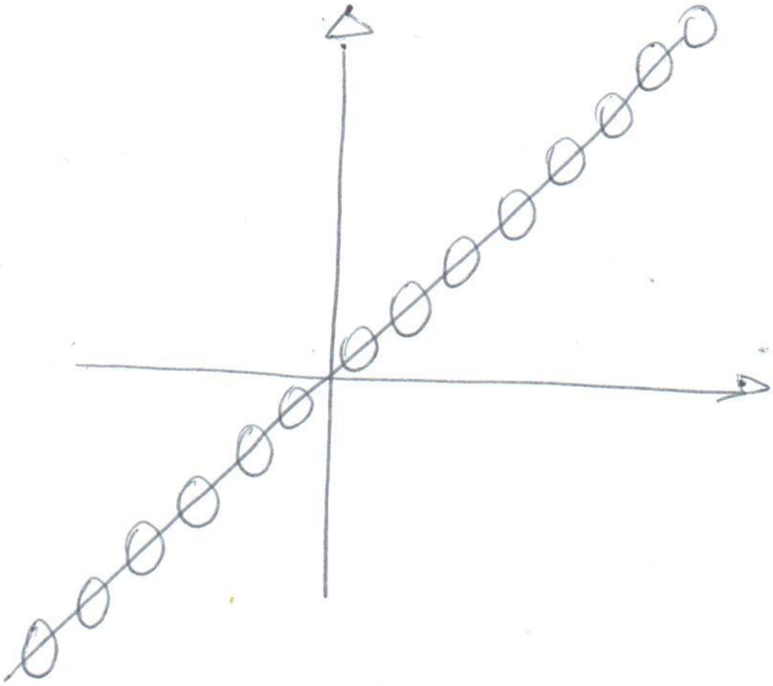
$$\Rightarrow (1+k^2)\int_0^\pi \cosh(x)\cos(kx)\,dx = (-1)^k \sinh(\pi).$$

$$\Rightarrow \underline{a_k = \frac{2}{\pi(1+k^2)}(-1)^k \sinh(\pi)}.$$

$$\Rightarrow f(x) = \frac{2}{\pi}\sinh(\pi)\sum_{k=1}^{+\infty}\frac{(-1)^k}{k^2+1} + \frac{\sinh(\pi)}{2\pi}$$

2) $$f(x,y) = \begin{cases} \log(x-y) & \text{se } y<x \\ \log(y-x) & \text{se } y>x \end{cases}$$

$$D = \{ y \neq x \}$$



In D, ovviamente, $f \in C^\infty(D)$.

$\Rightarrow \tilde{f} \in C^\infty(D)$

$$\frac{\partial \tilde{f}}{\partial x} = \frac{1}{x-y} \quad ; \quad \frac{\partial \tilde{f}}{\partial y} = \frac{1}{y-x}$$

$$\forall (x,y) \in D$$

lungo $\mathbb{R}^2 - D = \{y = x\}$: ④

$$\lim_{(x,y)\to(x_0,x_0)} \tilde{f}(x,y) = \lim_{(x,y)\to(x_0,x_0)} \log|x-y| = -\infty$$

$\Rightarrow \tilde{f}$ NON è continua lungo $y=x$.

$\Rightarrow \tilde{f}$ NON è differenziabile lungo $y=x$.

DERIVATE PARZIALI in (x_0, x_0):

$$\lim_{h\to 0^\pm} \frac{\log|\cancel{x_0}+h-\cancel{x_0}|}{h} = \mp\infty$$

$$\lim_{k\to 0^\pm} \frac{\log|\cancel{x_0}-(\cancel{x_0}+k)|}{k} = \mp\infty$$

$\nexists$ le der. parziali nei punti (x,x).

