rational arc-length parameterization is impossible

*Theorem.* It is impossible to parameterize any plane curve, other than a straight line, by rational functions of its arc length.

rational parameterization \( r(t) = (x(t), y(t)) \) \( \implies \) curve points can be exactly computed by a *finite sequence of arithmetic operations*

arc length parameterization \( r(t) = (x(t), y(t)) \) \( \implies \) equal parameter increments \( \Delta t \) generate *equidistantly spaced points along the curve*

simple result but subtle proof — Pythagorean triples of polynomials, integration of rational functions, and calculus of residues


T. Sakkalis, R. T. Farouki, and L. Vaserstein (2009), Non–existence of rational arc length parameterizations for curves in \( \mathbb{R}^n \), *J. Comp. Appl. Math.* 228, 494–497
arc length parameterization by rational functions?

rational parameterization

arc–length parameterization

\[ \Delta s = \Delta t \]
rational arc-length parameterization?

\[ x(t) = \frac{X(t)}{W(t)}, \quad y(t) = \frac{Y(t)}{W(t)} \quad \text{with} \quad \gcd(W, X, Y) = 1, \quad W(t) \neq \text{constant} \]

\[ x'^2(t) + y'^2(t) \equiv 1 \quad \Rightarrow \quad (WX' - W'X)^2 + (WY' - W'Y)^2 \equiv W^4 \]

Pythagorean triple \( \Rightarrow \) \((x', y') = \left( \frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2} \right)\)

are \( x(t) = \int \frac{u^2 - v^2}{u^2 + v^2} \, dt, \quad y(t) = \int \frac{2uv}{u^2 + v^2} \, dt \) both rational?

\[ \frac{f(t)}{g(t)} = \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{C_{ij}}{(t - z_i)^j} + \frac{\bar{C}_{ij}}{(t - \bar{z}_i)^j} \]

\[ C_{i1} = \text{residue}_{t=z_i} \frac{f(t)}{g(t)}, \quad \int \frac{f(t)}{g(t)} \, dt \quad \text{is rational} \quad \iff \quad C_{i1} = \bar{C}_{i1} = 0 \]
\[
\int_{-\infty}^{+\infty} \frac{f(t)}{g(t)} \, dt = 2\pi i \sum_{\text{Im}(z_i)>0} \text{residue } \frac{f(t)}{g(t)} \bigg|_{t=z_i}
\]

rational indefinite integral ⇔ zero definite integral

proof by contradiction:

\[x(t) = \int \frac{u^2 - v^2}{u^2 + v^2} \, dt, \quad y(t) = \int \frac{2uv}{u^2 + v^2} \, dt\]

assume both rational with \(u(t), v(t) \neq 0\) and \(\gcd(u, v) = 1\)

choose \(\alpha, \beta\) so that \(\deg(\alpha u + \beta v)^2 < \deg(u^2 + v^2)\)

\[
\int \frac{(\alpha u + \beta v)^2}{u^2 + v^2} \, dt = \frac{1}{2}(\alpha^2 - \beta^2) x(t) + \alpha \beta y(t) + \frac{1}{2}(\alpha^2 + \beta^2) t \quad \text{is rational}
\]

\[
\Rightarrow \int_{-\infty}^{+\infty} \frac{(\alpha u + \beta v)^2}{u^2 + v^2} \, dt = 0 \Rightarrow \frac{[\alpha u(t) + \beta v(t)]^2}{u^2(t) + v^2(t)} \equiv 0
\]

contradicts \(u(t), v(t) \neq 0\) or \(\gcd(u, v) = 1\)
parametric speed of curve \( r(\xi) \)

\[
\sigma(\xi) = |r'(\xi)| = \frac{ds}{d\xi} = \text{derivative of arc length } s \text{ w.r.t. parameter } \xi
\]

\[
= \sqrt{x'^2(\xi) + y'^2(\xi)} \quad \text{for plane curve}
\]

\[
= \sqrt{x'^2(\xi) + y'^2(\xi) + z'^2(\xi)} \quad \text{for space curve}
\]

\( \sigma(\xi) \equiv 1 \) — i.e., \( s \equiv \xi \) — for arc-length or “natural” parameterization, but impossible for any polynomial or rational curve except a straight line.

