SVOLGIMENTO ANALISI 1 del 10/9/2014

1) $$\lim_{x \to +\infty} x^3 \left[\operatorname{sen}\left(\frac{1}{x}\right) + x \log\left(1 - \frac{1}{x^2}\right) \right] =$$

$$\lim_{x \to +\infty} x^3 \left[\frac{1}{x} - \frac{1}{6x^3} + x\left(-\frac{1}{x^2} - \frac{1}{2x^4} + o\left(\frac{1}{x^4}\right)\right) + o\left(\frac{1}{x^3}\right) \right]$$

$$= \lim_{x \to +\infty} x^3 \left[\cancel{\frac{1}{x}} - \frac{1}{6x^3} - \cancel{\frac{1}{x}} - \frac{1}{2x^3} + o\left(\frac{1}{x^3}\right) \right]$$

$$= \left(-\frac{1}{6} - \frac{1}{2} \right) = -\frac{2}{3}.$$

2) $$\frac{i-3}{i-\frac{1}{2}} = \frac{(i-3)\left(-i-\frac{1}{2}\right)}{1+\frac{1}{4}} = \frac{1 - \frac{1}{2}i + 3i + \frac{3}{2}}{\frac{5}{4}}$$

$$= \frac{4}{5}\left(\frac{5}{2} + \frac{5}{2}i\right) = 2(1+i) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i \operatorname{sen}\left(\frac{\pi}{4}\right)\right]$$

$$z^8 = 4096\left[\cos(2\pi) + i \operatorname{sen}(2\pi)\right] = \cancel{16}\, 4096$$

$$\sqrt[4]{z} = \sqrt[8]{8}\left[\cos\left(\frac{\frac{\pi}{4}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{4}+2k\pi}{4}\right)\right] \quad (2)$$

$$w_0 = \sqrt[8]{8}\left[\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right]$$

$$w_1 = \sqrt[8]{8}\left[\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right]$$

$$w_2 = \sqrt[8]{8}\left[\cos\left(\frac{17}{16}\pi\right) + i\sin\left(\frac{17\pi}{16}\right)\right]$$

$$w_3 = \sqrt[8]{8}\left[\cos\left(\frac{25}{16}\pi\right) + i\sin\left(\frac{25}{16}\pi\right)\right]$$

3) ~~Eq~~ Equazione definita in $\mathbb{R}^2$.

$e^{-y} \neq 0 \quad \forall y \Rightarrow \nexists$ sol. singolari.

Teo. Esistenza e Unicità:

$$f(x,y) = xe^{x^2-y} = xe^{x^2} \cdot e^{-y}$$

$$A(x) = xe^{x^2} \in C^\infty(\mathbb{R}) \quad ; \quad e^{-y} \in C^\infty[\alpha,\beta]$$

$$\forall [\alpha,\beta] \subseteq \mathbb{R}$$

$\Rightarrow \exists!$ sol. di classe C^1 localmente.

$$\int \frac{dy}{e^{-y}} = \int x e^{x^2} dx$$

$$e^{y} = \frac{1}{2} e^{x^2} + C$$

$$y = \log\left(\frac{1}{2} e^{x^2} + C\right) \qquad \text{INT. GEN.}$$

$$y(0) = \underset{\log 1}{\underset{\parallel}{0}} = \log\left(\frac{1}{2} + C\right)$$

$$\Rightarrow 1 = \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\Rightarrow \exists!\ \text{sol.} \qquad y(x) = \log\left[\frac{1}{2}\left(e^{x^2} + 1\right)\right]$$

$$= \log\left(e^{x^2} + 1\right) - \log 2.$$

Si osservi che, nonostante il Teorema di Esistenza e Unicità sia di tipo locale, la soluzione è definita su tutto $\mathbb{R}$.

3_{bis}) $\quad f(x) \geq 0 \quad \forall x \in [1,4]$.

Dunque l'area del dominio corrisponde proprio a $\displaystyle\int_1^4 \frac{1}{x(x^2+4)}\,dx$.

Fratti semplici:

$$\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ 4A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=0 \end{cases}$$

$$\Rightarrow \int_1^4 f(x)\,dx = \frac{1}{4}\int_1^4 \left[\frac{1}{x} - \frac{x}{x^2+4}\right]dx$$

$$= \frac{1}{4}\left[\log(|x|) - \frac{1}{2}\log(x^2+4)\right]_1^4$$

$$= \frac{1}{4}\left[\log 4 - \frac{1}{2}\log(20) + \frac{1}{2}\log 5\right]$$

$$= \frac{1}{4}\left[\log 4 - \frac{1}{2}\log(4)\right] = \frac{1}{8}\log 4.$$

4) I_{def}: $\log x \geq 0 \implies x \geq 1$ (5)

$I_{def} = [1, +\infty)$.

$f(1) = 0$ $\qquad f \in C^0[1, +\infty)$.

Segno: $f(x) \geq 0 \quad \forall x \in I_{def}$.

$$f'(x) = \frac{1}{1+\log x} \cdot \frac{1}{2\sqrt{\log x}} \cdot \frac{1}{x} \geq 0$$

$\iff 1 + \log x > 0 \iff x > \frac{1}{e}$; $x \neq 1$

quindi $\forall x \in (1, +\infty)$.

f crescente $\forall x \in [1, +\infty)$.

