SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 30/3/2021

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1) Poiché
$$\sin x = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k$$

$$\Rightarrow \sin(t^2) = \sum_{k=0}^{+\infty} \frac{t^{4k+2}}{(2k+1)!}(-1)^k$$

CONV. TOTALE in ogni compatto

$$\Rightarrow \underline{\int_0^x \sin(t^2)\,dt} = \int_0^x \sum_{k=0}^{+\infty} \frac{t^{4k+2}}{(2k+1)!}(-1)^k\,dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^x t^{4k+2}\,dt =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\frac{1}{4k+3} t^{4k+3}\right]_0^x$$

$$= \underline{\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!\,(4k+3)} x^{4k+3}}$$

2) In $\mathbb{R}^2$ privato degli assi, $f(x,y) \in C^\infty$

$\Rightarrow$ differenziabile, derivabile, continua.

$$\frac{\partial f}{\partial x} = \frac{1}{y}\left[\frac{y\cos(xy)\,x - \operatorname{sen}(xy)}{x^2}\right]$$

Per simmetria: $$\frac{\partial f}{\partial y} = \frac{xy\cos(xy) - \operatorname{sen}(xy)}{xy^2}$$

Nell'origine:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy + o(xy)}{xy} = 1 = f(0,0)$$

Lungo gli assi: asse x

$$\lim_{(x,y)\to(x_0,0)} f(x,y) = \lim_{(x,y)\to(x_0,0)} \frac{\operatorname{sen}(xy)}{xy} = 1$$

e, per analogia, anche $\lim_{(x,y)\to(0,y_0)} f(x,y) = 1$

$\Rightarrow f \in C^0(\mathbb{R}^2)$

$$f(x,0) = f(0,y) = 1$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

lungo asse x: $\frac{\partial f}{\partial x}(x_0,0) = 0$

$$\frac{\partial f}{\partial y}(x_0,0) = \lim_{k\to 0}\left[\frac{\sin(x_0 k)}{x_0 k} - 1\right]\frac{1}{k}$$

$$= \lim_{k\to 0}\left[\frac{\cancel{x_0 k} - \frac{(x_0 k)^3}{3!} + o(k^3) - \cancel{x_0 k}}{x_0 k}\right]\frac{1}{k}$$

$$= \lim_{k\to 0}\left(\frac{-x_0^3}{6}\right)k = 0$$

Analogamente, $\frac{\partial f}{\partial y}(0,y_0) = \frac{\partial f}{\partial x}(0,y_0) = 0$

DERIVABILITÀ DIREZIONALE in $(0,0)$:

$$\lim_{t\to 0}\frac{\left[\frac{\sin(\alpha\beta t^2)}{\alpha\beta t^2} - 1\right]}{t} = \lim_{t\to 0}\left[\frac{\cancel{(\alpha\beta t^2)} - \frac{(\alpha\beta t^2)^3}{6} - \cancel{\alpha\beta t^2}}{\alpha\beta t^2}\right]\frac{1}{t}$$

$$= \frac{-(\alpha\beta)^3}{6\alpha\beta}\lim_{t\to 0}\frac{t^6}{t^3} = 0 \qquad \forall\, \alpha,\beta \neq 0$$

Già visto che per $\alpha=0$ $\frac{\partial f}{\partial y}(0,0)=0$

per $\beta=0$ $\frac{\partial f}{\partial x}(0,0)=0.$

DIFFERENZIABILITÀ in $(0,0)$:

A parte i limiti lungo gli assi:

$$\lim_{(h,k)\to(0,0)}\left|\frac{1}{\sqrt{h^2+k^2}}\left[\frac{\sin(hk)}{hk}-1\right]\right|=$$

$$\lim_{\substack{\cancel{(h,k)\to(0,0)} \\ \rho\to 0}} \cancel{\frac{1}{\sqrt{h^2+k^2}}}\left|\frac{1}{\rho}\left[\frac{\cancel{\rho^2\sin\vartheta\cos\vartheta}-\frac{1}{6}\rho^6\sin^3\vartheta\cos^3\vartheta}{\rho^2\cos\vartheta\sin\vartheta}\cancel{-\rho^2\cos\vartheta\sin\vartheta}\right]\right|$$

$$=\lim_{\rho\to 0}\left|-\frac{1}{6}\rho^3\left[\sin^3\vartheta\cos^3\vartheta\right]\right|$$

$$\leq\lim_{\rho\to 0}\frac{1}{6}\rho^3=0$$ UNIF. RISPETTO A ϑ.

