SVOLGIMENTO PROVA SCRITTA di ANALISI 2 del 13/4/2012

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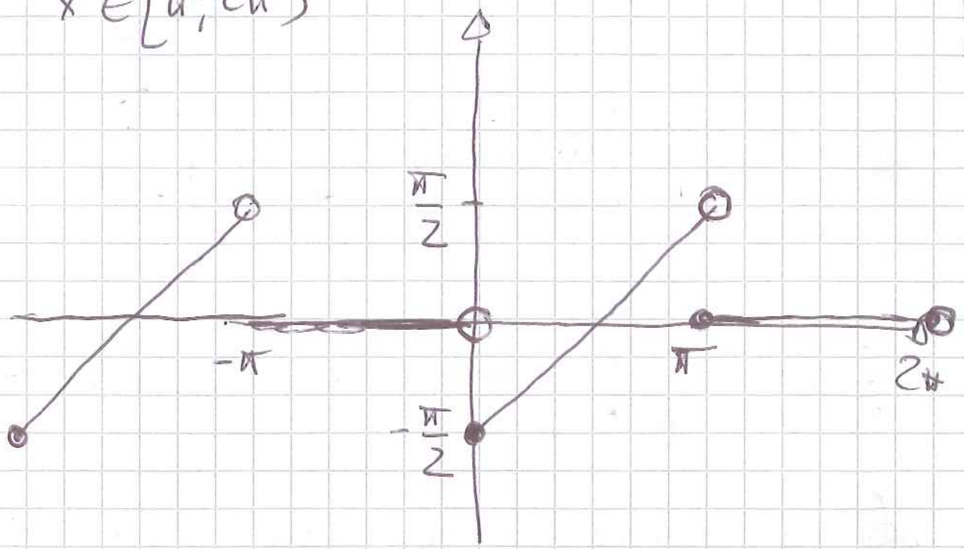
1)
$$f(x)=\begin{cases} x-\frac{\pi}{2} & x\in[0,\pi) \\ 0 & x\in[\pi,2\pi) \end{cases}$$



nè pari, nè dispari

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$
$$=\frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)dx$$
$$=\frac{1}{\pi}\left[\frac{x^2}{2}-\frac{\pi}{2}x\right]_0^{\pi}=\left(\frac{\pi}{2}-\frac{\pi}{2}\right)=0$$ (media integrale nulla).

$$a_k=\frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)\cos(kx)\,dx=\frac{1}{\pi k}\left[\sin(kx)\left(x-\frac{\pi}{2}\right)\Big|_0^{\pi}-\int_0^{\pi}\sin(kx)\,dx\right]=\frac{1}{\pi k^2}\Big[\cos(kx)\Big]_0^{\pi}$$

$$=\frac{1}{\pi k^2}\left[(-1)^k-1\right]=\begin{cases} 0 & \text{se } k=2m \\ -\frac{2}{\pi(2m+1)^2} & \text{se } k=2m+1 \end{cases}, \quad m\geq 0$$ $\cancel{\frac{1}{\pi k^2}\left[(-1)^k-1\right]}$

$$b_k = \frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)\sin(kx)\,dx = -\frac{1}{\pi k}\left[\left(x-\frac{\pi}{2}\right)\cos(kx) + \right.$$

$$\left. - \int_0^{\pi}\cos(kx)\,dx\right] = -\frac{1}{\pi k}\left[+\frac{\pi}{2}\cos(k\pi) + \frac{\pi}{2} - \frac{1}{k}\cancel{\sin(k\pi)}\right]_0^{\pi}$$

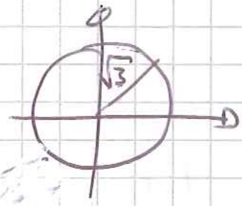
$$= -\frac{1}{2k}\left[\cos(k\pi)+1\right] = \begin{cases} 0 & \text{se } k=2m+1 \\ \dfrac{-1}{2m} & \text{se } k=2m\ ;\ m\geq 1 \end{cases}$$

$$f(x) \sim \sum_{m=0}^{\infty}\frac{2}{\pi(2m+1)^2}\cos\left[(2m+1)x\right] + \sum_{m=1}^{\infty}\frac{-1}{2m}\sin(2mx).$$

