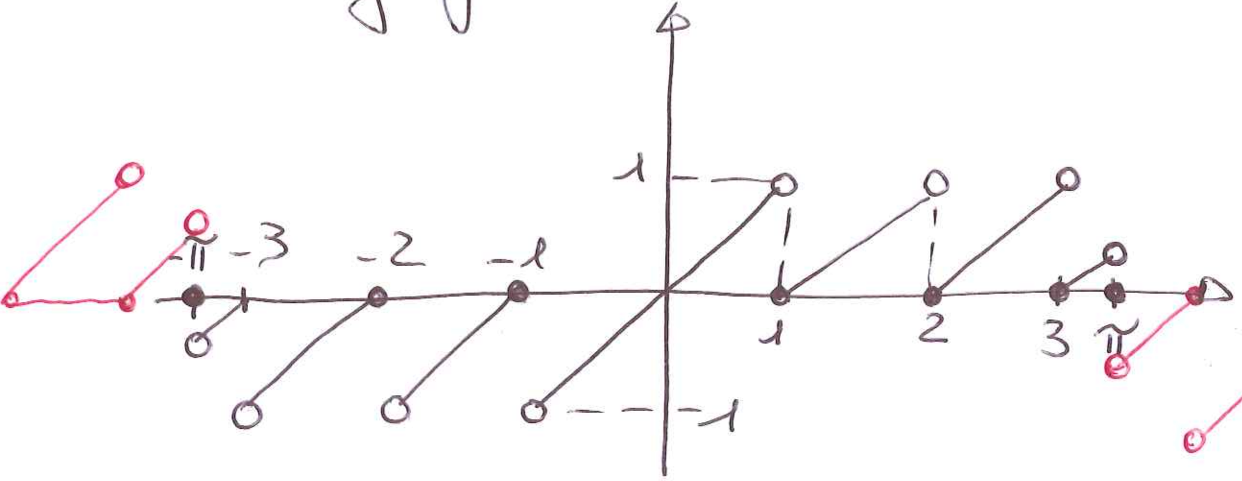
SVOLGIMENTI PROVA SCRITTA del 29/10/18
di ANALISI 2

1) La funzione prolungata per disparità ha come grafico



la funzione è regolare a tratti. Pertanto la serie converge puntualmente in $\mathbb{R}$ alla somma

$$S(x) = \begin{cases} f(x) & x \neq \pm 1, \pm 2, \pm 3, \pm \pi \\ \frac{1}{2} & x = 1, 2, 3 \\ -\frac{1}{2} & x = -1, -2, -3 \\ 0 & x = \pm \pi \end{cases}$$

e converge uniformemente in ogni intervallo chiuso e limitato di continuità di $f(x)$.

$f(x)$ è dispari $\Rightarrow a_k = 0 \quad \forall k \geq 0.$ (2)

$$f(x) = \begin{cases} x & \text{per } x \in [0,1) \\ x-1 & \text{per } x \in [1,2) \\ x-2 & \text{per } x \in [2,3) \\ x-3 & \text{per } x \in [3,\pi) \end{cases}$$

$$\Rightarrow b_k = \frac{2}{\pi}\left[\int_0^1 x \sin kx\,dx + \int_1^2 (x-1)\sin kx\,dx \right.$$

$$\left. + \int_2^3 (x-2)\sin kx\,dx + \int_3^\pi (x-3)\sin kx\,dx \right]$$

$$= \frac{2}{\pi}\left[\int_0^\pi x \sin kx\,dx - \int_1^2 \sin kx\,dx - 2\int_2^3 \sin kx\,dx \right.$$

$$\left. - 3\int_3^\pi \sin kx\,dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{k} x \cos kx \Big|_0^\pi + \frac{1}{k}\int_0^\pi \cos kx\,dx \right.$$

