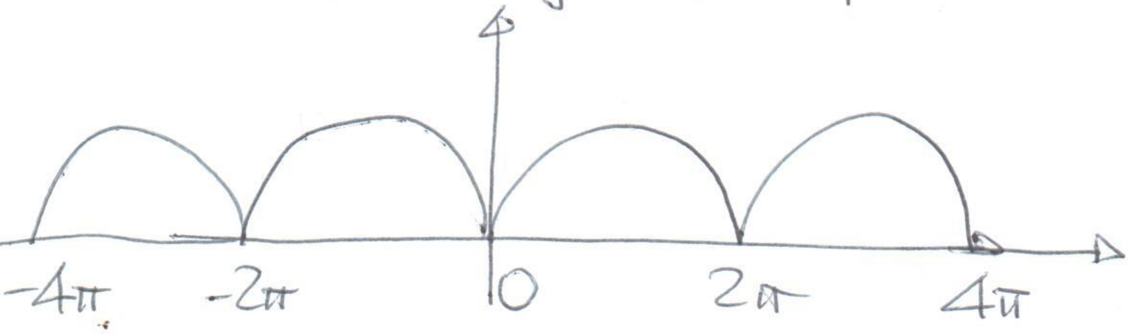
SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 29/4/2020.

1

1) $f(x) = \sin\left(\frac{x}{2}\right)$

in $[0, 2\pi)$:


$\Rightarrow$ la prolungata per periodicità è




$\Rightarrow f$ pari $\Rightarrow b_k = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right) dx = -\frac{4}{\pi}\cos\left(\frac{x}{2}\right)\Big|_0^{\pi}$$

$$= \frac{4}{\pi}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(kx)\, dx =$$

$$= \frac{2}{\pi}\left[-2\cos\left(\frac{x}{2}\right)\cos(kx)\Big|_0^{\pi} - 2k\int_0^{\pi}\cos\left(\frac{x}{2}\right)\sin(kx)\, dx\right]$$

$$= \cancel{\frac{4}{\pi}}\ \frac{2}{\pi}\left[2 - 2k\left[\cancel{2\sin\left(\frac{x}{2}\right)\sin(kx)\Big|_0^{\pi}} - 2k\int_0^{\pi}\sin\left(\frac{x}{2}\right)\cos(kx)\, dx\right]\right]$$

②

$$\Rightarrow \int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(kx)\,dx$$

$$= 2 + 4k^2 \int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(kx)\,dx$$

$$\Rightarrow \int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(kx)\,dx = \frac{2}{1-4k^2}$$

$$\Rightarrow a_k = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(kx)\,dx = \frac{4}{\pi(1-4k^2)}$$

$$\Rightarrow f(x) \sim \frac{2}{\pi} + \sum_{k=1}^{+\infty} \frac{4}{\pi(1-4k^2)}\cos(kx).$$

Poichè $f$ è CONTINUA e REGOLARE a TRATTI $\Rightarrow$ CONV. PUNTUALE UNIFORME TOTALE in $\mathbb{R}$.

$$f(x) = \frac{2}{\pi} + \sum_{k=1}^{+\infty} \frac{4}{\pi(1-4k^2)}\cos(kx).$$

2) $f$ è $C^\infty(\mathbb{R}^2-\{(0,0)\}) \Rightarrow C^1 \Rightarrow$ differenziabile $\Rightarrow C^0$

In $(0,0)$: lungo $y=x^2$

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{(e^{x^2+x^4}-1)^2}{2x^4} =$$

$$\lim_{x\to 0} \frac{[(x^2+x^4)+o(x^2+x^4)]^2}{2x^4} =$$

$$= \lim_{x\to 0} \frac{(x^2+o(x^2))^2}{2x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Ma $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{(e^{x^2}-1)^2}{x^4}$

$$= \lim_{x\to 0} \frac{(x^2)^2}{x^4} = 1$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$\Rightarrow f$ NON differenziabile in $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{(e^{h^2}-1)^2}{h^5} = \lim_{h\to 0} \frac{h^4}{h^5} =$$ 4

$$\lim_{h\to 0^\pm} \frac{1}{h} = \pm\infty \qquad \nexists \frac{\partial f}{\partial x}(0,0)$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{(e^{k^2}-1)^2}{k^3} = \lim_{k\to 0} \frac{k^4}{h^3} = 0$$

$$\frac{df}{d\vec{w}}(0,0) = \lim_{t\to 0} \frac{(e^{t^2}-1)^2}{t^3(\alpha^4 t^2+\beta^2)} =$$

$$= \lim_{t\to 0} \frac{t^{4}}{\cancel{t^3}(\alpha^4 t^2+\beta^2)} = \frac{0}{\beta^2} = 0 \quad \forall \beta \neq 0.$$

