Background

Let G be a discrete group and let $\lambda : G \longrightarrow \mathbb{U}(l^2(G))$ be its left regular representation (we will write λ_s to denote the unitary operator $\lambda(s)$, for every $s \in G$). There are several definitions of amenable group which are equivalent, but in this talk we need to give only the following three definitions. To this aim we recall that a function $\omega : G \longrightarrow \mathbb{C}$ is *positive definite* if for every $n \in \mathbb{N}$, $s_1, ..., s_n \in G$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$ we have

$$\sum_{1 \le i,j \le n} \alpha_i \overline{\alpha_j} \omega(s_i s_j^{-1}) > 0.$$

Definition 0.1. A discrete group G is said to be *amenable* if it satisfies one of the following equivalent conditions:

1. There exists a state μ on $l^\infty(G)$ which is invariant under the left traslation action: $\mu(f(st)) = f(t)$, for every $s \in G$.

2. There exists a net ω_α of finitely supported positive functions on G, with $\omega_\alpha(e) = 1$ for every α, such that $\omega_\alpha \to 1$ pointwise (e is the identity of G).

3. There are finitely supported unit vectors $\xi_\beta \in l^2(G)$ such that $\|\lambda_s(\xi_\beta) - \xi_\beta\|_2 \to 0$, for every $s \in G$.

As usual, we write $C_r^*(G)$ to denote the *reduced* C^*-algebra of G, i.e. the norm closure in $\mathbb{B}(l^2(G))$ of span$\{\lambda_s : s \in G\}$. We now give the general definition of nuclear C^*-algebra.

Definition 0.2. A C^*-algebra A is said to be *nuclear* if for every C^*-algebra B, one has $\|\cdot\|_{min} = \|\cdot\|_{max}$ on the algebraic tensorial product of A with B.

It can be shown that a discrete group G is amenable if and only if $C_r^*(G)$ is nuclear. For more general groups this equivalence is not true.

Definition 0.3. A Banach space E has the *metric approximation property* (m.p.a.) if there exists a net of finite rank contactions $T_\alpha \in \mathbb{B}(E)$ such that $\|T_\alpha x - x\| \to 0$, for every $x \in E$.

It can be shown that a nuclear C^*-algebra has the m.a.p. but the converse is not true. The C^*-algebra $\mathbb{B}(H)$ of bounded operator on a Hilbert space H is not nuclear and does not have the m.a.p.. Recall that a continuous linear map $F : A \longrightarrow B$ between C^*-algebras is *completely positive* if for every $n \in \mathbb{N}$, $(a_{i,j})_{1 \le i,j \le n} \in M_n(A)_+$, we have $(F(a_{i,j}))_{1 \le i,j \le n} \in M_n(B)_+$.

Theorem 0.4. *Let A be a C^*-algebra. Then A is nuclear if and only if it has the m.a.p. where every T_α is a completely positive map.*

The completely positivity is essential in the above characterization of nuclear C^*-algebras. We will see that the free group with n generators $\mathbb{F}_n$ has the m.a.p., although $C^*_r(\mathbb{F}_2)$ is not nuclear. In fact, for example, $\mathbb{F}_2$ is not amenable. For if, let $a, b \in \mathbb{F}_2$ be two generators of the group and define A^+ the set of all words starting with a, A^- the set of all words starting with a^{-1}, B^+ the set of all words starting with b, B^- the set of all words starting with b^{-1} and finally let C be the subgroup of $\mathbb{F}_2$ generated by b. It is easy to see that the following decompositions of the group holds:

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ - C) \sqcup (B^- \cup C) = A^+ \sqcup aA^- =$$

$$= b^{-1}(B^+ - C) \sqcup (B^+ \cup C).$$

Let f be a state on $l^\infty(\mathbb{F}_2)$ which is invariant under the left traslation action. We have

$$1 = f(\chi_{A^+} + \chi_{A^-} + \chi_{B^+-C} + \chi_{B^-\cup C}) = f(\chi_{A^+}) + f(\chi_{A^-}) + f(\chi_{B^+-C}) + f(\chi_{B^-\cup C}) =$$

$$= f(\chi_{A^+}) + f(\lambda_a(\chi_{A^-})) + f(\lambda_{b^{-1}}(\chi_{B^+-C})) + f(\chi_{B^-\cup C}) =$$

$$= f(\chi_{A^+} + \chi_{aA^-}) + f(\chi_{b^{-1}B^+-C} + \chi_{B^-\cup C}) = f(1) + f(1) = 2.$$

A contradiction.

The Main Result

Let G be a discrete group. We set $\delta_s := \chi_{\{s\}}$ and note that $\lambda_t(\delta_s) = \delta_{ts}$. Recall that for every positive definite function ω there is a triple $(H_\omega, \pi_\omega, \xi_\omega)$ (called *cyclic* emphrepresentation), where H_ω is an Hilbert space, π_ω is a unitary representation of G over H_ω and ξ_ω is a cyclic vector in H_ω such that $\omega(s) = \langle \xi_\omega, \pi_\omega(s)\xi_\omega \rangle$ for every $s \in G$.