La somma della serie $S(x)$ è

$$S(x) = \begin{cases} f(x) & \text{se } x \neq k\pi \\ -\dfrac{\pi}{4} & \text{se } x = 2m\pi \\ +\dfrac{\pi}{4} & \text{se } x = (2m+1)\pi \end{cases}$$

la serie converge UNIFORMEMENTE in ogni intervallo del tipo $[\alpha,\beta]$, con $k\pi < \alpha < \beta < (k+1)\pi$, $k \in \mathbb{Z}$.



2) $x^2+y^2 \leq 3 \Rightarrow$

$$\text{Area } S = \iint_{D(x,y)} \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$z_x = \frac{-x}{\sqrt{9-x^2-y^2}}\ ;\quad z_y = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$\Rightarrow \iint_{D(x,y)} \sqrt{1+\frac{x^2+y^2}{9-x^2-y^2}}\,dx\,dy$$

$$= \iint_{D(x,y)} \sqrt{\frac{9}{9-x^2-y^2}}\,dx\,dy$$

$$= \int_0^{2\pi} d\vartheta \int_0^{\sqrt{3}} \frac{3}{\sqrt{9-\rho^2}}\,\rho\,d\rho = 2\pi\left[-3\sqrt{9-\rho^2}\right]_0^{\sqrt{3}}$$

$$= -6\pi\left[\sqrt{6}-3\right] = 6\pi\left(3-\sqrt{6}\right).$$

3) a) In coordinate polari:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\rho\to 0} \rho^2 \log(\rho^2) = 0 \quad \text{(limite notevole).}$$

f è prolungabile per continuità:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

b) $$f_x = 2x\log(x^2+y^2) + \frac{(x^2+y^2)}{(x^2+y^2)}2x = 2x\left[1+\log(x^2+y^2)\right]$$

Analogamente

$$f_y = 2y\left[1+\log(x^2+y^2)\right]$$

c) In $(x,y)=(0,0)$: $$\tilde{f}_x = \lim_{x\to 0} \frac{f(x,0)-\tilde{f}(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x^2\log x^2}{x} = 0.$$

$$\tilde{f}_y = \lim_{y \to 0} \frac{\tilde{f}(0,y) - \tilde{f}(0,0)}{y} = \lim_{y \to 0} \frac{y^2 \log(y^2)}{y} = \lim_{y \to 0} y \log(y^2)$$

$$= 0.$$

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4)

$$X_y = \frac{(2y - 12x)(x^2-y^2)^{\cancel{2}} - (x^2+y^2-12xy)\,2\cancel{(x^2-y^2)}(-2y)}{(x^2-y^2)^{\cancel{4}\,3}}$$

$$= \frac{2yx^2 - 2y^3 - 12x^3 + 12xy^2 + 4x^2y + 4y^3 - 48xy^2}{(x^2-y^2)^3}$$

$$= \frac{-12x^3 + 6x^2y - 36xy^2 + 2y^3}{(x^2-y^2)^3}$$

$$Y_x = \frac{(12x-2y)(x^2-y^2)^{\cancel{2}} - (6x^2+6y^2-2xy)\,2\cancel{(x^2-y^2)}\,2x}{(x^2-y^2)^{\cancel{4}\,3}}$$