$$\left. + \frac{1}{k}\Big[\cos(2k) - \cos(k) + 2\cos(3k) - 2\cos(2k) + 3\cos(k\pi) - 3\cos(3k) \Big] \right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{k}\pi\cos(k\pi) + \frac{1}{k^2}\sin(kx)\Big|_0^{\pi} + \frac{1}{k}\Big[-\cos(k) - \cos(2k) - \cos(3k) + 3\cos(k\pi)\Big]\right]$$ ③

$$= \frac{2}{\pi k}\Big[(3-\pi)(-1)^k - \cos(k) - \cos(2k) - \cos(3k)\Big]$$

$$\Rightarrow \quad f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx).$$

2) f è definita in tutto $\mathbb{R}^2$.

È ovviamente $C^\infty(\mathbb{R}^2 - \{x=0\})$.

Consideriamo ora i punti $(0, y_0)$:

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \begin{cases} \displaystyle\lim_{(x,y)\to(0,y_0)} \frac{\sin(xy)}{x} & \text{in } \mathbb{R}^2 - \{x=0\} \\ \displaystyle\lim_{y\to y_0} y = \underset{f(0,y_0)}{\underbrace{y_0}} & \text{lungo asse } y \end{cases}$$

$$\lim_{(x,y)\to(0,y_0)} \frac{\sin xy}{x} = \lim_{(x,y)\to(0,y_0)} \frac{xy}{x} = \lim_{(x,y)\to(0,y_0)} y$$

$$= y_0 = f(0, y_0)$$

$\Rightarrow$ f CONTINUA ANCHE LUNGO ASSE y.

DERIVABILITÀ:

In $\mathbb{R}^2 - \{x=0\}$

$$f_x(x,y) = \frac{xy\cos(xy) - \sin(xy)}{x^2}$$

$$f_y(x,y) = \frac{1}{x} x\cos(xy) = \cos(xy)$$

~~Orig~~ Nell'origine: $f(x,0)=0$

$f(0,y)=y$

$$\Rightarrow f_x(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{k-0}{k} = 1$$

Negli altri punti dell'asse y:

$$f_y(0,y_0) = \lim_{k\to 0} \frac{\cancel{y_0}+k-\cancel{y_0}}{k} = 1$$

$$f_x(0,y_0) = \lim_{h\to 0} \frac{\frac{\sin(hy_0)}{h} - y_0}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{1}{h}\left[\cancel{hy_0} - \frac{(h^3y_0^3)}{6} + o(h^3)\right] - \cancel{y_0}}{h} = -\frac{1}{6}\lim hy_0^3$$

$$= 0$$

DIFFERENZIABILITÀ:

In $\mathbb{R}^2 - \{x=0\}$ immediato, perchè f è C^∞.

Nell'origine: $\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(h,k)-k}{\sqrt{h^2+k^2}} =$

$$\begin{cases} \displaystyle\lim_{(h,k)\to(0,0)} \frac{\frac{\sin(hk)}{h}-k}{\sqrt{h^2+k^2}} \overset{(*)}{=} \cancel{\lim_{(h,k)\to(0,0)} \frac{1}{h}\left[hk-\frac{(h^3k^3)}{6}+\right.} & \text{al di fuori dell'asse } y \\ \displaystyle\lim_{k\to 0} \frac{k-k}{\sqrt{k^2}} = 0 & \text{lungo asse } y \end{cases}$$

$$(*) = \lim_{(h,k)\to(0,0)} \frac{\frac{1}{h}\left[\cancel{hk}-\frac{(h^3k^3)}{6}+o(h^3k^3)\right]\cancel{-k}}{\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0} -\frac{1}{6}\left[\frac{\rho^5\cos^2\vartheta\sin^3\vartheta}{\rho}\right] = -\frac{1}{6}\lim_{\rho\to 0}\rho^4\left[\cos^2\vartheta\sin^3\vartheta\right]$$

$$\lim_{\rho\to 0}\left|\frac{\Delta f - df}{\rho}\right| \le \frac{1}{6}\lim \rho^4 = 0$$