LUNGO GLI ASSI: $f(h,0)-f(0,0)=0$

~~$\Rightarrow$ DIFFERENZIABILE in $(0,0)$~~

$=f(0,k)-f(0,0)$

$$\Rightarrow\lim_{(h,k)\to(0,0)}\frac{1}{\sqrt{h^2+k^2}}\left[\Delta f-df\right]=\lim_{(h,k)\to(0,0)}\frac{0}{\sqrt{h^2+k^2}}$$

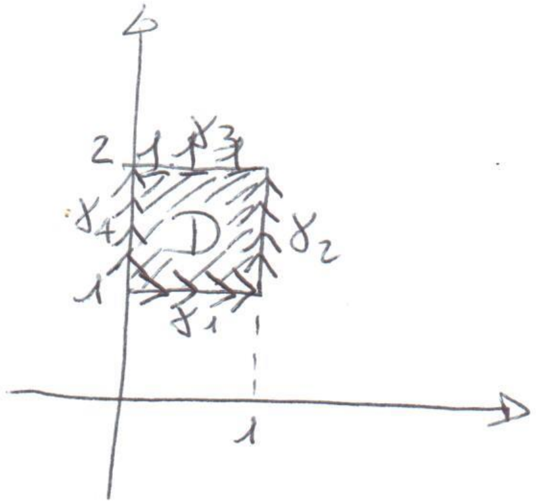
$$=\lim_{(h,k)\to(0,0)}0=0$$ $\Rightarrow$ DIFF. BILE in $(0,0)$

3) PUNTI STAZIONARI:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[1 + \frac{y^2-4}{4+x^2}\right] = \frac{(4-y^2)}{(4+x^2)^2}\,2x$$

$$\frac{\partial f}{\partial y} = \frac{1}{4+x^2}\,\frac{\partial}{\partial y}\left(x^2+y^2\right) = \frac{2y}{4+x^2}$$

$$\vec{\nabla} f = \vec{0} \iff (x,y) = (0,0) \in D.$$



SULLA FRONTIERA:

$$f\big|_{\gamma_1} = f(x,1) = \frac{x^2+1}{x^2+4}$$

$$\left(f\big|_{\gamma_1}\right)' = \frac{6x}{(x^2+4)^2} \geq 0 \quad \forall x \geq 0$$

$$f\big|_{\gamma_2} = f(1,y) = \frac{1+y^2}{5} \; ; \quad \left(f\big|_{\gamma_2}\right)' = \frac{2}{5}y \geq 0 \quad \forall y \geq 0$$

$$f\big|_{\gamma_3} = f(x,2) = \frac{4+x^2}{4+x^2} = 1 \quad \text{costante}$$

$$f\big|_{\gamma_4} = f(0,y) = \frac{y^2}{4} \; ; \quad \left(f\big|_{\gamma_4}\right)' = \frac{1}{2}y \geq 0 \quad \forall y \geq 0$$

MIN. ASS. in $(0,1)$:

$$f(0,1) = \frac{1}{4}$$

MAX. ASS. in TUTTI i punti $\in \gamma_3$:

$$(x,2)\ ;\quad x \in [0,1].$$

$$f(x,2) = 1.$$

In $\mathbb{R}^2$: $\quad f(x,y) \geq 0$

$$f(x,y) = 0 \iff (x,y) = (0,0)$$

MIN. ASS.

$$f(0,y) = \frac{y^2}{4} \xrightarrow[y \to \pm\infty]{} +\infty \qquad \nexists \text{ MAX. ASS.}$$

4) $$\begin{cases} x(t) = \sin^2 t = \dfrac{1-\cos 2t}{2} \\ y(t) = \cos^2 2t \\ z(t) = \sin t \cos t = \dfrac{1}{2} \sin 2t \end{cases}$$

$$\vec{r}(0) = (0,1,0)\ ; \quad \vec{r}(2\pi) = (0,1,0)$$

CURVA CHIUSA

$$\vec{r}(\pi) = (0,1,0) = \vec{r}(0)$$

CURVA NON SEMPLICE

$$\begin{cases} x'(t) = 2\sin t\cos t = \sin 2t \\ y'(t) = -4\cos 2t\,\sin 2t = -2\sin 4t \\ z'(t) = \cos 2t \end{cases}$$