Irrational nature of \( \sigma(\xi) \) has unfortunate computational implications:

- arc length must be computed approximately by numerical quadrature
- unit tangent \( t \), normal \( n \), curvature \( \kappa \), etc, not rational functions of \( \xi \)
- offset curve \( r_d(\xi) = r(\xi) + d\, n(\xi) \) at distance \( d \) must be approximated
- requires approximate real-time CNC interpolator algorithms, for motion along \( r(\xi) \) with given speed (feedrate) \( V = ds/dt \)
curves with “simple” parametric speed

Although $\sigma(\xi) = 1$ is impossible, we can gain significant advantages by considering curves for which the argument of $\sqrt{x'^2(\xi) + y'^2(\xi)}$ or $\sqrt{x'^2(\xi) + y'^2(\xi) + z'^2(\xi)}$ is a perfect square — i.e., polynomial curves whose hodograph components satisfy the Pythagorean conditions

$$x'^2(\xi) + y'^2(\xi) = \sigma^2(\xi) \quad \text{or} \quad x'^2(\xi) + y'^2(\xi) + z'^2(\xi) = \sigma^2(\xi)$$

for some polynomial $\sigma(\xi)$. To achieve this, the Pythagorean structure must be built into the hodograph \textit{a priori}, by a suitable algebraic model.

planar PH curves — Pythagorean structure of $r'(t)$ achieved through complex variable model

spatial PH curves — Pythagorean structure of $r'(t)$ achieved through quaternion or Hopf map models

higher dimensions or Minkowski metric — Clifford algebra formulation
\[ a, b, c = \text{real numbers} \]

choose any \( a, b \to c = \sqrt{a^2 + b^2} \)

\[ a, b, c = \text{integers} \]

\[ a^2 + b^2 = c^2 \iff \begin{cases} a = (u^2 - v^2)w \\ b = 2uvw \\ c = (u^2 + v^2)w \end{cases} \]

\[ a(t), b(t), c(t) = \text{polynomials} \]

\[ a^2(t) + b^2(t) \equiv c^2(t) \iff \begin{cases} a(t) = [u^2(t) - v^2(t)]w(t) \\ b(t) = 2u(t)v(t)w(t) \\ c(t) = [u^2(t) + v^2(t)]w(t) \end{cases} \]

hodograph of curve \( r(t) = \) derivative \( r'(t) \)

Pythagorean structure: \( x'^2(t) + y'^2(t) = \sigma^2(t) \) for some polynomial \( \sigma(t) \)
**Pythagorean-hodograph (PH) curves**

\[ \mathbf{r}(t) \text{ is a PH curve in } \mathbb{R}^n \iff \text{coordinate components of } \mathbf{r}'(t) \]
are elements of a “Pythagorean \((n + 1)\)-tuple of polynomials”

PH curves exhibit **special algebraic structures** in their hodographs

- rational offset curves  
  \[ \mathbf{r}_d(t) = \mathbf{r}(t) + d \mathbf{n}(t) \]

- polynomial arc-length function  
  \[ s(t) = \int_0^t |\mathbf{r}'(\tau)| \, d\tau \]

- closed-form evaluation of energy integral  
  \[ E = \int_0^1 \kappa^2 \, ds \]

- real–time CNC interpolators, rotation-minimizing frames, etc.

**generalize** PH curves to non-Euclidean metrics & other functional forms
planar offset curves

plane curve \( r(t) = (x(t), y(t)) \) with unit normal \( n(t) = \frac{(y'(t), -x'(t))}{\sqrt{x'^2(t) + y'^2(t)}} \)

offset at distance \( d \) defined by \( r_d(t) = r(t) + d n(t) \)

- defines center–line tool path, in order to cut a desired profile
- defines tolerance zone characterizing allowed variations in part shape
- defines erosion & dilation operators in mathematical morphology, image processing, geometrical smoothing procedures, etc.
- offset curves typically approximated in CAD systems
- PH curves have exact rational offset curve representations
offsets to Pythagorean–hodograph (PH) curves
Left: untrimmed offsets obtained by sweeping a normal vector of length $d$ around the original curve (including appropriate rotations at vertices).