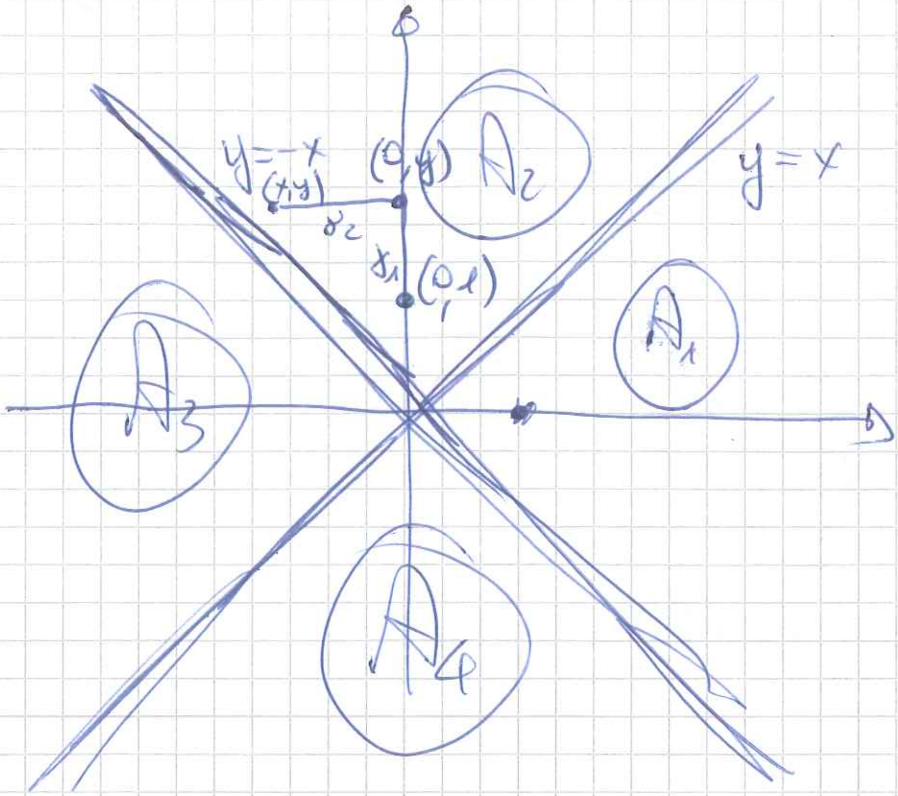
$$= \frac{12x^3 - 12xy^2 - 2x^2y + 2y^3 - 24x^3 - 24xy^2 + 8x^2y}{(x^2-y^2)^3}$$

$$= \frac{-12x^3 + 6x^2y - 36xy^2 + 2y^3}{(x^2-y^2)^3}$$

$X_y = Y_x$ FORMA CHIUSA

$$I_{def} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \neq 0\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid |x| \neq |y|\}$$



Forme esatte in ogni componente connessa (A_1, A_2, A_3, A_4)

Calcolo delle primitive in $A_2 \ni (0,1)$:

mi muovo lungo la curva $\gamma_1 \cup \gamma_2$

lungo γ_1: $\begin{cases} y=t & t\in[1,y] \\ x=0 & dx=0 \end{cases}$

$$\int_{\gamma_1} \omega = \int_1^y \frac{6t^2}{t^4}\,dt = \int_1^y \frac{6}{t^2}\,dt = -\frac{6}{t}\Big|_1^y = -6\left(\frac{1}{y}-1\right)$$

γ_2: $\begin{cases} x=t & t\in[0,x] \\ y \text{ costante} & dy=0 \end{cases}$

$$\int_{\gamma_2} \omega = \int_0^x \frac{t^2+y^2-12ty}{(t^2-y^2)^2}\,dt$$

$$= \int_0^x \left[\frac{t^2-y^2+2y^2-12ty}{(t^2-y^2)^2}\right] dt$$

$$= \int_0^x \left[\frac{1}{t^2-y^2} + \right.$$

$$\frac{t^2+y^2-12ty}{(t^2-y^2)^2} = \frac{A}{t+y} + \frac{B}{(t+y)^2} + \frac{C}{(t-y)} + \frac{D}{(t-y)^2}$$

$$= \frac{A(t+y)(t-y)^2 + B(t-y)^2 + C(t-y)(t+y)^2 + D(t+y)^2}{(t+y)^2(t-y)^2}$$

$$= \frac{A(t+y)(t^2-2ty+y^2) + B(t^2+y^2-2ty) + C(t-y)(t^2+2ty+y^2) + D(t^2+2yt+y^2)}{(\quad)(\quad)}$$

$$\Rightarrow \begin{cases} A + C = 0 \\ -2Ay + Ay + B + 2yC - Cy + D = 1 \\ Ay^2 - 2Ay^2 - 2yB + Cy^2 - 2Cy^2 + 2Dy = -12y \\ Ay^3 + By^2 - Cy^3 + Dy^2 = y^2 \end{cases}$$