Per $\beta=0 \implies \frac{df}{d\vec{w}}(0,0) = \frac{\partial f}{\partial x}(0,0) \nexists$

In $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{2(e^{x^2+y^2}-1)e^{x^2+y^2}2x\cdot(x^4+y^2) - (e^{x^2+y^2}-1)^2 4x^3}{(x^4+y^2)^2}$$

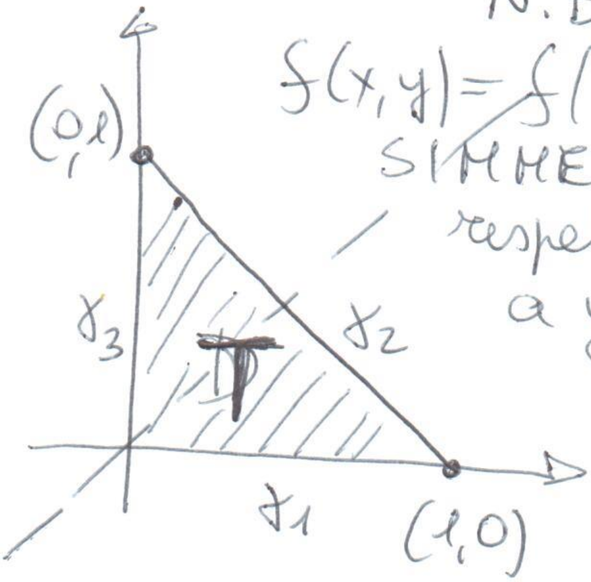
$$= \frac{4(e^{x^2+y^2}-1)x\left[e^{x^2+y^2}(x^4+y^2) - (e^{x^2+y^2}-1)x^2\right]}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2(e^{x^2+y^2}-1)\,e^{x^2+y^2}\,2y\,(x^4+y^2) - (e^{x^2+y^2}-1)^2\,2y}{(x^4+y^2)^2}$$

$$= \frac{2y(e^{x^2+y^2}-1)}{(x^4+y^2)^2}\left[2e^{x^2+y^2}(x^4+y^2)-(e^{x^2+y^2}-1)\right]$$

3)

In $\mathring{T}$:




N.B.: $f(x,y)=f(y,x)$ SIMMETRIA rispetto a $y=x$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(\quad)^2}\left[4x+2(x-1)\right]=0$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(\quad)^2}\cdot\left[4y+2(y-1)\right]=0$$

$$\begin{cases} 3x-1=0 \\ 3y-1=0 \end{cases} \Rightarrow (x,y)=\left(\frac{1}{3},\frac{1}{3}\right).$$

$$f\left(\frac{1}{3},\frac{1}{3}\right)= \operatorname{arctg}\left(\frac{4}{9}+\frac{4}{9}+\frac{4}{9}\right)=\operatorname{arctg}\left(\frac{4}{3}\right)$$

Sia $\partial \mathbb{T} = \gamma_1 \cup \gamma_2 \cup \gamma_3$

$$\gamma_1 : \begin{cases} x = t \\ y = 0 \end{cases} ;\ t \in [0,1] \qquad f|_{\gamma_1} = f(t,0) = \operatorname{arctg}\left(2t^2 + (t-1)^2\right)$$

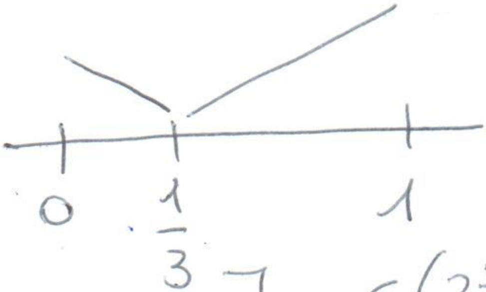
$$\gamma_2 : \begin{cases} x = t \\ y = 1-t \end{cases} ,\ t \in [0,1] \qquad f|_{\gamma_2} = f(t,1-t) = \operatorname{arctg}\begin{pmatrix} 2t^2 + 2(1-t)^2 \\ + (t-1)^2 + (-t)^2 \end{pmatrix}$$

$$= \operatorname{arctg}\left(3t^2 + 3(t-1)^2\right)$$

$$\gamma_3 : \begin{cases} x = 0 \\ y = t \end{cases} ,\ t \in [0,1] = \operatorname{arctg}\left(2t^2 + (t-1)^2\right)$$

