ON SCHUBERT DECOMPOSITIONS OF QUIVER GRASSMANNIANS

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ABSTRACT. In this paper, we introduce Schubert decompositions for quiver Grassmannians and investigate example classes of quiver Grassmannians with a Schubert decomposition into affine spaces. The main theorem puts the cells of a Schubert decomposition into relation to the cells of a certain simpler quiver Grassmannian. This allows us to extend known examples of Schubert decompositions into affine spaces to a larger class of quiver Grassmannians. This includes the class of monomial representations of forests as well as string modules and ordered tree modules. Finally, we draw conclusions on the Euler characteristics and the cohomology of quiver Grassmannians.

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INTRODUCTION

In 1994, Lusztig published his seminal book [12] on the existence of canonical bases for Lie algebras. This was the starting point of vivid research that aimed for a better understanding of canonical bases. Despite being hard to compute, much insight was gained into the general structure of canonical bases during the last years.

A major contribution to the subject was the introduction of cluster algebras by Fomin and Zelevinsky in 2002, see [8] and their subsequent publications. An important feature of the theory of cluster algebras is the mutation operation that associates to a quiver $Q$, by recursion, a set of so-called cluster variables, which generates the associated cluster algebra. In 2006, Caldero and Chapoton found an explicit formula that expresses the cluster variables in terms of the Euler characteristics of the quiver Grassmannians $\text{Gr}_e(M)$ for the rigid representations $M$ of the quiver $Q$, see [1].

The Caldero-Chapoton formula drew attention to quiver Grassmannians and, in particular, to their Euler characteristics. In [2], Caldero and Reineke established many basic properties
of quiver Grassmannians for acyclic quivers, e.g. its smoothness in the case of a rigid representation $M$. They determine the Euler characteristic of $\text{Gr}_e(M)$ if $M$ is an indecomposable representation of the Kronecker quiver, and they remark that Schubert decompositions of quiver Grassmannians might help to compute their Euler characteristics.

Many other publications followed. Cerulli and Esposito inspect in [4] quiver Grassmannians of Kronecker type in further detail and apply this to the canonical basis of cluster algebras of types $A_1^{(1)}$ and $A_2^{(1)}$. In particular, they describe a decomposition of $\text{Gr}_e(M)$ into affine spaces in case $M$ is a regular representation (cf. Example 2.6). Szántó establishes in [17] a counting polynomial of the $F_q$-rational points of quiver Grassmannians of Kronecker type, which hints that there exists a Schubert decomposition into affine spaces for other types of representations of the Kronecker quiver as well.

Rupel conjectures in [15] the positivity of acyclic seeds for cluster algebras. This conjecture implies that quiver Grassmannians of rigid representations have a counting polynomial in the acyclic case, which in turn implies the positivity of their Euler characteristics if the quiver Grassmannian is not empty. This conjecture was partially proven by Qin in [13], followed by a complete proof by Rupel in [16].

In [5], [6] and [7], Cerulli, Feigin and Reineke realize degenerate flag varieties as quiver Grassmannians of Dynkin type. A particular result of interest for the present paper is the existence of a Schubert decomposition into affine spaces (cf. Examples 5.8 and 6.13).

The two papers that essentially inspired the results of this paper are [3] and [10]. In [3], Cerulli gives a formula for the Euler characteristics of quiver Grassmannians of orientable string modules. In [10], Haupt extends the results of [3] to the class of tree modules and also provides a formula for the Euler characteristic of quiver Grassmannians of band modules. The method of both papers is to construct a weighted diagonal action of the one-dimensional torus $T = \mathbb{G}_m$ on the representation $M$ in question. This divides the quiver Grassmannian $X = \text{Gr}_e(M)$ into the locally closed subscheme $X^T$ of fixed points and its complement $Z = X - X^T$, which yields

$$
\chi(\text{Gr}_e(M)) = \chi(X^T) + \chi(Z) = \chi(X^T) + \chi(T) \cdot \chi(Z//T) = \chi(X^T).
$$

After applying this trick several times, the fixed point set $X^T$ is finite and can be identified with the number of subrepresentations of a certain quiver representation $\hat{M}$ that is simpler than $M$. This means, in particular, that these Euler characteristics are positive if the quiver Grassmannian is not empty.

