SVOLGIMENTO PROVA SCRITTA AN. 1
del 22/3/2018

1) a) OMOGENEA ASSOCIATA:

$$9\alpha^2-6\alpha+1=(3\alpha-1)^2=0$$

$$\Rightarrow \alpha_1=\alpha_2=\frac{1}{3}$$

$$\Rightarrow y_0(x)=C_1 e^{\frac{1}{3}x}+C_2 x e^{\frac{1}{3}x}$$

$$=C_1\sqrt[3]{e^x}+C_2 x\sqrt[3]{e^x}$$

NON OMOGENEA:

$$f(x)=e^{\frac{1}{3}x} \Rightarrow y_p(x)=x^2e^{\frac{1}{3}x}A$$

$$y_p'(x)=\left(\frac{1}{3}x^2+2x\right)e^{\frac{1}{3}x}A$$

$$y_p''(x)=\left(\frac{2}{3}x+2+\frac{1}{9}x^2+\frac{2}{3}x\right)e^{\frac{1}{3}x}A$$

$$=\left(\frac{1}{9}x^2+\frac{4}{3}x+2\right)e^{\frac{1}{3}x}A$$

$$\Rightarrow A\left(\cancel{x^2}+\cancel{12x}+18-\cancel{2x^2}-\cancel{12x}+\cancel{x^2}\right)e^{\frac{1}{3}x}=e^{\frac{1}{3}x}$$

$$\Rightarrow A=\frac{1}{18}$$

$$\Rightarrow y(x)=C_1 e^{\frac{x}{3}}+C_2 x e^{\frac{x}{3}}+\frac{1}{18}x^2e^{\frac{1}{3}x}$$

$$=\left(C_1+C_2x+\frac{1}{18}x^2\right)e^{\frac{x}{3}}$$

b) Poiché $\lim_{x \to -\infty} x^n e^{\frac{x}{3}} = 0 \quad \forall n \in \mathbb{N}$ ②

$\Rightarrow$ TUTTE LE SOL. SONO INFINITESIME

per $x \to -\infty$

c) $y(x) \underset{x \to +\infty}{\sim} \frac{1}{18} x^2 e^{\frac{x}{3}} \longrightarrow +\infty$

$\Rightarrow$ NON CI SONO SOLUZIONI

2) $0 \leq \frac{\operatorname{arctg} x}{x^2} < \underbrace{\frac{\pi}{2x^2}}_{\text{integrabile in } [1,+\infty)}$

$$\int_1^{+\infty} \frac{\operatorname{arctg} x}{x^2} dx = -\frac{1}{x} \operatorname{arctg} x \Big|_1^{+\infty}$$

$$+ \int_1^{+\infty} \frac{1}{x(1+x^2)} dx$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}$$

$$\Rightarrow \begin{cases} B = -A = -1 \\ C = 0 \\ A = 1 \end{cases}$$

$$\int_1^{+\infty} f(x)\,dx = \operatorname{arctg} 1 + \int_1^{+\infty}\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$$= \frac{\pi}{4} + \left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^{+\infty}$$

$$= \frac{\pi}{4} + \left[\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right]_1^{+\infty} = \frac{\pi}{4} - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$+ \underbrace{\lim_{x\to+\infty} \ln\left(\frac{x}{\sqrt{1+x^2}}\right)}_{= \ln 1 = 0} = \frac{\pi}{4} + \frac{1}{2}\ln 2$$

3) $z \neq 2$

$$|z+2| = 3\,|z-2|$$

$$|x+2+iy|^2 = 9\,|x-2+iy|^2$$

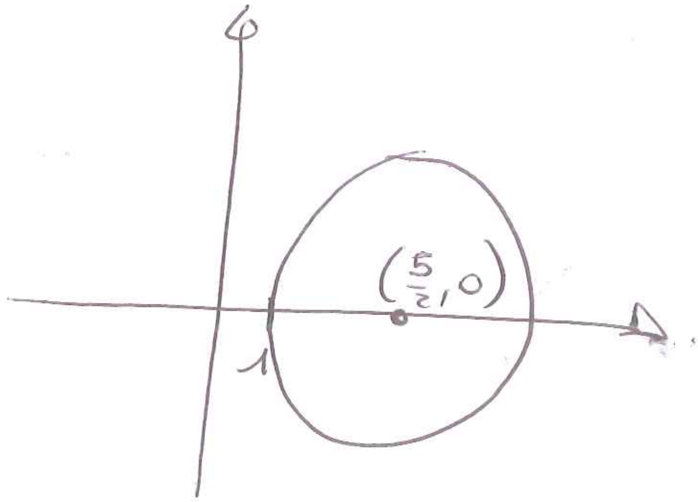
$$(x+2)^2 + y^2 = 9\left[(x-2)^2 + y^2\right]$$