Right: trimmed offsets, obtained by deleting certain segments of the untrimmed offsets, that are not globally distance $d$ from the given curve.
offset curve trimming procedure

Left: **self-intersections** of the untrimmed offset. Right: trimmed offset, after discarding segments between these points that fail the **distance test**.
Bezier control polygons of rational offsets

Offsets exact at any distance
intricate topology of parallel (offset) curves

"innocuous" curve

$y = x^4$

offset distance = 1
4 cusps, 6 self-intersections

offset distance $< d_{\text{crit}} = d_{\text{crit}} > d_{\text{crit}}$

offset curve geometry governed by Huygens’ principle (geometrical optics)
polynomial arc length function $s(\xi)$

for a planar PH curve of degree $n = 2m + 1$ specified by

$$r'(\xi) = (x'(\xi), y'(\xi)) = (u^2(\xi) - v^2(\xi), 2u(\xi)v(\xi))$$

where

$$u(\xi) = \sum_{k=0}^{m} u_k \left(\binom{m}{k}\right)(1 - \xi)^{m-k}\xi^k, \quad v(\xi) = \sum_{k=0}^{m} v_k \left(\binom{m}{k}\right)(1 - \xi)^{m-k}\xi^k,$$

the parametric speed can be expressed in Bernstein form as

$$\sigma(\xi) = |r'(\xi)| = u^2(\xi) + v^2(\xi) = \sum_{k=0}^{2m} \sigma_k \left(\binom{2m}{k}\right)(1 - \xi)^{2m-k}\xi^k,$$

where

$$\sigma_k = \sum_{j=\max(0,k-m)}^{\min(m,k)} \binom{m}{j} \binom{m}{k-j} \frac{(n-1)}{k} (u_j u_{k-j} + v_j v_{k-j}).$$
The cumulative arc length $s(\xi)$ is then the polynomial function

$$s(\xi) = \int_0^\xi \sigma(\tau) \, d\tau = \sum_{k=0}^n s_k \binom{n}{k} (1 - \xi)^{n-k} \xi^k,$$

of the curve parameter $\xi$, with Bernstein coefficients given by

$$s_0 = 0 \quad \text{and} \quad s_k = \frac{1}{n} \sum_{j=0}^{k-1} \sigma_j, \quad k = 1, \ldots, n.$$

Hence, the total arc length $S$ of the curve is simply

$$S = s(1) - s(0) = \frac{\sigma_0 + \sigma_1 + \cdots + \sigma_{n-1}}{n},$$

and the arc length of any segment $\xi \in [a, b]$ is $s(b) - s(a)$. The result is exact, as compared to the approximate numerical quadrature required for “ordinary” polynomial curves.
inversion of arc length function — find parameter value \( \xi_* \) at which arc length has a given value \( s_* \) — i.e., solve equation

\[
s(\xi_*) = s_*
\]

note that \( s \) is \textit{monotone-increasing} with \( \xi \) (since \( \sigma = ds/dt \geq 0 \)) and hence this polynomial equation has just one (simple) real root — easily computed to machine precision by Newton-Raphson iteration

Example: \textit{uniform rendering} of a PH curve — for given arc-length increment \( \Delta s \), find parameter values \( \xi_1, \ldots, \xi_N \) such that

\[
s(\xi_k) = k \Delta s, \quad k = 1, \ldots, N.
\]

With initial approximation \( \xi_k^{(0)} = \xi_{k-1} + \Delta s / \sigma(\xi_{k-1}) \), use Newton iteration

\[
\xi_k^{(r+1)} = \xi_k^{(r)} - \frac{s(\xi_k^{(r)})}{\sigma(\xi_k^{(r)})}, \quad r = 0, 1, \ldots
\]

Values \( \xi_1, \ldots, \xi_N \) define motion at \textit{uniform speed} along a curve — simplest case of a broader class of problems addressed by real–time interpolator algorithms for digital motion controllers.
exact arc lengths

S = 8

S = 22/3

uniform arc-length rendering

Δs = constant

Δt = constant
planar PH curves — complex variable model

\[ x''^2(t) + y''^2(t) = \sigma^2(t) \iff \begin{cases} 
  x'(t) = h(t) \left[ u^2(t) - v^2(t) \right] \\
  y'(t) = 2 h(t) u(t) v(t) \\
  \sigma(t) = h(t) \left[ u^2(t) + v^2(t) \right] 
\end{cases} \]

usually choose \( h(t) = 1 \) to define a primitive hodograph

\[ \gcd(u(t), v(t)) = 1 \iff \gcd(x'(t), y'(t)) = 1 \]

if \( \deg(u(t), v(t)) = m \), defines planar PH curve of odd degree \( n = 2m + 1 \)

planar PH condition automatically satisfied using complex polynomials

\[ w(t) = u(t) + i v(t) \text{ maps to } r'(t) = w^2(t) = u^2(t) - v^2(t) + i 2 u(t)v(t) \]