DERIVATE DIREZIONALI in (x_0, x_0):

$$\lim_{t\to 0^\pm} \frac{\log|\cancel{x_0}+\alpha t-(\cancel{x_0}+\beta t)|}{t}$$ se $(\alpha,\beta) \neq \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$= \lim_{t\to 0^\pm} \frac{\log|(\alpha-\beta)t|}{t} = \mp\infty.$$

Se $(\alpha, \beta) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (cioè lungo $y = x$)

$f(x,x) \equiv 0 \implies \frac{df}{d\vec{v}} = 0$ (UNICA DER. DIR. in (x_0, x_0)).

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3) $I_{def} = \mathbb{R}^2 - \{(0,0)\}$

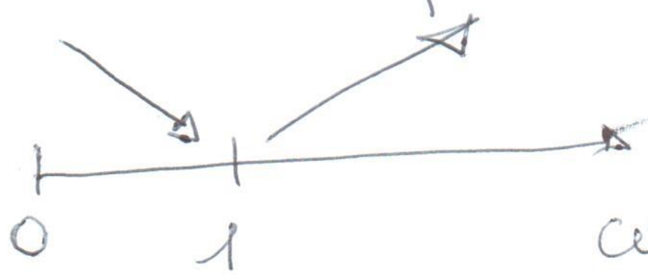
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\rho\to 0^+} \rho + \frac{1}{\rho} = +\infty$$

$\implies f$ è ILLIMITATA SUP.

$\cancel{f(x,y) \geq 0 \quad \lim_{\rho\to+\infty}\left(\rho + \frac{1}{\rho}\right)}$

$\tilde{f}(\rho) = \rho + \frac{1}{\rho}$ (FUNZIONE RADIALE)

$\tilde{f}'(\rho) = 1 - \frac{1}{\rho^2} = 0 \iff \rho = 1$



Quindi tutti i punti sulla circonferenza $\{x^2 + y^2 = 1\}$ sono di MIN. REL. e ASS.

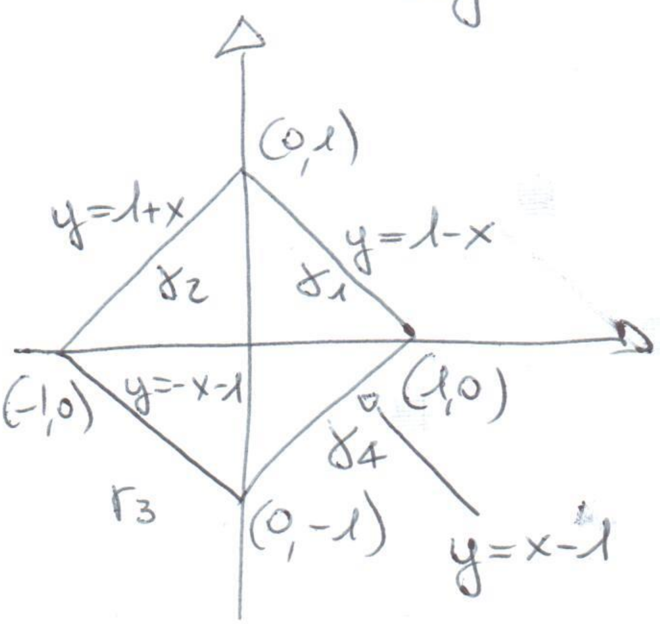
$$f(x^2+y^2=1) = \tilde{f}(1) = 2.$$

Altrimenti, con le der. parziali: (6)

$$\begin{cases} f_x = \dfrac{x}{\sqrt{x^2+y^2}} - \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}} = \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}(x^2+y^2-1) = 0 \\[2ex] f_y = \dfrac{y(x^2+y^2-1)}{(x^2+y^2)^{3/2}} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y(y^2-1) = 0 \end{cases} \cup \begin{cases} x^2+y^2=1 \\ // \end{cases}$$

$$\Rightarrow \cancel{\begin{cases} x = 0 \\ y = 0 \end{cases}} \notin I_{def} \quad \cup \begin{cases} x = 0 \\ y = \pm 1 \end{cases} \cup \{x^2+y^2=1\}$$



N.B.: poiché la funzione è radiale e ammette MIN. ASS. lungo $\{x^2+y^2=1\}$

$\Rightarrow f\big|_D$ ammetterà

MINIMO ASS. nei punti del quadrato tali che $\rho = 1$ (7)

$\Rightarrow$ MIN. ASS. in $(0, \pm 1)$ e $(\pm 1, 0)$.