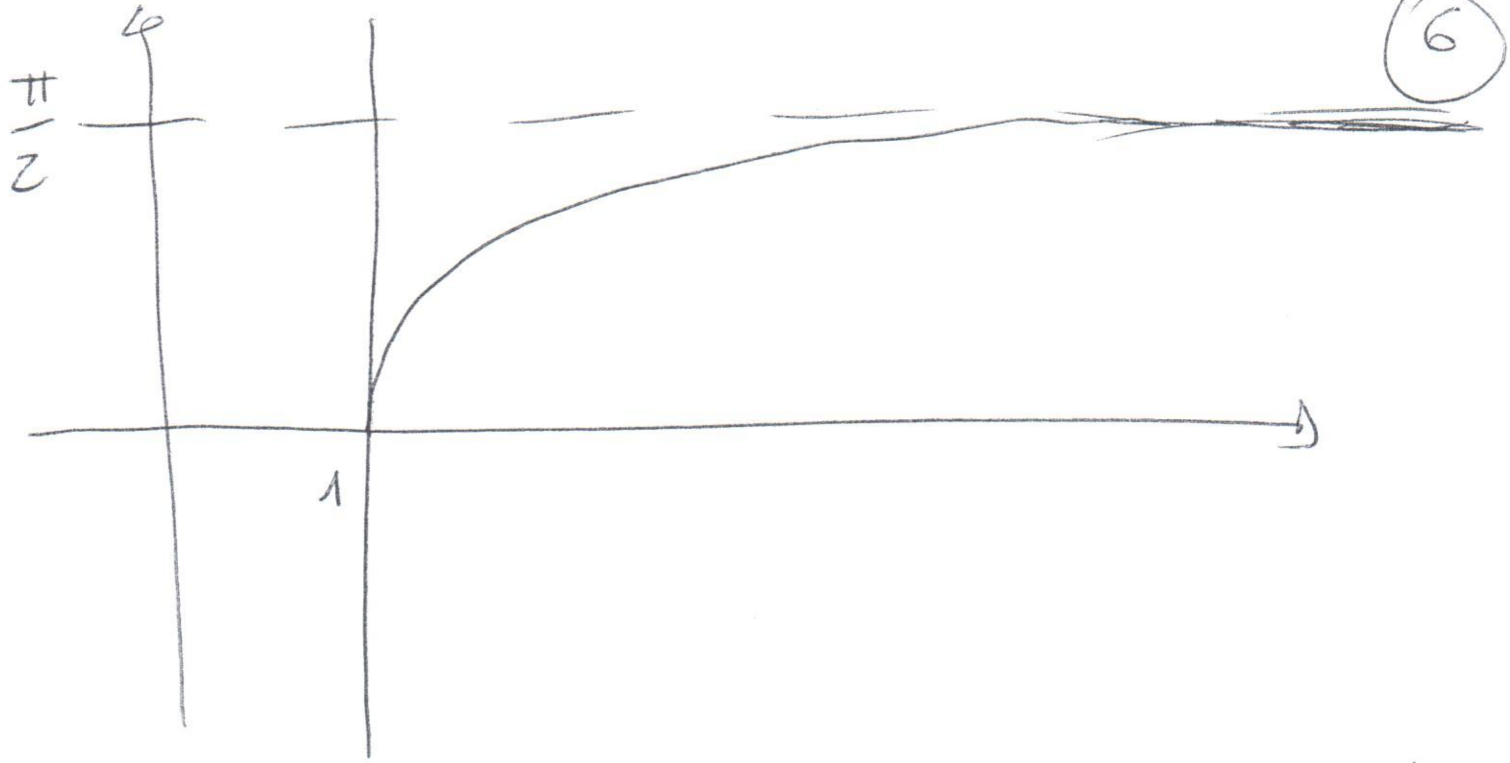
f' NON è definita in $x = 1$:

$$\lim_{x \to 1^+} f'(x) = \frac{1}{2 \cdot 0^+} = +\infty$$

$$\lim_{x \to +\infty} f(x) = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$; $y = \frac{\pi}{2}$

AS. ORIZZONTALE per $x \to +\infty$.

Poichè la funzione ha tangente verticale in $x = 1$, si può disegnare il grafico senza flessi:



5) $a_n = n^3 \log\left(\frac{n^4+1}{n^4}\right) = n^3 \log\left(1+\frac{1}{n^4}\right)$

<u>Primo metodo</u>:

$\cancel{a_n = \log\left[\left(1+\frac{1}{n^4}\right)^{n^3}\right] = \log[}$

$a_n = \frac{1}{n} n^4 \log\left(1+\frac{1}{n^4}\right) = \frac{1}{n} \log\left[\left(1+\frac{1}{n^4}\right)^{n^4}\right]$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \log\left[\left(1+\frac{1}{n^4}\right)^{n^4}\right]$

$= \log(e) \cdot \lim_{n\to\infty} \frac{1}{n} = 0$

Quindi $a_n \sim \frac{1}{n}$.

Secondo metodo:

$$a_n \sim \cancel{\frac{1}{n}}\, n^3 \cdot \frac{1}{n^4} = \frac{1}{n} \longrightarrow 0$$

Studio di $\sum a_n$:

Poichè $a_n \sim \frac{1}{n}$

$\Rightarrow$ la serie diverge, per confronto asintotico con la serie armonica.