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$$\begin{cases} C = -A \\ -Ay + Cy + B + D = 1 \\ -Ay^2 - Cy^2 - 2\cancel{y}B + 2\cancel{y}D = -12\cancel{y} \\ \cancel{Ay^3} + Ay^3 - Cy^3 + B\cancel{y^2} + D\cancel{y^2} = \cancel{y^2}1 \end{cases}$$

$$\begin{cases} C = -A \\ -2Ay + B + D = 1 \\ -B + D = -6 \\ 2Ay + B + D = 1 \end{cases} \qquad \begin{cases} C = -A \\ B + D = 1 \\ -B + D = -6 \\ 4Ay = 0 \end{cases} \qquad \begin{cases} A = 0 \\ C = 0 \\ D = -\frac{5}{2} \\ B = \frac{7}{2} \end{cases}$$

$$= \int_0^x \left[\frac{7}{2(t+y)^2} - \frac{5}{2(t-y)^2} \right] dt$$

$$\boxed{\text{Infatti: } \frac{7(t-y)^2 - 5(t+y)^2}{2(t^2-y^2)^2} = \frac{7t^2 - 14ty + 7y^2 - 5t^2 - 10ty - 5y^2}{2(\quad)^2} = \frac{2t^2 + 2y^2 - 24ty}{2(t^2-y^2)^2} = \frac{t^2+y^2-12ty}{(t^2-y^2)^2}}$$

$$= \frac{-7}{2} \cdot \frac{1}{(t+y)} \Big|_0^x + \frac{5}{2(t-y)} \Big|_0^x =$$

$$= \frac{-7}{2}\left[\frac{1}{x+y} - \frac{1}{y}\right] + \frac{5}{2}\left[\frac{1}{x-y} + \frac{1}{y}\right]$$

$$= -\frac{7}{2}\left(\frac{1}{x+y}\right) + \frac{5}{2}\left(\frac{1}{x-y}\right) + \frac{6}{y}$$

$$\Rightarrow \quad V(x,y) = \cancel{\frac{-6}{y}} + 6 - \frac{7}{2}\left(\frac{1}{x+y}\right) + \frac{5}{2}\left(\frac{1}{x-y}\right) + \cancel{\frac{6}{y}} + C$$

$$= -\frac{7}{2}\left(\frac{1}{x+y}\right) + \frac{5}{2}\left(\frac{1}{x-y}\right) + K$$

$$V(0,1) = -\frac{7}{2} - \frac{5}{2} + K = K - 6 = 0$$

$$\Rightarrow K = 6$$

$$\Rightarrow \boxed{V(x,y)} = -\frac{7}{2}\left(\frac{1}{x+y}\right) + \frac{5}{2}\left(\frac{1}{x-y}\right) + 6$$

$$= \frac{-7(x-y) + 5(x+y) + 12(x^2-y^2)}{2(x^2-y^2)}$$

$$= \frac{-2x + 12y + 12(x^2-y^2)}{2(x^2-y^2)}$$

$$= \boxed{\frac{-x+6y+6(x^2-y^2)}{(x^2-y^2)}} = \frac{-x+6y}{x^2-y^2} + 6$$

N.B.. $$V_x = \frac{(-1+12x)(x^2-y^2) - (-x+6y+6x^2-6y^2)2x}{(x^2-y^2)^2}$$

$$= \frac{-x^2+y^2+\cancel{12x^3}-\cancel{12xy^2}+2x^2-12xy-\cancel{12x^3}+\cancel{12xy^2}}{(x^2-y^2)^2} = \frac{x^2+y^2-12xy}{(x^2-y^2)^2} = X(x,y)$$