Poichè $\lim_{\rho \to 0^+} \tilde{f}(\rho) = +\infty \Rightarrow \nexists$ MAX. ASS.

Altrimenti, con le restrizioni sulla frontiera:

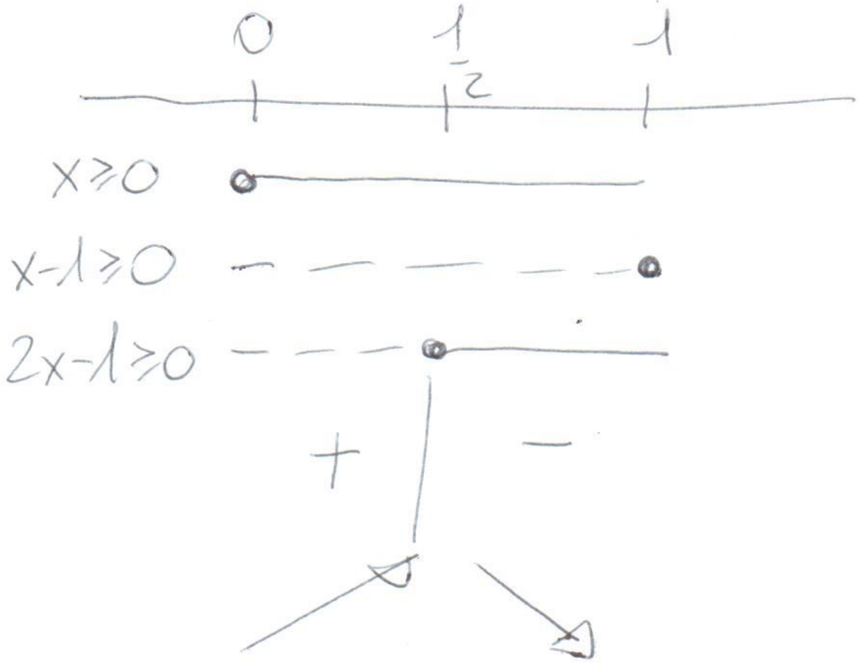
Osserviamo che f è SIMMETRICA rispetto a entrambi gli assi (nonchè rispetto all'origine). Quindi basta studiare $f\big|_{\gamma_1}$.

$$\gamma_1 : \begin{cases} x \in [0,1] \\ y = 1 - x \end{cases} \qquad f\Big|_{y=1-x} = \left[x^2 + (1-x)^2\right]^{\frac{1}{2}} + \frac{1}{\left[x^2+(1-x)^2\right]^{1/2}}$$

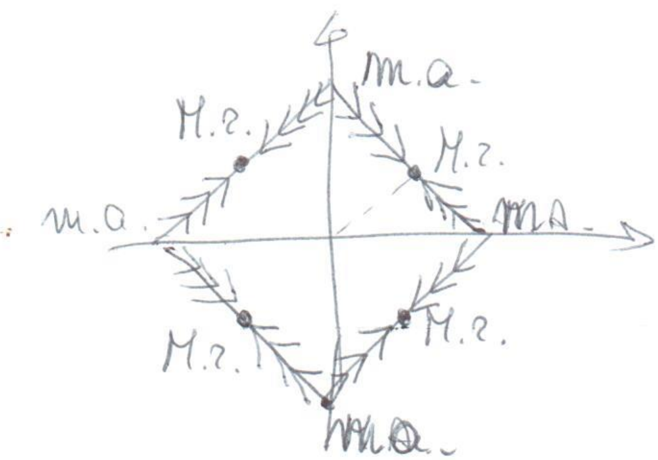
$$= \left(2x^2 - 2x + 1\right)^{\frac{1}{2}} + \frac{1}{\left(2x^2-2x+1\right)^{1/2}}$$