$\Rightarrow$ f differenziabile nell'origine

$$\Rightarrow \frac{df}{d\vec{v}}(0,0) = \alpha\cdot 0 = \beta\cdot 1 = \beta.$$

lungo asse y:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,y_0+k)-f(0,y_0)-k}{\sqrt{h^2+k^2}}$$

$$= \begin{cases} \displaystyle\lim_{(h,k)\to(0,0)} \frac{\frac{\sin[h(y_0+k)]}{h} - y_0 - k}{\sqrt{h^2+k^2}} \quad (**) & \text{fuori dall'asse } y \\ \displaystyle\lim_{k\to 0} \frac{y_0+k-y_0-k}{\sqrt{h^2}} = 0 & \text{lungo l'asse } y \end{cases}$$

$$(**) = \lim_{(h,k)\to(0,0)} \frac{\frac{1}{h}\left[\cancel{h(y_0+k)} - \frac{(h^3(y_0+k)^3)}{6} + o(h^3(y_0+k)^3)\right] \cancel{- y_0 - k}}{\sqrt{h^2+k^2}}$$

$$= \lim_{\rho\to 0}\left[-\frac{1}{6}\frac{\rho^{\cancel{3}}\cos^2\vartheta\,[y_0+\rho\sin\vartheta]^3}{\cancel{\rho}}\right]$$

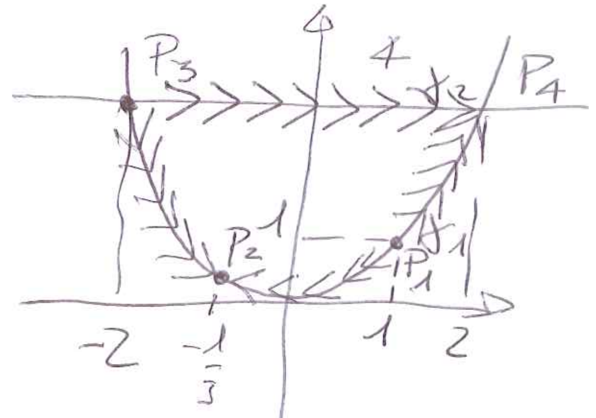
$$\lim_{\rho\to 0}\left|\frac{\Delta f - df}{\rho}\right| = \frac{1}{6}\lim_{\rho\to 0}\rho|\cos^2\vartheta||y_0+\rho\sin\vartheta|^3$$

$$\leq \frac{1}{6}\lim_{\rho\to 0}\rho|y_0+\rho\sin\vartheta|^3$$

$$= \frac{1}{6}|y_0|\cdot\lim_{\rho\to 0}\rho = 0$$

$\Rightarrow f$ DIFFERENZIABILE anche in ogni punto $x=0$.

3) Punti stazionari in $\mathring{D}$:



(7)

$$\begin{cases} f_x = y-1 \\ f_y = x-1 \end{cases} \Rightarrow \vec{\nabla} f = 0 \iff P_1 \equiv (1,1)$$

che però è un punto della frontiera.

Sulla frontiera: $\partial D = \gamma_1 \cup \gamma_2$

$$\gamma_1: \begin{cases} x \in [-2,2] \\ y = x^2 \end{cases} \qquad f\big|_{\gamma_1} = 1 + x^3 - x^2 - x$$

$$\left(f\big|_{\gamma_1}\right)' = 3x^2 - 2x - 1 = 0 \iff x_{1,2} = \frac{1 \pm \sqrt{4}}{3}$$

$$\Rightarrow \left(x_1 = 1 \,;\, y_1 = 1\right)^{\equiv P_1} ; \left(x_2 = -\frac{1}{3} \,;\, y_2 = \frac{1}{9}\right)^{\equiv P_2}$$

$f\big|_{\gamma_1}$ cresce dal punto $P_3 \equiv (-2,4)$ fino al punto P_2; decresce fra P_2 e P_1; cresce da P_1 al punto $P_4 \equiv (2,4)$

$f\big|_{\gamma_2} = f(x,4) = 3x - 1$ crescente (è una retta)

Dal disegno si evince che P_3 punto di MIN. REL.; P_4 punto di MAX. REL.; P_2 punto di MAX. REL.; P_1 punto di MIN. REL.