During the attempt to understand the geometry of the quiver Grassmannians considered in [3] and [10], it turned out that in many cases, quiver Grassmannians have a decomposition into affine spaces. A systematic study of these decompositions led to the results of this paper. Though the methods of this paper are completely different, we will obtain formulas for the Euler characteristics of quiver Grassmannians in a class that has a large intersection with the class of cases treated in [3] and [10]. The existence of Schubert decompositions into affine spaces allows us further to extract information about the cohomology. For instance, if the representation $M$ is rigid, then the Schubert cells determine an additive basis for the cohomology ring.
Results. The quiver Grassmannian $\text{Gr}_e(M)$ of subrepresentations $V$ of $M$ with dimension vector $e$ is defined as a closed subscheme of the usual Grassmannian $\text{Gr}(e, m)$ where $e$ is the dimension of $V$ and $m$ is the dimension of $M$ over the ground field. The intersection of $\text{Gr}_e(M)$ with a Schubert decomposition of $\text{Gr}(e, m)$ defines a Schubert decomposition of $\text{Gr}_e(M)$. In general, this is not a decomposition into affine spaces, and the isomorphism type of the Schubert cells is not independent of the choices that define the Schubert decomposition for $\text{Gr}(e, m)$.

The results of this paper concentrate on establishing cases of quiver Grassmannians that have a Schubert decomposition into affine spaces. The main result Theorem 4.2 roughly says the following: let $F : T \to Q$ be a morphism of quivers satisfying certain conditions and $S \subset T$ a subquiver such that the quotient $T/S$ is a tree. Let $M$ be a representation of $T$. Then the Schubert cell $C_{\beta, \gamma}^{F, M}$ of the push-forward $F_* M$ of $M$ equals the product $A^n \times C_{\beta, \gamma}^{F, M_S}$ of an affine space with the corresponding Schubert cell for the push-forward of the restriction $M_S$ of $M$ to $S$.

While this main result is too technical to be explained in brevity, we will illustrate a number of its consequences. We will explain all the terminology that is used below in the main text of this paper, close-by the mentioned result.

(i) Let $T$ be a tree and $M$ a monomial representation of $T$. Then $\text{Gr}_e(M)$ has a Schubert decomposition into affine spaces (see Thm. 5.4). If all linear maps defining $\text{Gr}_e(M)$ are isomorphisms, then the quiver Grassmannian decomposes into a series of fibre bundles whose fibres are usual Grassmannians (see Thm. 3.3).

(ii) Let $Q$ be an arbitrary quiver and $M$ a string module of $Q$. Then $\text{Gr}_e(M)$ has a Schubert decomposition into affine spaces (see Cor. 5.7).

(iii) Let $Q$ be an arbitrary quiver and $M$ an ordered tree module. Then $\text{Gr}_e(M)$ has a Schubert decomposition into affine spaces (see Cor. 5.9).

(iv) We re-obtain the Schubert decompositions of Cerulli and Esposito in [1] (see Example 2.6) and Cerulli, Feigin and Reineke in [3] (see Examples 5.8 and 6.13).

(v) If $\text{Gr}_e(M, C) = \bigcup_{i \in I} X_i(C)$ is a decomposition into complex affine spaces $X_i(C)$, then the Euler characteristic of $\text{Gr}_e(M)$ is $\chi(\text{Gr}_e(M)) = \# I$. By (ii) and (iii), this reproduces the formulas in [3] and [10] (for strictly ordered tree modules) in terms of the combinatorics of the Schubert cells (see Section 6.1). We extend these results to a larger class of quiver Grassmannians.

(vi) If $\text{Gr}_e(M, \mathbb{C}) = \bigcup_{i \in I} X_i(\mathbb{C})$ is a regular decomposition into complex affine spaces such that all irreducible components are smooth, then the cohomology of $\text{Gr}_e(M)$ is concentrated in even degrees and the classes of the closures $\overline{X_i}$ of the Schubert cells form an additive basis for the cohomology ring (see Thm. 6.6).

Next to these results, the reader will find numerous side results, remarks and examples, which shall illustrate certain effects of the theory of Schubert decompositions of quiver Grassmannians.

Open questions. There are some open questions that might be of interest for further investigations.
(i) Is the technical condition that the winding $F : T \to Q$ is ordered necessary in the main result Theorem 4.2 (see Remark 4.3)? If this condition could be removed, we would re-obtain the complete result on tree modules in [10].

(ii) Is there a similar theorem as Theorem 4.2 for band modules (see Remark 4.4)? A affirmative answer would explain the result on band modules in [10] in terms of Schubert decompositions.

(iii) Can the assumptions of the regularity of the Schubert decomposition and of the smoothness of the irreducible components of $\text{Gr}_e(M)$ be weakened in Theorem 6.6? In case the regularity plays a crucial role, it is interesting to establish regular Schubert decompositions for larger classes of quiver Grassmannians, e.g. for monomial representations of trees or of tree extensions of subquivers that admit a regular Schubert decomposition, cf. Conjectures 6.14 and 6.15.