$$\Rightarrow \left(f\Big|_{y=1-x}\right)' = \left[\frac{1}{2(\quad)^{1/2}} - \frac{1}{2(\quad)^{3/2}}\right](4x-2)$$

$$= \frac{(2x^2 - 2x \cancel{+1})(2x-1)}{(\quad)^{3/2}} = \frac{2x(x-1)(2x-1)}{(\quad)^{3/2}}$$



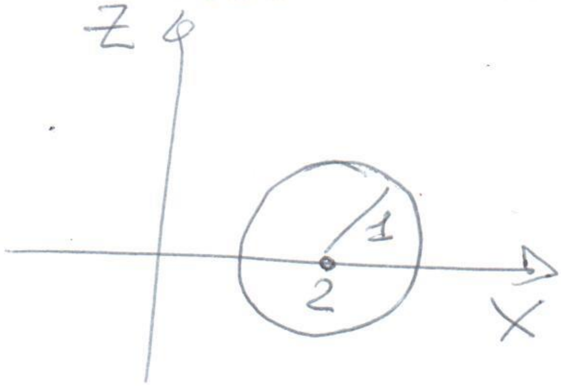
$f|_{\gamma_1}$ cresce da $(0,1)$ a $\left(\frac{1}{2}, \frac{1}{2}\right)$, per poi decrescere da $\left(\frac{1}{2}, \frac{1}{2}\right)$ a $(1,0)$



$$f\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right) = \sqrt{\frac{1}{2}} + \sqrt{2} = \frac{3}{\sqrt{2}}$$

La presenza dei MAX. REL. di $f|_{\gamma_i}$ non è in contraddizione con l'altro metodo perché i punti $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$ sono di MAX. REL. solo per $f|_{\gamma_i}$, NON per f.
Infatti le curve di livello sono circonferenze e su di essi f decresce fino alla circonferenza $\{x^2+y^2=1\}$ per poi crescere

4) La superficie è un TORO, ottenuto dalla rotazione della circonferenza in figura:



$\operatorname{div}\vec{F} = 2 + 1 = 3$

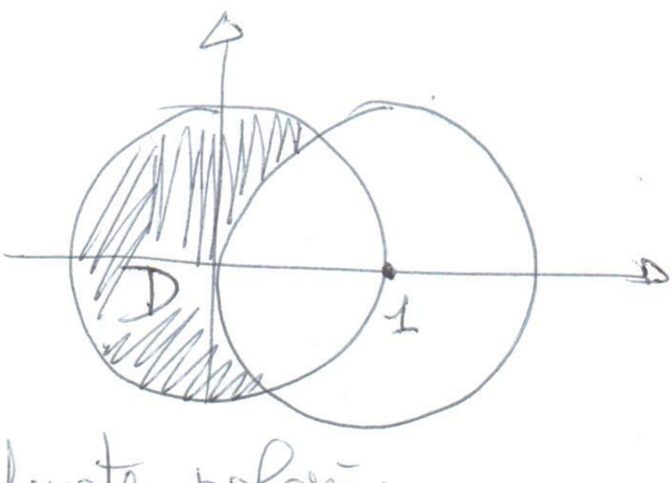
$$\Rightarrow \quad \Phi_S(\vec{F}) = \iiint_T \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$= 3\,\mathrm{vol}\,T = 3\cdot 2\pi\cdot 2\cdot\pi = 12\pi^2$$

DOMINIO TOROIDALE

$3\cdot 2\pi x_B \cdot$ Area

5)