Si può scegliere a proprio piacimento di riportare tutto a funzioni di $2t$ e/o funzioni di $4t$.

$$v(t) = \sqrt{\sin^2 2t + 4\sin^2 4t + \cos^2 2t}$$

$$= \sqrt{1 + 4\sin^2 4t} \geq 1 > 0$$ CURVA REGOLARE

$\cancel{\vec{r}' \wedge \vec{r}'' = \vec{i}}$

$$\begin{cases} x'' = 2\cos 2t \\ y'' = -8\cos 4t \\ z'' = -2\sin 2t \end{cases} ; \quad \begin{cases} x''' = -4\sin 2t \\ y''' = 32\sin 4t \\ z''' = -4\cos 2t \end{cases}$$

$$\vec{r}' \wedge \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin 2t & -2\sin 4t & \cos 2t \\ 2\cos 2t & -8\cos 4t & -2\sin 2t \end{vmatrix}$$

$$= \vec{i}\left[4\sin 4t\sin 2t + 8\cos 2t\cos 4t\right] + \vec{j}\left[2\sin^2 2t + 2\cos^2 2t\right] + \vec{k}\left[-8\cos 4t\sin 2t + 4\sin 4t\cos 2t\right]$$

$$= \vec{i}\left[8\cos 2t \operatorname{sen}^2 2t + 8\cos 2t\left(\cos^2 2t - \operatorname{sen}^2 2t\right)\right]$$

$$+ 2\vec{j} + \vec{k}\left[-8\operatorname{sen} 2t\left(\cos^2 2t - \operatorname{sen}^2 2t\right) + 8\cos^2 2t \operatorname{sen} 2t\right]$$

$$= 8\cos^3 2t\,\vec{i} + 2\vec{j} - 8\operatorname{sen}^3 2t\,\vec{k}$$

$$\Rightarrow \quad \|\vec{r}\,' \wedge \vec{r}\,''\| = 2\sqrt{1 + 16\left(\cos^6 2t + \operatorname{sen}^6 2t\right)}$$

Si osservi che, perché $(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$$\Rightarrow \cos^6 2t + \operatorname{sen}^6 2t = \left(\cos^2 2t + \operatorname{sen}^2 2t\right)\cdot\left(\cos^4 2t - \cos^2 2t \operatorname{sen}^2 2t + \operatorname{sen}^4 2t\right)$$

$$= \left[\left(1-\operatorname{sen}^2 2t\right)^2 - \left(1-\operatorname{sen}^2 2t\right)\operatorname{sen}^2 2t + \operatorname{sen}^4 2t\right]$$

$$= 1 + \operatorname{sen}^4 2t - 2\operatorname{sen}^2 2t - \operatorname{sen}^2 2t + \operatorname{sen}^4 2t + \operatorname{sen}^4 2t$$

$$= 3\operatorname{sen}^4 2t - 3\operatorname{sen}^2 2t + 1 = 3\operatorname{sen}^2 2t\left(\operatorname{sen}^2 2t - 1\right) + 1$$

$$= -3\operatorname{sen}^2 2t \cos^2 2t + 1 = 1 - \frac{3}{4}\operatorname{sen}^2 4t$$

$$\Rightarrow \quad \|\vec{r}\,' \wedge \vec{r}\,''\| = \cancel{2\sqrt{1 - 12\operatorname{sen}^2 4t}}$$