Content overview. The paper is structured as follows. In Section 1 we recall the definition of quiver Grassmannians and cite some basic facts.

In Section 2, we define Schubert cells for quiver Grassmannians. In 2.1, we explain the connection between the definition of this paper and the one given in [2] for acyclic quiver. In 2.2 we identify the $K$-rational points of a Schubert cell with certain matrices of generating vectors with prescribed pivot element. In 2.3 we describe some examples of Schubert decompositions.

In Section 3, we introduce the notion of a tree extension $T$ of a quiver $S$. In 3.1, we state the main results for tree extension that connect a quiver Grassmannian $\text{Gr}_e(M)$ of a representation $M$ of $T$ to the quiver Grassmannian $\text{Gr}_{eS}(M_S)$ of the restriction $M_S$ of $M$ to $S$. In particular, there is a smooth projective morphism $\text{Gr}_e(M) \to \text{Gr}_{eS}(M_S)$ and the Schubert cells of $\text{Gr}_e(M)$ are a product of a Schubert cell of $\text{Gr}_{eS}(M_S)$ with an affine space.

In Section 4, we introduce push-forwards of quiver representations along morphisms of quivers. In 4.1 we describe the equations that are satisfied by the $K$-rational points of Schubert cells when we push-forward a representation. In 4.2 we introduce comparison morphisms between a Schubert cell and the corresponding Schubert cell for the push-forward. In 4.3 we state and prove the main theorem of this paper (see Results above).

In Section 5, we list some consequences of the main theorem. First of all, we give a general condition for a quiver Grassmannian to have a Schubert decomposition into affine spaces. In 5.1 we explain a result for the quiver Grassmannian of a direct sum of representations. In 5.2 we show that monomial representations of forests yield quiver Grassmannians with a Schubert decomposition into affine spaces (see Results (i)). In 5.3 we prove a result for the push-forward of a monomial representation of a forest and establish Schubert decompositions into affine spaces for string modules (see Results (ii)) and ordered tree modules (see Results (iii)).

In Section 6, we draw conclusions on the cohomology of a quiver Grassmannian that has a Schubert decompositions into affine spaces. In 6.1 we describe formulas for the Euler characteristics for a certain classes of quiver Grassmannians. In 6.2 we generalize the known results for the cohomology of smooth projective schemes with a decomposition into affine spaces to projective schemes with several smooth irreducible components. In 6.3 we describe certain
example classes of quiver Grassmannians with regular Schubert decompositions and formulate two conjectures on the existence of regular Schubert decompositions.

**Remark.** As pointed out to me by Giovanni Cerulli Irelli and Grégoire Dupont, the formulas for the Euler characteristics in [3] and [10] count subrepresentations that look like “$\mathbb{F}_1$-rational points” (cf. Szczesny’s paper [13] on quiver representation over $\mathbb{F}_1$). That the number of $\mathbb{F}_1$-rational points equals the Euler characteristic is one of the main concepts in $\mathbb{F}_1$-geometry. Therefore, there is the hope that a better understanding of the geometry of quiver Grassmannians over $\mathbb{F}_1$ will help to compute their Euler characteristics. The connection of quiver Grassmannians with Schubert decompositions to $\mathbb{F}_1$-geometry will be the topic of a subsequent paper. This is the reason why we work over an arbitrary base ring $k$ in this paper.

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1. **Background**

A *quiver* is a finite directed graph with possibly multiple edges and loops. We formalize a quiver as a quadruple $Q = (Q_0, Q_1, s, t)$ where $Q_0$ is a finite set of vertices, $Q_1$ is a finite set of arrows, $s : Q_1 \to Q_0$ associates to each arrow its source or tail and $t : Q_1 \to Q_0$ associates to each arrow its target or head.

During the major part of this paper, we fix a ring $k$. We will only specify to the case $k = \mathbb{C}$ in some parts of Section 6. But for many applications, it is enough to keep the case $k = \mathbb{C}$ in mind.

The *path algebra* of $Q$ over $k$ is the $k$-algebra $k[Q]$ that is freely generated as a $k$-module by all oriented paths in $Q$. In particular, there is a path $\epsilon_p = [p|p]$ of length 0 at every vertex $p$ of $Q$. The multiplication is defined by composition of paths if possible, and 0 otherwise. The elements $\epsilon_p$ are idempotent, and $1 = \sum_{p \in Q_0} \epsilon_p$ is the identity of $k[Q]$. As a $k$-algebra, $k[Q]$ is generated by the idempotents $\epsilon_p$ and the paths of length 1, i.e. by the arrows $\alpha$ of $Q$.