Coordinate polari:

$$\begin{cases} \rho^2 \le 1 \\ (\rho\cos\vartheta - 1)^2 + \rho^2\sin^2\vartheta \ge 1 \end{cases} \Rightarrow \begin{cases} 0 \le \rho \le 1 \\ \rho^{\cancel{2}} - 2\cancel{\rho}\cos\vartheta \ge 0 \end{cases}$$

$$\Rightarrow \begin{cases} 0 \le \rho \le 1 \\ \rho \ge 2\cos\vartheta \end{cases}$$

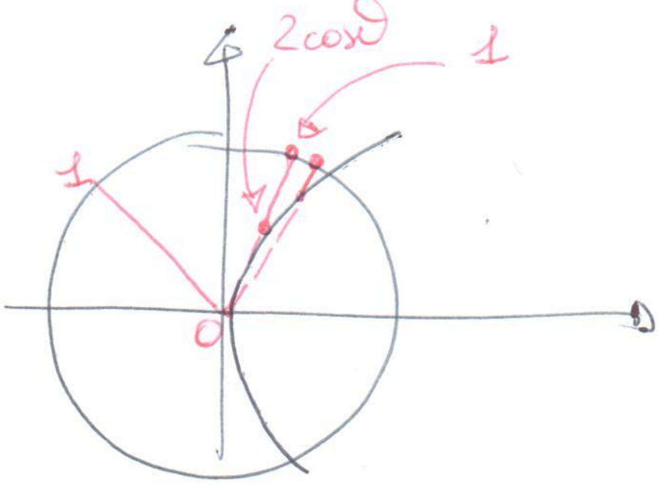
COMPATIBILITÀ:

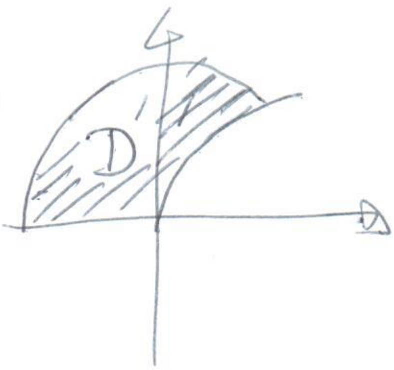
$$2\cos\vartheta \le 1 \Rightarrow \cos\vartheta \le \frac{1}{2} \Rightarrow \vartheta \in \left[\frac{\pi}{3}; \frac{5}{3}\pi\right]$$

Inoltre, avremo due situazioni:

$$\begin{cases} 2\cos\vartheta \ge 0 \\ \Rightarrow 2\cos\vartheta \le \rho \le 1 \end{cases} \cup \begin{cases} 2\cos\vartheta \le 0 \\ 0 \le \rho \le 1 \end{cases}$$

$$\begin{cases} \vartheta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \cup \left[\frac{3}{2}\pi, \frac{5}{3}\pi\right] \\ 2\cos\vartheta \le \rho \le 1 \end{cases} \cup \begin{cases} \vartheta \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right] \\ \rho \in [0, 1] \end{cases}$$





Inoltre, $f(x,-y) = f(x,y)$

$$\Rightarrow \iint_D x|y| \, dx\,dy = 2 \iint_{D'} xy \, dx\,dy$$

$$= 2\left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\vartheta \int_{2\cos\vartheta}^{1} \rho^3 \cos\vartheta \sin\vartheta \, d\vartheta + \int_{\frac{\pi}{2}}^{\pi} d\vartheta \int_{0}^{1} \rho^3 \cos\vartheta \sin\vartheta \, d\vartheta\right]$$

$$= 2\left\{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\vartheta \sin\vartheta \left[\frac{\rho^4}{4}\right]_{2\cos\vartheta}^{1} d\vartheta + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2\vartheta}{2}\, d\vartheta \left[\frac{\rho^4}{4}\right]_{0}^{1}\right\}$$