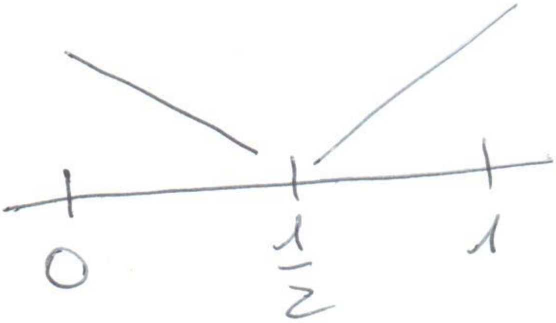
$$\left(f|_{\gamma_1}\right)' = \frac{1}{1+[\ \ ]^2}\left(4t + 2(t-1)\right) = \frac{1}{1+[\ ]^2}(6t-2) > 0$$

$$\iff t > \frac{1}{3}$$

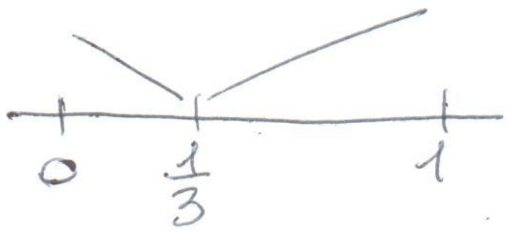


$$\left(f|_{\gamma_2}\right)' = \frac{3}{1+[\ \ ]^2}\left[2t + 2(t-1)\right] = \frac{6(2t-1)}{1+[\ \ ]^2} > 0$$

$$\iff t > \frac{1}{2}$$



$$\left(f\big|_{\gamma_3}\right)' = \frac{2}{1+[\quad]^2}(3t-1) > 0 \iff t > \frac{1}{3}$$

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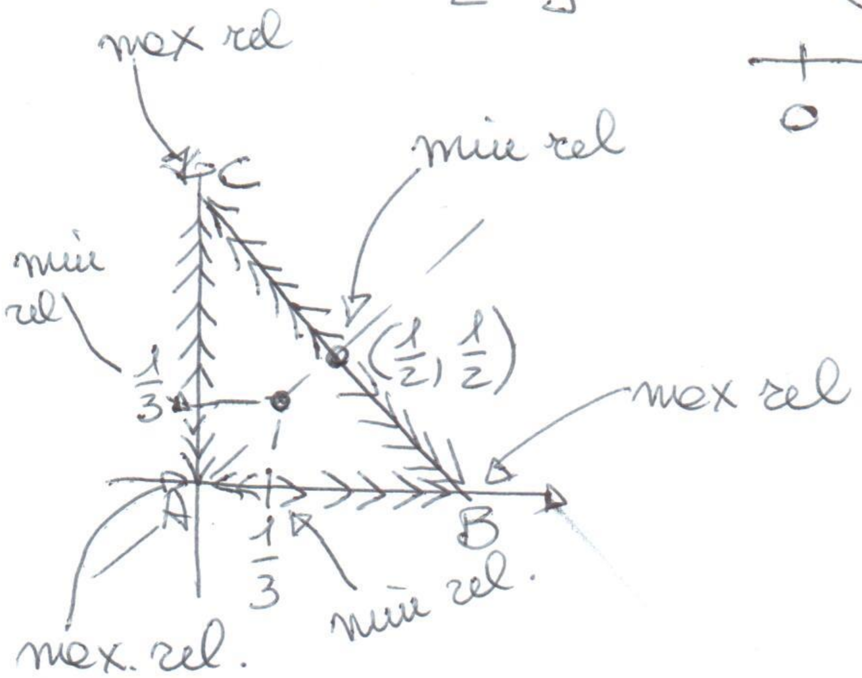