A *representation* of $Q$ over $k$ or, for short, a *$Q$-module* is a free $k[Q]$-module $M$ of finite rank. Equivalently, we can consider $M$ as a collection of free $k$-modules $M_p = \epsilon_p M$ for $p \in Q_0$ together with the collection of $k$-linear maps $M_\alpha : M_p \to M_q$, defined by $f_\alpha(\epsilon_p m) = \alpha m$ for every arrow $\alpha$ from $p$ to $q$. Then $M = \bigoplus_{p \in Q_0} M_p$, and the $k[Q]$-algebra structure is determined by the $k$-linear maps $M_\alpha$. The *dimension vector* $\text{dim} M$ of $M$ is the tuple $m = (m_p)_{p \in Q_0}$ where $m_p$ is the rank of $M_p$ over $k$.

In the following, we will relax the language a bit. We assume that the base ring is fixed and do not mention $k$ if the context is clear. We will further identify $M$ with both $\bigoplus M_p$ and $(\{M_p\}_{p \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$, and switch viewpoints where this is helpful.

A submodule $V$ of $M$ can be identified with a collection of sub-$k$-modules $V_p = \epsilon_p V$ of $M_p$ for every $p \in Q_0$ such that $M_\alpha(V_p) \subseteq V_q$ for every arrow $\alpha : p \to q$ in $Q_1$. Let $\underline{e} = (e_p)_{p \in Q_0}$ be a dimension vector smaller or equal to $m$, i.e. $e_p \leq m_p$ for all $p \in Q_0$. Define $\text{Gr}_\underline{e}(m)$ as the product $\prod_{p \in Q_0} \text{Gr}(e_p, m_p)$ and $R_m(Q)$ as the product $\prod_{\alpha \in Q_1} \text{Hom}(k^{m_\alpha}, k^{m_\alpha})$, which we consider as
a scheme by identifying the homomorphism sets with affine spaces over $k$ of adequate dimensions. Then $\text{Gr}_e(m) \times R_m(Q)$ is a reduced scheme over $k$. The universal Grassmannian $\text{Gr}_e^Q(m)$ is the closed reduced subscheme of $\text{Gr}_e(m) \times R_m(Q)$ whose $K$-rational points are described as the set
\[
\left\{ \left( (V_p \subset k^{m_p})_{p \in Q_0}, (f_{\alpha})_{\alpha \in Q_1} \right) \mid f_{\alpha}^{-1}(V_p) \subset V_q \text{ for all } \alpha : p \to q \text{ in } Q_1 \right\}
\]
for any field extension $K$ of $k$. For a $k$-rational point $M$ of $R_m(Q)$—which is nothing else than a $Q$-module over $k$, together with a fixed basis,—, the quiver Grassmannian $\text{Gr}_e(M)$ is defined as the fibre of $\text{pr}_2 : \text{Gr}_e^Q(m) \to R_m(Q)$ over $M$. See Sections 2.2 and 2.3 in [5] for more details on the definition of $\text{Gr}_e(M)$.

Note that the isomorphism type of $\text{Gr}_e(M)$ does not depend on the choice of basis for $M$, which allows us to define $\text{Gr}_e(M)$ for any $Q$-module $M$. Note further that $\text{Gr}_e(M)$ is in general not reduced. For a field extension $K$ of $k$, the set of $K$-rational points of $\text{Gr}_e(M)$ coincides with the set
\[
\left\{ V \subset M_K \mid M_{\alpha}(V_p) \subset V_q \text{ for all } \alpha : p \to q \text{ in } Q_1 \right\}
\]
where $M_K = M \otimes_k K$.

The quiver Grassmannian $\text{Gr}_e(M)$ is a closed subscheme of the product $\prod \text{Gr}(e_v, m_v)$ of the usual Grassmannians over all vertices $v$ of $Q$. We cite two general facts about quiver Grassmannians.

**Theorem 1.1** (Reineke, [14]). Every projective $k$-scheme is isomorphic to a quiver Grassmannian.

A $Q$-module $M$ is rigid or exceptional if it has no self-extensions, i.e. $\text{Ext}^1(M, M) = 0$. For instance, indecomposable $Q$-modules are rigid.

**Theorem 1.2** (Caldero and Reineke, [2]). If $M$ is a rigid $Q$-module, then $\text{Gr}_e(M)_k$ is a smooth $k$-scheme.