→ formulation of efficient complex arithmetic algorithms for the construction and analysis of planar PH curves
summary of planar PH curve properties

• planar PH cubics are scaled, rotated, reparameterized segments of a unique curve, Tschinhausen’s cubic (caustic for reflection by parabola)

• planar PH cubics characterized by intuitive geometrical constraints on Bézier control polygon, but not sufficiently flexible for free-form design

• planar PH quintics are excellent design tools — can inflect, and match first–order Hermite data by solving system of three quadratic equations

• select “good” interpolant from multiple solutions using shape measure — arc length, absolute rotation index, elastic bending energy

• generalizes to $C^2$ PH quintic splines smoothly interpolating sequence of points $p_0, \ldots, p_N$ — efficient complex arithmetic algorithms

• theory & algorithms for planar PH curves have attained a mature stage of development
curves with two-sided rational offsets


parabola $r(t) = (t, t^2)$ is simplest example

$$t = \frac{s^2 - 16}{16s} : \begin{cases} s \in (-\infty, 0) \quad \text{if} \quad s \in (0, +\infty) \end{cases} \rightarrow t \in (-\infty, +\infty)$$

defines a “doubly-traced” rational re-parameterization

two-sided offset $r_d(s) = r(s) \pm d\mathbf{n}(s) = \left( \frac{X_d(s)}{W_d(s)}, \frac{Y_d(s)}{W_d(s)} \right)$ is rational:

$$X_d(s) = 16(s^4 + 16d s^3 - 256d s - 256)s,$$

$$Y_d(s) = s^6 - 16s^4 - 2048d s^3 - 256s^2 + 4096,$$

$$W_d(s) = 256(s^2 + 16)s^2.$$
offset is algebraic curve of degree 6 with implicit equation

\[ f_d(x, y) = 16(x^2 + y^2)x^4 - 8(5x^2 + 4y^2)x^2y 
- (48d^2 - 1)x^4 - 32(d^2 - 1)x^2y^2 + 16y^4 
+ [2(4d^2 - 1)x^2 - 8(4d^2 + 1)y^2]y 
+ 4d^2(12d^2 - 5)x^2 + (4d^2 - 1)^2y^2 
+ 8d^2(4d^2 + 1)y - d^2(4d^2 + 1)^2 = 0 \]

\[ \text{genus} = 0 \Rightarrow \frac{1}{2}(n - 1)(n - 2) = 10 \text{ double points} \]

one affine node + six affine cusps

“non–ordinary” double point at infinity with double points in first & second neighborhoods
generalized complex form (Lü 1995)

\[ h(t) = \text{real polynomial}, \quad w(t) = u(t) + iv(t) = \text{complex polynomial} \]

polynomial PH curve  \quad r(t) = \int h(t) w^2(t) \, dt

two-sided rational offset curve  \quad r(t) = \int (kt + 1) h(t) w^2(t) \, dt

\[ h(t) = 1 \quad \text{and} \quad w(t) = 1 \quad \rightarrow \quad \text{parabola} \]

\[ h(t) \quad \text{linear and} \quad w(t) = 1 \quad \rightarrow \quad \text{cuspidal cubic} \]

\[ k = 0 \quad \text{and} \quad h(t) = 1 \quad \rightarrow \quad \text{regular PH curve} \]

describes all polynomial curves with rational offsets
characterization of spatial Pythagorean hodographs


\[ x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \iff \left\{ \begin{array}{l} x'(t) = u^2(t) - v^2(t) - w^2(t) \\ y'(t) = 2u(t)v(t) \\ z'(t) = 2u(t)w(t) \\ \sigma(t) = u^2(t) + v^2(t) + w^2(t) \end{array} \right. \]

only a sufficient condition — not invariant with respect to rotations in \( \mathbb{R}^3 \)


\[ x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \iff \left\{ \begin{array}{l} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2[u(t)q(t) + v(t)p(t)] \\ z'(t) = 2[v(t)q(t) - u(t)p(t)] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{array} \right. \]
spatial PH curves — quaternion & Hopf map models

quaternion model \((\mathbb{H} \rightarrow \mathbb{R}^3)\) \(\mathcal{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k}\)

\[
\rightarrow r'(t) = A(t) i A^*(t) = [u^2(t) + v^2(t) - p^2(t) - q^2(t)] \mathbf{i} \\
+ 2 [u(t)q(t) + v(t)p(t)] \mathbf{j} + 2 [v(t)q(t) - u(t)p(t)] \mathbf{k}
\]