$f(0,0) = \operatorname{arctg} 2$

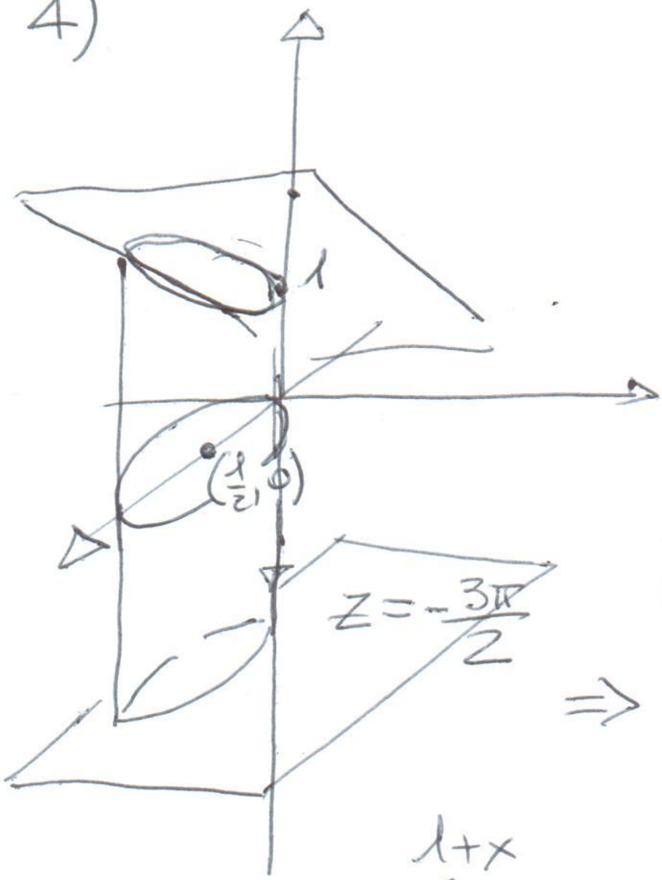
$f\left(\frac{1}{3}, 0\right) = f\left(0, \frac{1}{3}\right) = \operatorname{arctg}\left(\frac{5}{3}\right)$

$f(1,0) = f(0,1) = \operatorname{arctg} 3$ ← MAX. ASS.

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \operatorname{arctg}\left(\frac{3}{2}\right)$

$f\left(\frac{1}{3}, \frac{1}{3}\right) = \operatorname{arctg}\left(\frac{4}{3}\right)$ ← MIN. ASS.

4)




$$x^2 - x + y^2 = 0$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{-\sin z}{\cos(1+x)}$$

La superficie è chiusa

$$\Rightarrow \Phi_{\substack{S \\ = \partial D}}(\vec{F}) = \iiint_D \vec{\nabla} \cdot \vec{F} \, dx\, dy\, dz$$

$$= \iint_{T(x,y)} dx\, dy \int_{-\frac{3\pi}{2}}^{1+x} \frac{-\sin z}{\cos(1+x)} \, dz$$

$$= \iint_{T(x,y)} dx\, dy \, \frac{1}{\cos(1+x)} \Big[\cos z\Big]_{-\frac{3\pi}{2}}^{1+x} =$$

$$= \iint_{T(x,y)} dx\, dy \, \frac{1}{\cancel{\cos(1+x)}} \Big[\cancel{\cos(1+x)} - \underbrace{\cancel{\cos\left(-\frac{3\pi}{2}\right)}}_{=0}\Big]$$

$$= \text{Area } T(x,y) = \pi \frac{1}{4} = \frac{\pi}{4}$$

5)

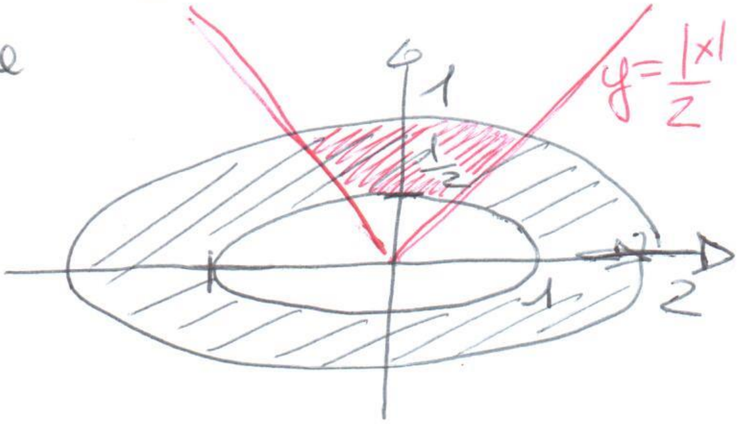
N.B.: $e^{-x^4-16y^4-8x^2y^2} = e^{-(x^2+4y^2)^2}$

$$1-3y^2 \le x^2+y^2 \le 4-3y^2$$

$$\Rightarrow \begin{cases} x^2+4y^2 \ge 1 \\ x^2+4y^2 \le 4 \end{cases} \Rightarrow 1 \le x^2+4y^2 \le 4$$