Lemma 0.5. *For every positive definite function ω, there is a unique completely positive map M_ω from $C^*_r(G)$ to itself such that $M_\omega(\lambda_s) = \omega(s)\lambda_s$ and $\|M_\omega\| = \omega(e)$.*

Proof. Let $(H_\omega, \pi_\omega, \xi_\omega)$ be the cyclic representation associated to ω and e_i, $i \in I$, a basis for H_ω. Define $a_i(s) = \langle e_i, \pi_\omega(s)^* \xi_\omega \rangle$ and note that $a_i \in l^\infty(G)$ and $\sum |a_i(s)|^2 < \infty$ for every $s \in G$. Moreover, using the definition, it is easy to see that $\sum a_i(s)\overline{a_i(t)} = \omega(st^{-1})$. Let

N_{a_i} be the multiplication operator associated to the function a_i and set $M_\omega x := \sum N_{a_i} x N_{a_i}^*$, $x \in C_r^*(G)$. We have $\sum N_{a_i} N_{a_i}^* = \omega(e)1$ and

$$M_\omega(\lambda_s)\delta_t = \sum N_{a_i}\lambda_s N_{a_i}^*\delta_t = \sum N_{a_i}\lambda_s \overline{a_i(t)}\delta_t = \sum \overline{a_i(t)} N_{a_i}\delta_{st} =$$

$$= \sum \overline{a_i(t)} a_i(st)\delta_{st} = \omega(s)\lambda_s\delta_t,$$

for every $s, t \in G$. Since δ_t is a basis for $l^2(G)$ we have $M_\omega(\lambda_s) = \omega(s)\lambda_s$. Finally $\|M_\omega\| = \|m_\omega(1)\| = \omega(1)$ (because ω is the coefficient of π_ω, i.e. $\omega(s) = \langle \xi_\omega, \pi_\omega(s)\xi_\omega \rangle$), then M_ω is a bounded linear operator on a dense subspace of $C_r^*(G)$. Clearly M_ω is completely positive. □

Corollary 0.6. *If G is amenable, then $C_r^*(G)$ is nuclear.*

Proof. Using the second definition of amenable group and the previous lemma we can find a net of completely positive contractions of finite rank $M_{\omega_\alpha} : C_r^*(G) \longrightarrow C_r^*(G)$, with $M_{\omega_\alpha}\lambda_s = \omega_\alpha(s)\lambda_s$, for every $s \in G$. Since $\omega_\alpha(e) = 1$ and $\omega_\alpha(s) \to 1$ for every $s \in G$ we have that $\|M_{\omega_\alpha} x - x\| \to 0$, $x \in C_r^*(G)$.

□

From now on, we suppose that G is a free group with finitely many generators $a_1, ..., a_n$, $n \geq 2$ (i.e. $G = \mathbb{F}_n$), then every element a of G can be expressed as a finite product of a_i, a_j^{-1} which does not contain two adiacent inverse factors (the *word* of a). We write $|a|$ to denote the lenght of the word of a. Since $|a^{-1}| = |a|$, the function $d(a, b) = |b^{-1}a|$ is a left invariant metric on G. Moreover it is easy to see that $|ab| \leq |a| + |b|$ (if the number of factors deleted is $2l$, with $0 \leq l \leq \min(|a|, |b|)$, then the word of ab starts with the first $|a| - l$ factors of the word of a and ends with the last $|b| - l$ factors of b). Let E_m be the subset of G which contains all the element $a \in G$ with $|a| = m$. It is easy to see that $|E_0| = 1$ and $|E_m| = n(n-1)^{m-1}$, $m > 0$.

Lemma 0.7. *For every $t > 0$, the function $a \longrightarrow e^{-t|a|}$ is a positive definite function on G.*

Proof. Uffe Haagerup "An Example of Non Nuclear C^*-Algebras which has the m.a.p.".

□

Definition 0.8. We say that a function $\omega : G \longrightarrow \mathbb{C}$ *multiplies* $C_r^*(G)$ into itself if there exists a unique map $M_\omega : C_r^*(G) \longrightarrow C_r^*(G)$ such that $M_\omega(\lambda_s) = \omega(s)\lambda_s$, for every $s \in G$, or equivalently $M_\omega\lambda(f) = \lambda(\omega f)$, for every $f \in \mathbb{C}[G]$ and where $\lambda(f) := \sum_{s \in G} f(s)\lambda_s \in \mathbb{B}(l^2(G))$.

From a previous lemma we know that every positive definite function on G multiplies $C_r^*(G)$ into itself. To prove the main Theorem we need the next lemma.

Lemma 0.9. *Let ω be a function on G such that*

$$\sup_{s\in G} |\omega(s)|(1+|s|)^2 < \infty,$$

then ω multiplies $C_r^*(G)$ into itself and

$$\|M_\omega\| \leq 2\sup_{s\in G} |\omega(s)|(1+|s|)^2.$$

Theorem 0.10. *There exists a net ω_α of finitely supported functions on G such that $\|M_{\omega_\alpha}\| \leq 1$, for every α, and $\|M_{\omega_\alpha}x - x\| \to 0$, $x \in C_r^*(G)$. In particular $C_r^*(G)$ has the m.a.p..*

Proof. Uffe Haagerup "An Example of Non Nuclear C^*-Algebras which has the m.a.p.".

□