$f(P_1) = 0$

$f(P_2) = (1-x)(1-y)\Big|_{\substack{x=-\frac{1}{3} \\ y=\frac{1}{9}}} = \frac{4}{3} \cdot \frac{8}{9} = \frac{32}{27}$

$f(P_3) = 3 \cdot (-3) = -9$

$f(P_4) = -(-3) = 3$

$\Rightarrow$ MAX. ASS. in P_4 ; MIN. ASS. in P_3.

Poiché ad esempio $\lim_{x \to \pm\infty} f(x,0) = \lim_{x \to \pm\infty} (1-x) = \mp\infty$,

allora f è ILLIMITATA.

Poiché $\dfrac{1}{f(x,y)} = \dfrac{1}{(x-1)(y-1)}$,

$\dfrac{1}{f}$ non è definita lungo le rette $x=1$ e $y=1$.

Inoltre ad esempio $\lim_{x \to 1^\pm} \dfrac{1}{f(0,x)} = \lim_{x \to 1^\pm} \dfrac{1}{1-x} = \mp\infty$

$\dfrac{1}{f}$ è ILLIMITATA.

4) La superficie è un cilindro circolare retto.

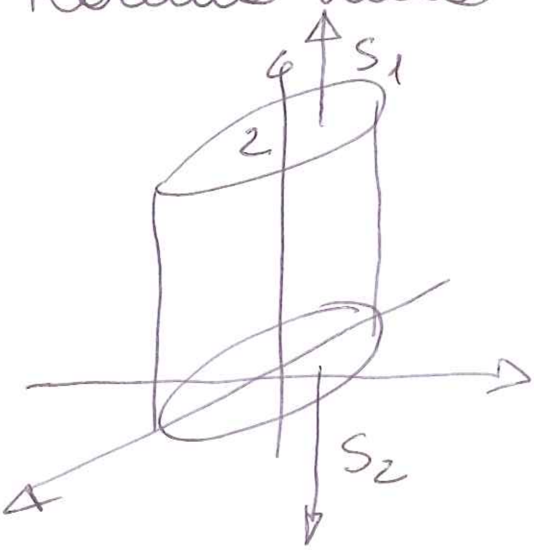
$L = 3\cos\vartheta \; ; \; M = 3\,\mathrm{sen}\,\vartheta \; ; \; N = 0.$

$$\Phi_S(\vec{F}) = \int_0^{2\pi} d\vartheta \int_0^{2} dt \left[81 \cos^2\vartheta\, \mathrm{sen}\,\vartheta\, (t+1) \cos\vartheta + 3\,\mathrm{sen}\,\vartheta\, t \right]$$

$$= \int_0^{2} dt \int_0^{2\pi} \left[81 \cos^3\vartheta\, \mathrm{sen}\,\vartheta\, (t+1) + 3\,\mathrm{sen}\,\vartheta\, t \right] d\vartheta$$

$$= \int_0^{2} dt \left[-81 \frac{\cos^4\vartheta}{4} (t+1) - 3\cos\vartheta \cdot t \right]_0^{2\pi} = 0$$