Hopf map model \((\mathbb{C}^2 \rightarrow \mathbb{R}^3)\) \(\alpha(t) = u(t) + i v(t), \beta(t) = q(t) + i p(t)\)

\[
\rightarrow (x'(t), y'(t), z'(t)) = (|\alpha(t)|^2 - |\beta(t)|^2, 2 \text{Re}(\alpha(t)\overline{\beta}(t)), 2 \text{Im}(\alpha(t)\overline{\beta}(t)))
\]

equivalence — identify “i” with “\(i\)” and set \(\mathcal{A}(t) = \alpha(t) + k \beta(t)\)

both forms invariant under general spatial rotation by \(\theta\) about axis \(n\)
summary of spatial PH curve properties

- All spatial PH cubics are **helical curves** — satisfy \( \mathbf{a} \cdot \mathbf{t} = \cos \alpha \) (where \( \mathbf{a} \) = axis of helix, \( \alpha \) = pitch angle) and \( \kappa / \tau = \text{constant} \)

- Spatial PH cubics characterized by **intuitive geometrical constraints** on Bézier control polygons

- Spatial PH quintics well-suited to free-from design applications — two-parameter family of interpolants to first-order Hermite data

- Optimal choice of free parameters is a rather subtle problem — one parameter controls curve shape, the other total arc length

- Generalization to spatial \( C^2 \) PH quintic splines is problematic — too many free parameters!

- Many interesting **subspecies** — helical polynomial curves, “double” PH curves, rational rotation-minimizing frame curves, etc.
special classes of spatial PH curves

helical polynomial space curves satisfy $\mathbf{a} \cdot \mathbf{t} = \cos \alpha$ ($\mathbf{a} =$ axis, $\alpha =$ pitch angle) and $\kappa/\tau = \tan \alpha$

all helical polynomial curves are PH curves (implied by $\mathbf{a} \cdot \mathbf{t} = \cos \alpha$)

all spatial PH cubics are helical, but not all PH curves of degree $\geq 5$

“double” Pythagorean–hodograph (DPH) curves

$\mathbf{r}'(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ both have Pythagorean structures — have rational Frenet frames $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and curvatures $\kappa$

all helical polynomial curves are DPH — not just PH — curves

all DPH quintics are helical, but not all DPH curves of degree $\geq 7$
rational rotation–minimizing frame (RRMF) curves

rational frames \((t, u, v)\) with **angular velocity** satisfying \(\omega \cdot t \equiv 0\)

RRMF curves are of minimum degree 5 (proper subset of PH quintics) identifiable by quadratic (vector) constraint on quaternion coefficients

useful in **spatial motion planning and rigid–body orientation control**
rational Pythagorean-hodograph curves


- employs dual representation — plane curve regarded as envelope of tangent lines, rather than point locus
- offsets to a rational PH curve are of the same degree as that curve
- admist natural generalization to rational surfaces with rational offsets
- parametric speed, but not arc length, is a rational function of curve parameter (rational functions do not, in general, have rational integrals)
- geometrical optics interpretation — rational PH curves are caustics for reflection of parallel light rays by rational plane curves
- Laguerre geometry model — oriented contact of lines & circles
rational unit normal to planar curve \( \mathbf{r}(t) = \left( \frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)} \right) \)

\[
\begin{align*}
n_x(t) &= \frac{2a(t)b(t)}{a^2(t) + b^2(t)}, \\
n_y(t) &= \frac{a^2(t) - b^2(t)}{a^2(t) + b^2(t)}
\end{align*}
\]

equation of tangent line at point \((x, y)\) on rational curve

\[
\ell(x, y, t) = n_x(t)x + n_y(t)y - \frac{f(t)}{g(t)} = 0
\]

envelope of tangent lines — solve \( \ell(x, y, t) = \frac{\partial \ell}{\partial t}(x, y, t) = 0 \) for \((x, y)\) and set \( x = X(t)/W(t), y = Y(t)/W(t) \) to obtain