$$x^2 + 4x + 4 + y^2 = 9x^2 - 36x + 36 + 9y^2$$

$$-8x^2 - 8y^2 + 40x - 32 = 0$$

$$x^2+y^2-5x+4=0$$

$$\left(x-\frac{5}{2}\right)^2+y^2=\frac{9}{4}$$

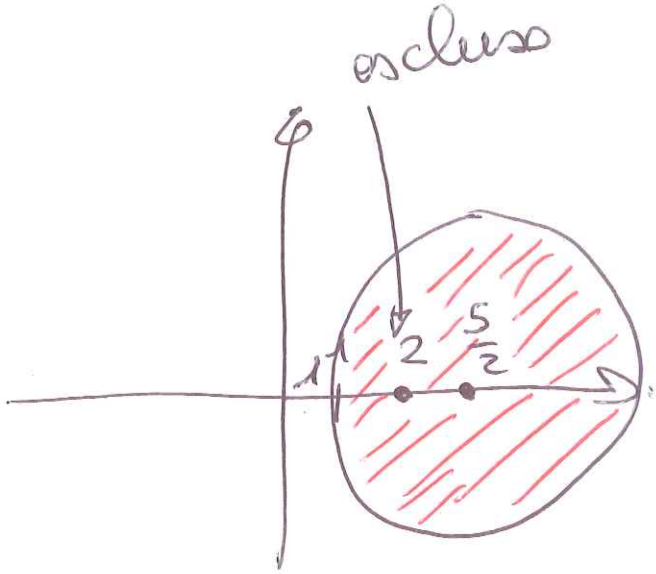


$$\left|\frac{z+2}{z-2}\right|>3$$

$$\Rightarrow \begin{cases} -8x^2-8y^2+40x-32>0 \\ z\neq 2 \end{cases}$$

$$\begin{cases} x^2+y^2-\cancel{40x}\,5x+4<0 \\ z\neq 2 \end{cases}$$

$$\begin{cases} \left(x-\frac{5}{2}\right)^2+y^2<\frac{9}{4} \\ z\neq 2 \end{cases}$$



4) $$\cos\frac{1}{n}+\frac{1}{2n^2}=1-\cancel{\frac{1}{2n^2}}+\frac{1}{4!\,n^4}+\cancel{\frac{1}{2n^2}}+o\left(\frac{1}{n^4}\right)$$

$$\sim 1+\frac{1}{24n^4}$$

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$$\Rightarrow \log\left(\cos\frac{1}{n}+\frac{1}{2n^2}\right)\sim\frac{1}{24n^4}$$

$$\Rightarrow \sum_{n=1}^{+\infty}a_n \approx \frac{1}{24}\sum\frac{1}{n^4}$$

CONVERGENTE

5) a) $D=\{x\neq 0\}$

$f(x)=0 \iff x=1$

$f(x)\geq 0 \iff x\geq 0$

$\nexists f(0)$ NO INTERSEZIONE ASSE y.

$$\lim_{x\to 0^\pm} f(x)=\frac{1}{0^\pm}=\pm\infty$$

AS. VERT. DX e SX

$x=0$.

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^{\frac{2}{3}}}{x} = \lim_{x \to \pm\infty} \frac{1}{x^{\frac{1}{3}}} = 0^{\pm}$$

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$y = 0$ AS. ORIZZ.

per $x \to \pm\infty$

b) $$f'(x) = \frac{-\frac{2}{3}(1-x)^{-\frac{1}{3}}x - (1-x)^{\frac{2}{3}}}{x^2}$$

$$= \frac{-\frac{2}{3}\frac{x}{(1-x)^{\frac{1}{3}}} - (1-x)^{\frac{2}{3}}}{x^2} = \frac{-1}{3x^2(1-x)^{\frac{1}{3}}}\left[2x + 3(1-x)\right]$$

$$= \frac{-1}{3x^2(1-x)^{\frac{1}{3}}}(3-x) > 0$$

$$\Longleftrightarrow \frac{3-x}{1-x} < 0 \Longleftrightarrow \frac{x-3}{x-1} < 0$$