In alternativa, applicando il Teorema della Divergenza:



$$\Phi_S(\vec{F}) = \iiint_D \mathrm{div}\,\vec{F}\, dx\, dy\, dz - \Phi_{S_1}(\vec{F}) - \Phi_{S_2}(\vec{F})$$

$$= \int_0^{2} dz \int_0^{2\pi} d\vartheta \int_0^{3} d\rho\, \rho \left[\left(2xy(t+1) + y \right) \right]_{\substack{x=\rho\cos\vartheta \\ y=\rho\,\mathrm{sen}\,\vartheta}} - \Phi_{S_1} - \Phi_{S_2}$$

$$= \int_0^{2} dz \int_0^{2\pi} d\vartheta \int_0^{3} d\rho \left[2\rho^3 \cos\vartheta\, \mathrm{sen}\,\vartheta\, (t+1) + \rho^2\, \mathrm{sen}\,\vartheta \right] - \Phi_{S_1} - \Phi_{S_2}$$

$$= \int_0^2 dz \int_0^3 d\rho \left[-\rho^3 \cos^2\vartheta (t+1) - \rho^2 \cos\vartheta \right]_0^{2\pi} - \Phi_{S_1} - \Phi_{S_2}$$

$$= 0 - \Phi_{S_1} - \Phi_{S_2}$$

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Su S_1 : $\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = 2 \end{cases}$

$$\vec{F} \cdot \vec{n}_e = 3\rho^3 \cos^2\vartheta \sin\vartheta \cdot 0 + 2 \cdot 0 + 2\rho\sin\vartheta \cdot 1$$

$$\Rightarrow \Phi_{S_1}(\vec{F}) = \int_0^{2\pi} d\vartheta \int_0^3 2\rho^2 \sin\vartheta \, d\vartheta = 0$$

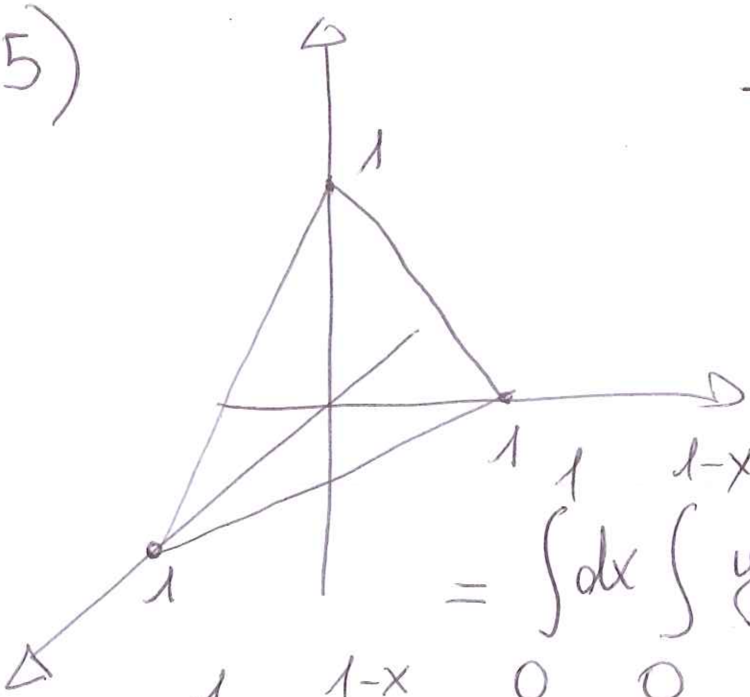
Su S_2 : $\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = 0 \end{cases}$

$$\vec{F} \cdot \vec{n}_e = \rho^3 \cos^2\vartheta \sin\vartheta \cdot 0 + 0 \cdot 0 + 0 \cdot (-1) = 0$$

$$\Rightarrow \Phi_{S_2}(\vec{F}) = 0.$$

$$\Rightarrow \Phi_S(\vec{F}) = 0,$$

5)



$$\Omega = \left\{ 0 \le x \le 1;\ 1 \le y \le 1-x;\ 0 \le z \le 1-x-y \right\}$$