$$= 2\left[\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-16\cos^4\vartheta)\cos\vartheta \sin\vartheta\, d\vartheta - \frac{\cos 2\vartheta}{4}\Big|_{\frac{\pi}{2}}^{\pi} \cdot \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\left(-\frac{\cos^2\vartheta}{2} + \frac{16}{6}\cos^6\vartheta\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \frac{1}{4}\left[1+1\right]\right]$$

$$= \frac{1}{2}\left[-\left(-\frac{1}{8} + \frac{\cancel{8}}{3}\cdot\frac{1}{\cancel{2^6}\,8}\right) - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{8}\left(1-\frac{1}{3}\right) - \frac{1}{2}\right] = \frac{1}{2}\left(\frac{1}{12} - \frac{1}{2}\right) = \boxed{\frac{-5}{24}}$$

Altrimenti, con coordinate polari decentrate:

$$\begin{cases} x = 1 + \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \Rightarrow \begin{cases} (1+\rho\cos\vartheta)^2 + \rho^2\sin^2\vartheta \le 1 \\ \rho^2 \ge 1 \end{cases}$$

$$\Rightarrow \begin{cases} \rho^{\cancel{2}} + 2\cancel{\rho}\cos\vartheta \le 0 \\ \rho \ge 1 \end{cases}$$

(12)

$$\Rightarrow \left\{ 1 \leq \rho \leq -2\cos\vartheta \right\}$$

CONDIZIONE COMPATIBILITÀ: $1 \leq -2\cos\vartheta$

$$\Rightarrow \cos\vartheta \leq \frac{-1}{2} \Rightarrow \vartheta \in \left[\frac{2\pi}{3}, \frac{4}{3}\pi\right]$$

SIMMETRIA di f_ϑ

$$\Rightarrow 2 \int_{\frac{2\pi}{3}}^{\pi} d\vartheta \int_{1}^{-2\cos\vartheta} \rho\,(1+\rho\cos\vartheta)\,\rho\sin\vartheta\, d\rho$$

$$= 2 \int_{\frac{2}{3}\pi}^{\pi} d\vartheta \sin\vartheta \int_{1}^{-2\cos\vartheta} \left[\rho^2 + \rho^3\cos\vartheta\right] d\rho$$

$$= 2 \int_{\frac{2}{3}\pi}^{\pi} d\vartheta \sin\vartheta \left[\frac{\rho^3}{3} + \frac{\rho^4}{4}\cos\vartheta\right]_{1}^{-2\cos\vartheta}$$

$$= 2 \int_{\frac{2}{3}\pi}^{\pi} \sin\vartheta \left[\frac{-8\cos^3\vartheta}{3} - \frac{1}{3} + \frac{\cancel{16}^{4}\cos^5\vartheta}{\cancel{4}_{1}} - \frac{1}{4}\cos\vartheta\right]$$

$$= 2 \left[\frac{\cancel{8}^{2}}{\cancel{12}_{3}}\cos^4\vartheta + \frac{1}{3}\cos\vartheta - \frac{\cancel{4}^{2}}{\cancel{6}_{3}}\cos^6\vartheta + \frac{1}{8}\cos^2\vartheta\right]_{\frac{2}{3}\pi}^{\pi}$$

$$= 2 \left[\cancel{\frac{2}{3}} - \frac{1}{3} - \cancel{\frac{2}{3}} + \frac{1}{8} - \left(\frac{\cancel{2}^{1}}{3}\cdot\frac{1}{\cancel{16}_{8}} - \frac{1}{3}\cdot\frac{1}{2} - \frac{\cancel{2}}{3}\cdot\frac{1}{2^{\cancel{6}5}} + \frac{1}{8}\cdot\frac{1}{4}\right)\right]$$

$$= 2\left[-\frac{1}{3}+\frac{1}{8}-\frac{1}{24}+\frac{1}{6}+\frac{1}{96}-\frac{1}{32}\right]$$

$$= \frac{2}{96}\left[-32+12-4+16+1-3\right] = \frac{-20}{96} = \frac{-5}{24}.$$