f cresce in $(1, 3)$

f decresce in $(-\infty, 0) \cup (0, 1) \cup (3, +\infty)$

$x = 3$ punto di MAX. REL. : $f(3) = \frac{2^{2/3}}{3}$

$x = 1$ punto di MIN. REL. : $f(1) = 0$

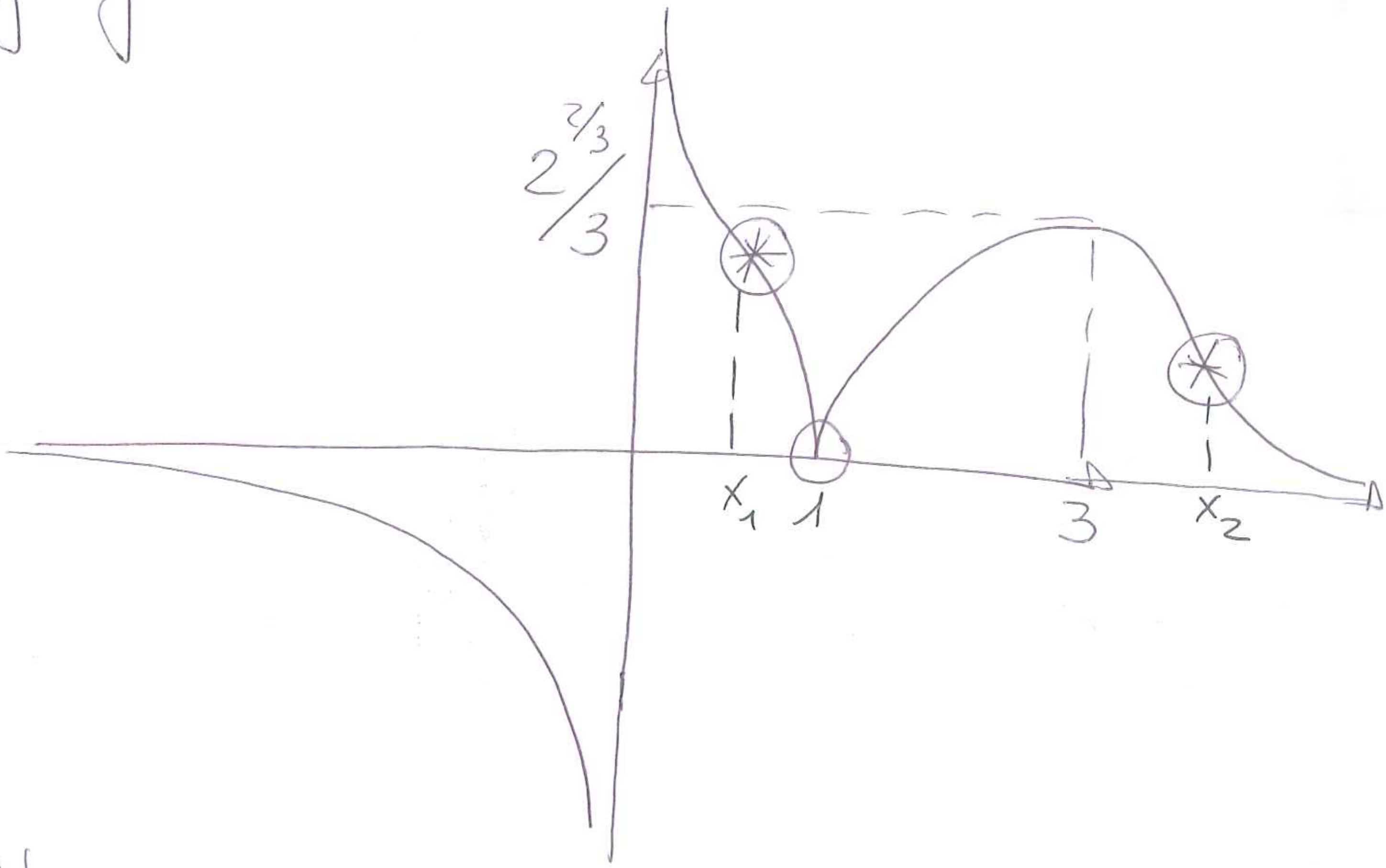
Poiché $\lim_{x \to 0^{\pm}} f(x) = \pm\infty \Rightarrow \nexists$ MAX. - MIN. ASS.

c) $\lim_{x \to 1^{\pm}} f'(x) = \frac{-1}{3(0^{\mp})^{1/3}} \cdot 2$

$= \pm\infty$

$\Rightarrow$ Il punto $(1, f(1)=0)$ è un punto di CUSPIDE.