$$V_y = \frac{(6-12y)(x^2-y^2) - (-x+6y+6x^2-6y^2)(-2y)}{(x^2-y^2)^2} =$$

$$= \frac{6x^2-6y^2-\cancel{12yx^2}+\cancel{12y^3}-2xy+12y^2+\cancel{12yx^2}-\cancel{12y^3}}{(x^2-y^2)^2} = \frac{6x^2+6y^2-2xy}{(x^2-y^2)^2} = Y(x,y)$$

5) $\begin{cases} \cancel{\sqrt{1-y^2}}\ 1-y^2 \geq 0 \\ x \neq 0 \\ -1 \leq \frac{y}{x} \leq 1 \end{cases} \Rightarrow \begin{cases} |y| \leq 1 \\ x > 0 \\ -x \leq y \leq x \end{cases} \cup \begin{cases} |y| \leq 1 \\ x < 0 \\ -x \geq y \geq x \end{cases}$

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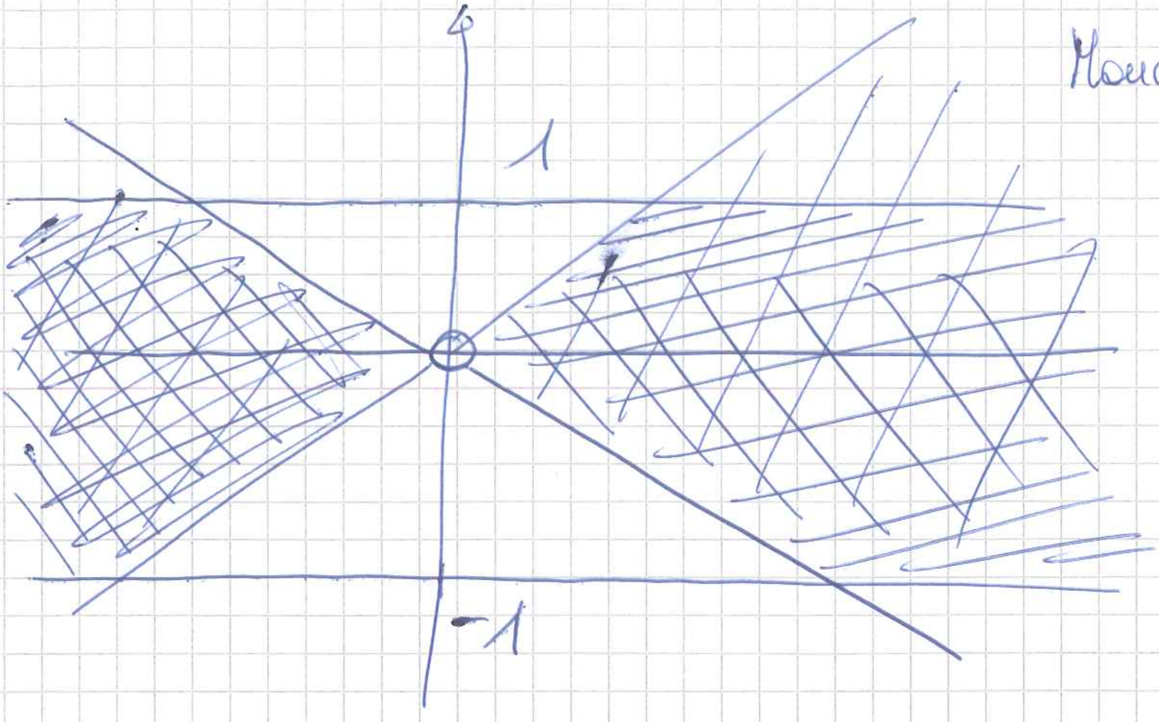

Mancando l'origine, l'INSIEME È ILLIMITATO, SCONNESSO, NÉ APERTO NÉ CHIUSO.