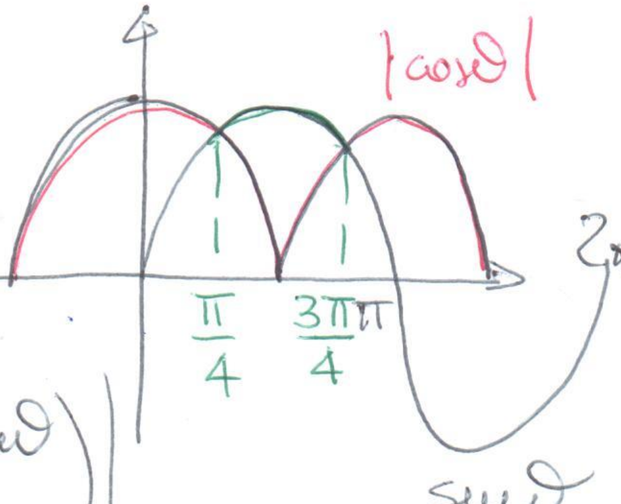
Coordinate ellittiche

$$\begin{cases} x = \rho\cos\vartheta \\ y = \frac{\rho}{2}\sin\vartheta \end{cases}$$




$$\Rightarrow \begin{cases} 1 \le \rho^2 \le 4 \\ \cancel{\rho}\sin\vartheta \ge \cancel{\rho}|\cos\vartheta| \end{cases}$$

$$\Rightarrow \begin{cases} 1 \le \rho \le 2 \\ \vartheta \in \left[\frac{\pi}{4}, \frac{3}{4}\pi\right] \end{cases}$$




$$|J| = \left| \cancel{\cos\vartheta}\ \det\begin{pmatrix} \cos\vartheta & \frac{1}{2}\sin\vartheta \\ -\rho\sin\vartheta & \frac{\rho}{2}\cos\vartheta \end{pmatrix} \right|$$

$$= \left| \cancel{\det}\ \frac{\rho}{2}(\cos^2\vartheta + \sin^2\vartheta) \right| = \frac{\rho}{2}$$

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_1^2 \frac{\rho}{2}\left[\rho^2 e^{-\rho^4}\right] d\rho =$$ 10

$$\left(\frac{3}{4}\pi - \frac{\pi}{4}\right) \int_1^2 \frac{4\rho^3}{8} e^{-\rho^4} d\rho =$$

$$= \frac{\pi}{16}\left[-e^{-\rho^4}\right]_1^2 = \frac{\pi}{16}\left(e^{-1} - e^{-16}\right)$$

6) $\omega$ è definito per $y \neq 0$
Due semipiani (sempl. linearmente connessi).

$$\frac{\partial X}{\partial y} = \frac{-x}{y^2(x^2+y^2)}\left(y\sqrt{x^2+y^2}\right)_y$$

$$= \frac{-x}{y^2(x^2+y^2)}\left[\sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}\right]$$

$$= \frac{-x}{y(x^2+y^2)^{3/2}}\left[x^2+2y^2\right]$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2}{y^2\sqrt{x^2+y^2}} + \frac{1}{y^2}\right] =$$

$$\frac{1}{y^2}\left[\frac{2x\sqrt{x^2+y^2} - \frac{x^3}{\sqrt{x^2+y^2}}}{(x^2+y^2)}\right] = \frac{1}{y^2}\left[\frac{2x(x^2+y^2)-x^3}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{x}{y^2}\left[\frac{x^2+2y^2}{(x^2+y^2)^{3/2}}\right] \qquad Y_x = X_y$$

$\Rightarrow \omega$ CHIUSA in ogni semipiano

$\Rightarrow \omega$ ESATTA in ogni semipiano

$$V(x,y) = \int X(x,y)\,dx = \frac{1}{y}\int \frac{x}{\sqrt{x^2+y^2}}\,dx$$

$$= \frac{1}{y}\sqrt{x^2+y^2} + \varphi(y)$$

$$\frac{\partial V}{\partial y} = \frac{\frac{y}{\sqrt{x^2+y^2}}\cdot y - \sqrt{x^2+y^2}}{y^2} + \varphi'(y)$$

$$= \frac{y^2-(x^2+y^2)}{y^2\sqrt{x^2+y^2}} + \varphi'(y) = \underline{\frac{-x^2}{y^2\sqrt{x^2+y^2}} + \varphi'(y)}$$

$$= Y = \underline{\frac{-x^2}{y^2\sqrt{x^2+y^2}} - \frac{1}{y^2}}$$

$$\Rightarrow \varphi'(y) = -\frac{1}{y^2} \qquad \Rightarrow \varphi(y) = \frac{1}{y} + C$$

$$\Rightarrow \underline{V(x,y) = \frac{1}{y}\sqrt{x^2+y^2} + \frac{1}{y} + C}$$