$$\Rightarrow \iiint_{\Omega} y^2 \, dx\, dy\, dz$$

$$= \int_0^1 dx \int_0^{1-x} y^2 dy \left[1-x-y\right]$$

$$= \int_0^1 dx \int_0^{1-x} \left[(1-x)\frac{y^3}{3} - \frac{y^4}{4}\right]_0^{1-x} =$$

$$= \int_0^1 dx \left[\frac{(1-x)^4}{3} - \frac{(1-x)^4}{4}\right] dx = \int_0^1 \frac{1}{12}\left[(1-x)^4\right] dx$$

$$= -\frac{1}{12} \cdot \frac{(1-x)^5}{5} = \frac{1}{60}.$$

6)

$$\vec{r}'(t) = \begin{cases} x'(t) = \sqrt{2}\cos t \\ y'(t) = (1-\sin t) \\ z'(t) = (1+\sin t) \end{cases}$$

$$\vec{r}''(t) = \begin{cases} x''(t) = -\sqrt{2}\sin t \\ y''(t) = -\cos t \\ z''(t) = \cos t \end{cases} ; \quad \vec{r}'''(t) = \begin{cases} x'''(t) = -\sqrt{2}\cos t \\ y'''(t) = \sin t \\ z'''(t) = -\sin t \end{cases}$$

$$v(t) = \|\vec{r}\,'(t)\| = \sqrt{2\cos^2 t + 1 + \sin^2 t - \cancel{2\sin t} + 1 + \sin^2 t + \cancel{2\sin t}}$$

$$= \sqrt{2(\cos^2 t + \sin^2 t) + 2} = 2$$

$$\vec{r}\,'(t) \wedge \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2}\cos t & (1-\sin t) & (1+\sin t) \\ -\sqrt{2}\sin t & -\cos t & \cos t \end{vmatrix}$$

$$= 2\cos t\,\vec{i} - \left[\sqrt{2}(1+\sin t)\right]\vec{j} + \left[\sqrt{2}(\sin t - 1)\right]\vec{k}$$

$$\Rightarrow \|\vec{r}\,'(t) \wedge \vec{r}\,''(t)\| = \sqrt{4\cos^2 t + 2(1+\sin t)^2 + 2(1-\sin t)^2} = \sqrt{2}\,v(t)$$

$$\Rightarrow \varkappa(t) = \frac{\|\vec{r}\,'(t) \wedge \vec{r}\,''(t)\|}{v^3(t)} = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$$

$$\left(\vec{r}\,'(t) \wedge \vec{r}\,''(t)\right) \cdot \vec{r}\,'''(t) = \begin{vmatrix} -\sqrt{2}\cos t & \sin t & -\sin t \\ \sqrt{2}\cos t & (1-\sin t) & (1+\sin t) \\ -\sqrt{2}\sin t & -\cos t & \cos t \end{vmatrix}$$

$\mathrm{I}^{\underline{a}} + \mathrm{II}^{\underline{a}}$

$$= \begin{vmatrix} -\sqrt{2}\cos t & \sin t & -\sin t \\ 0 & 1 & 1 \\ -\sqrt{2}\sin t & -\cos t & \cos t \end{vmatrix}$$

$II^a + III^a$

$$= \begin{vmatrix} -\sqrt{2}\cos t & 0 & -\sin t \\ 0 & 2 & 1 \\ -\sqrt{2}\sin t & 0 & \cos t \end{vmatrix} = 2\left[-\sqrt{2}(\cos^2 t + \sin^2 t)\right]$$

$$= -2\sqrt{2}$$

$$\Rightarrow \tau(t) = \frac{\left(\vec{r}\,'(t) \wedge \vec{r}\,''(t)\right) \cdot \vec{r}\,'''(t)}{\|\vec{r}\,'(t) \wedge \vec{r}\,''(t)\|^2} = \frac{-2\sqrt{2}}{8} = \frac{-\sqrt{2}}{4}$$

Poichè la curva possiede curvatura e torsione costanti, essa è una elica cilindrica, per quanto non in forma canonica.