\[
\begin{align*}
W &= (a^2 + b^2)(a'b - ab')g^2, \\
X &= 2ab(a'b - ab')f g - \frac{1}{2}(a^4 - b^4)(f'g - fg'), \\
Y &= (a^2 - b^2)(a'b - ab')f g + ab(a^2 + b^2)(f'g - fg').
\end{align*}
\]
dual representation in line coordinates $K(t), L(t), M(t)$ is simpler

define set of all tangent lines to rational PH curve by

$$K(t) W + L(t) X + M(t) Y = 0$$

line coordinates are given in terms $a(t), b(t)$ and $f(t), g(t)$ by

$$K : L : M = -(a^2 + b^2)f : 2abg : (a^2 - b^2)g$$

for rational PH curves, class (= degree of line representation) is less than order (= degree of point representation)

dual Bézier representation — control points replaced by control lines

rational offsets constructed by parallel displacement of control lines
medial axis transform of planar domain

medial axis = locus of centers of maximal inscribed disks, touching domain boundary in at least two points; medial axis transform (MAT) = medial axis + superposed function specifying radii of maximal disks
Minkowski Pythagorean-hodograph (MPH) curves


\[(x(t), y(t), r(t)) = \text{medial axis transform (MAT) of planar domain } \mathcal{D}\]

characterizes domain \( \mathcal{D} \) as union of one-parameter family of circular disks \( \mathcal{C}(t) \) with centers \((x(t), y(t))\) and radii \(r(t)\)

recovery of domain boundary \( \partial \mathcal{D} \) as envelope of one–parameter family of circular disks specified by the MAT \((x(t), y(t), r(t))\)

\[x_e(t) = x(t) - r(t) \left( \frac{r'(t)x'(t) \pm \sqrt{x'^2(t) + y'^2(t) - r'^2(t)y'(t)}}{x'^2(t) + y'^2(t)} \right),\]

\[y_e(t) = y(t) - r(t) \left( \frac{r'(t)y'(t) \mp \sqrt{x'^2(t) + y'^2(t) - r'^2(t)x'(t)}}{x'^2(t) + y'^2(t)} \right).\]
for parameterization to be rational, MAT hodograph must satisfy

\[ x'{}^2(t) + y'{}^2(t) - r'{}^2(t) = \sigma^2(t) \]

— this is a Pythagorean condition in the Minkowski space \( \mathbb{R}^{(2,1)} \)

metric of Minkowski space \( \mathbb{R}^{(2,1)} \) has signature \(+ + -\) rather than usual signature \(+ + +\) for metric of Euclidean space \( \mathbb{R}^3 \)

Moon (1999): sufficient–and–necessary characterization of Minkowski Pythagorean hodographs in terms of four polynomials \( u(t), v(t), p(t), q(t) \)

\[
\begin{align*}
x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t) , \\
y'(t) &= 2 \left[ u(t)p(t) - v(t)q(t) \right] , \\
r'(t) &= 2 \left[ u(t)v(t) - p(t)q(t) \right] , \\
\sigma(t) &= u^2(t) - v^2(t) + p^2(t) - q^2(t) .
\end{align*}
\]
interpretation of Minkowski metric

originates in special relativity: distance \(d\) between events with space–time coordinates \((x_1, y_1, t_1)\) and \((x_2, y_2, t_2)\) is defined by

\[
\begin{align*}
    d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 - c^2(t_2 - t_1)^2 \\
\end{align*}
\]

space-like if \(d\) real, light-like if \(d = 0\), time-like if \(d\) imaginary

distance between circles \((x_1, y_1, r_1)\) and \((x_2, y_2, r_2)\) as points in \(\mathbb{R}^{(2,1)}\)

\[
\begin{align*}
    d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 - (r_2 - r_1)^2 \\
\end{align*}
\]
rational boundary reconstructed from MPH curve
• **advantages** of PH curves: rational offset curves, exact arc–length computation, real-time CNC interpolators, exact rotation–minimizing frames, bending energies, etc.

• **applications** of PH curves in digital motion control, path planning, robotics, animation, computer graphics, etc.

• **investigation** of PH curves involves a wealth of concepts from **algebra** and **geometry** with a long and fascinating history

• many **open problems** remain: optimal choice of degrees of freedom, \( C^2 \) spline formulations, control polygons for design of PH splines, deeper geometrical insight into quaternion representation, etc.
Durham, England 1962 — spot incipient PH affliction?