$$= 2 \cdot \sqrt{1 + 16\left(1 - \frac{3}{4}\operatorname{sen}^2 4t\right)}$$

$$\boxed{= 2\sqrt{17 - 12\operatorname{sen}^2 4t}}$$

$$\Rightarrow \quad k(t) = \frac{\|\vec{r}\,' \wedge \vec{r}\,''\|}{v^3(t)}$$

$$= \frac{2\sqrt{17 - 12\sin^2 4t}}{(1 + 4\operatorname{sen}^2 4t)^{3/2}}$$

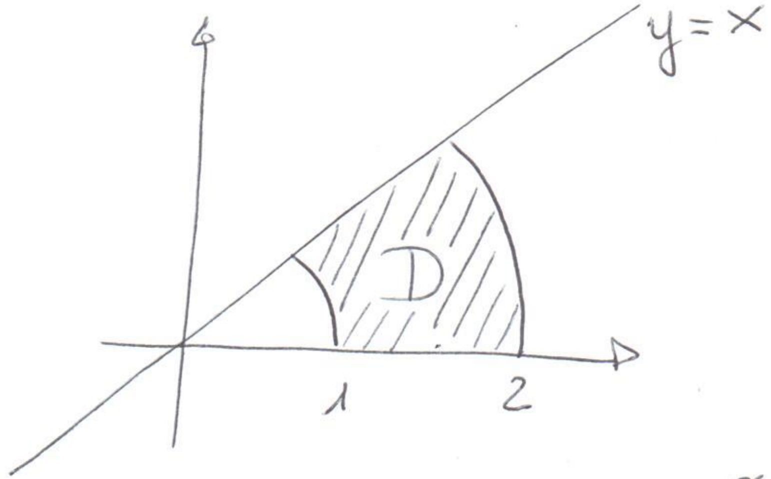
$$\tau(t) = \frac{-\vec{r}\,''' \cdot (\vec{r}\,' \wedge \vec{r}\,'')}{\|\vec{r}\,' \wedge \vec{r}\,''\|^2}$$

$$\vec{r}\,''' \cdot (\vec{r}\,' \wedge \vec{r}\,'') = \begin{vmatrix} -4\operatorname{sen} 2t & 32\operatorname{sen} 4t & -4\cos 2t \\ \operatorname{sen} 2t & -2\sin 4t & \cos 2t \\ 2\cos 2t & -8\cos 4t & -2\operatorname{sen} 2t \end{vmatrix}$$

$$= -16\operatorname{sen} 4t \operatorname{sen}^2 2t + 64\cos^2 2t \operatorname{sen} 4t$$
$$\cancel{+32\cos 2t \operatorname{sen} 2t \cos 4t} - 16\cos^2 2t \sin 4t$$
$$\cancel{-32\cos 2t \operatorname{sen} 2t \cos 4t} + 64\operatorname{sen}^2 2t \sin 4t$$

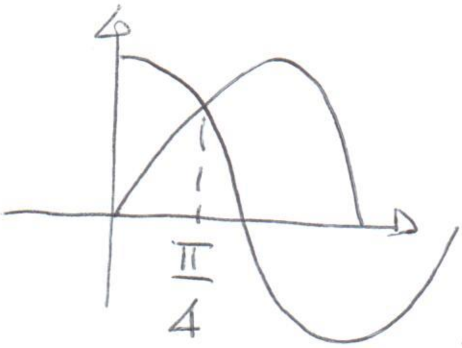
$$= 64\operatorname{sen} 4t - 16\operatorname{sen} 4t = 48\operatorname{sen} 4t$$

$$\Rightarrow \tau(t) = \frac{-48\operatorname{sen} 4t}{4\left[17 - 12\operatorname{sen}^2 4t\right]} = \frac{-12\operatorname{sen} 4t}{17 - 12\sin^2 4t}$$

5) 

In coordinate polari: $\begin{cases} \cancel{\rho^2 \leq 4} \; 1 \leq \rho^2 \leq 4 \\ 0 \leq \rho \sin\vartheta \leq \rho\cos\vartheta \end{cases}$

$\Rightarrow \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \sin\vartheta \leq \cos\vartheta \end{cases} \Rightarrow \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \vartheta \leq \frac{\pi}{4} \end{cases}$



$$\Rightarrow \int_0^{\frac{\pi}{4}} d\vartheta \int_1^2 \frac{\cancel{\rho^3}\cos\vartheta\sin\vartheta}{\cancel{\rho^3}}\, d\rho$$

$$= \left[\frac{1}{2}\sin^2\vartheta\right]_0^{\frac{\pi}{4}} \cdot \left[\rho\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{2}\right] = \frac{1}{4}.$$