Grafico:



Numero minimo di flessi: 2

Anche se non richiesto, verifichiamo l'esistenza dei flessi esplicitamente.

$$f''(x) = \frac{1}{3}\left[\frac{x-3}{x^2(1-x)^{1/3}}\right]' = \frac{1}{3}\left[\frac{\left[x^2(1-x)^{1/3}\right] - (x-3)\left[x^2(1-x)^{1/3}\right]'}{x^4(1-x)^{2/3}}\right]$$

$$= \frac{1}{3}\left\{\frac{x^{\cancel{2}}(1-x)^{1/3} - (x-3)\left[2\cancel{x}(1-x)^{1/3} - \frac{1}{3}x^{\cancel{2}}\frac{1}{(1-x)^{2/3}}\right]}{x^{\cancel{4}3}(1-x)^{2/3}}\right\}$$

$$= \frac{1}{9}\left[\frac{3x(1-x) - (x-3)\left[6(1-x) - x\right]}{x^3(1-x)^{4/3}}\right]$$

$$= \frac{1}{9}\left[\frac{3x - 3x^2 - (x-3)(6-7x)}{x^3(1-x)^{4/3}}\right]$$

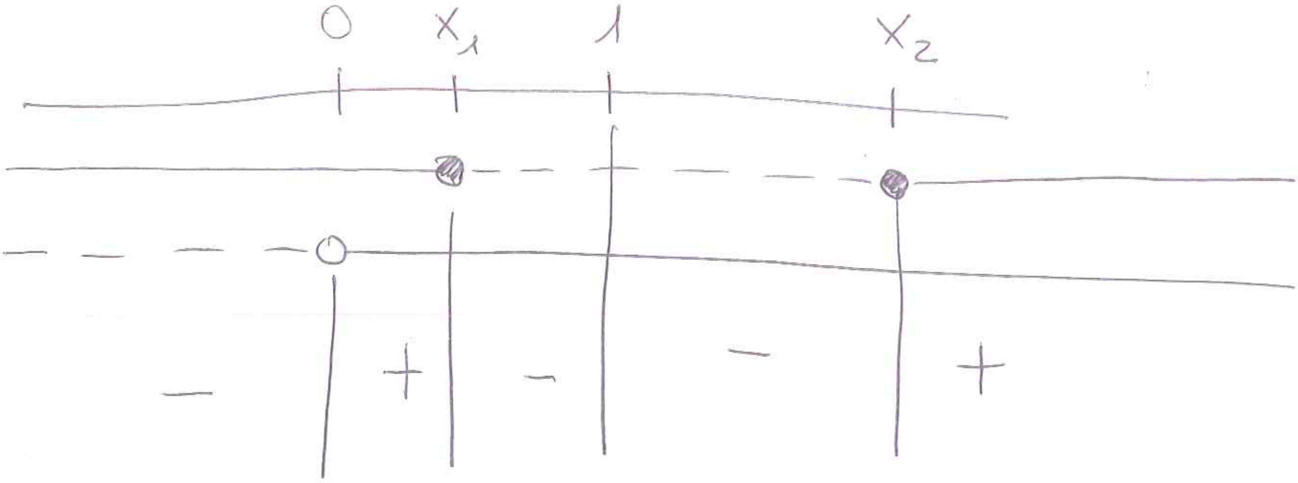
$$= \frac{1}{9}\left[\frac{3x - 3x^2 - 6x + 7x^2 + 18 - 21x}{x^3(1-x)^{4/3}}\right]$$

$$= \frac{1}{9}\left[\frac{4x^2 - 24x + 18}{x^3(1-x)^{4/3}}\right] =$$

$$= \frac{2}{9}\left[\frac{2x^2 - 12x + 9}{x^3(1-x)^{4/3}}\right] \qquad x_{1,2} = \frac{6 \pm 3\sqrt{2}}{2}$$

$$0 < x_1 = \frac{6 - 3\sqrt{2}}{2} < 1$$

Segno di $f''(x)$



f concava in $(-\infty, 0)$.

f convessa in $(0, x_1)$

f concava in $(x_1, 1)$

f concava in $(1, x_2)$

f convessa in $(x_2, +\infty)$

In x_1 flesso obliquo discendente

In x_2 flesso obliquo ascendente.