2. Schubert cells

Let $M$ be a free $k$-module and $e \leq \text{rk} M$ a non-negative integer. Then the choice of a (linearly) ordered basis $B$ of $M$ over $k$ defines a Schubert cell decomposition of the usual Grassmannian $\text{Gr}_e(M)$ into affine spaces. In case of a $Q$-module $M$ with dimension vector $\underline{m}$ and $e \leq \underline{m}$, the quiver Grassmannian $\text{Gr}_e(M)$ is a subscheme of the usual Grassmannian $\text{Gr}_{|\underline{e}|}(M)$ via the closed embedding
\[
\iota : \text{Gr}_e(M) \longrightarrow \prod_{p \in Q_0} \text{Gr}(e_p, m_p) \longrightarrow \text{Gr}_{|\underline{e}|}(M)
\]
where $|\underline{e}| = \sum_{p \in Q_0} e_p$. This allows to define the Schubert decomposition of $\text{Gr}_e(M)$ w.r.t. $B$ as the pull-back of the Schubert decomposition of $\text{Gr}_{|\underline{e}|}(M)$.

We will explain this definition in more detail, assuming the following general hypothesis that will be valid throughout the paper unless explicitly mentioned otherwise. Let $Q$ be a quiver and $M$ a $Q$-module with basis $B$ (as a $k$-module). Then we assume the following property.
(H) The intersection $\mathcal{B}_p = \mathcal{B} \cap M_p$ is a basis of $M_p$ for every $p \in Q_0$. In other words, $\mathcal{B} = \bigsqcup_{p \in Q_0} \mathcal{B}_p$.

For a subset $\beta$ of $\mathcal{B}$, we define $\beta_p = \beta \cap \mathcal{B}_p$. The type of $\beta$ is the dimension vector $e = (e_p)$ with $e_p = \#\beta_p$. If $\mathcal{B}$ is an ordered basis of $M$, $p \in Q_0$ and $\beta$ and $\gamma$ are subsets of $\mathcal{B}$ of the same type $e$, then we write $\beta_p \leq \gamma_p$ if we have $b_{p,l} \leq c_{p,l}$ for all $l \in \{1, \ldots, e_p\}$ where we write $\beta_p = \{b_{p,1}, \ldots, b_{p,e_p}\}$ and $\gamma = \{c_{p,1}, \ldots, c_{p,e_p}\}$, ordered by size. We write $\beta \leq \gamma$ if $\beta_p \leq \gamma_p$ for all $p \in Q_0$.

Let $\mathcal{B}$ be a basis of $M$. The Plücker coordinates of the product Grassmannian

$$\text{Gr}_\mathcal{E}(m) = \prod_{p \in Q_0} \text{Gr}(e_p, m_p) \subset \prod_{p \in Q_0} \mathbb{P}^{(m_p)}$$

are the $e_p \times e_p$-minors

$$\Delta_{\beta_p} : v^p \mapsto \det (v^p_{i,j})_{i,j=1,\ldots,e_p}$$

of $m_p \times e_p$-matrices $v^p = (v^p_{i,j})_{i \in \beta_p, j=1,\ldots,e_p}$ where $p$ varies through $Q_0$ and $\beta$ through the subsets of $\mathcal{B}$ of type $e$. We denote by $U_\beta$ the canonical open subset of $\text{Gr}_\mathcal{E}(m)$ with $\Delta_{\beta_p} = 1$ for all $p \in Q_0$.

Let $\mathcal{B}$ be an ordered basis of $M$ and $\beta \subset \mathcal{B}$ a subset. Then define the Schubert cell $C_{\beta}(m)$ of $\text{Gr}_\mathcal{E}(m)$ as the intersection of $U_\beta$ with the vanishing set of all $e_p \times e_p$-minors $\Delta_{\gamma_p}$ with $\gamma_p \preceq \beta_p$, seen as a locally closed and reduced subscheme of $\text{Gr}_\mathcal{E}(m)$. We define the Schubert cell $C^M_{\beta}$ of $\text{Gr}_\mathcal{E}(M)$ as the pull-back of $C_{\beta}(m)$ along the closed embedding $\text{Gr}_\mathcal{E}(M) \hookrightarrow \text{Gr}_\mathcal{E}(m)$. Then $C^M_{\beta}$ is a locally closed subscheme of $\text{Gr}_\mathcal{E}(M)$. Note that $C^M_{\beta}$ is in general not reduced (cf. Example 2.5). Sometimes we refer to the reduced subscheme $C^M_{\beta, \text{red}} = (C^M_{\beta})^\text{red}$ as a reduced Schubert cell.

By the Schubert decomposition of usual Grassmannians, $\text{Gr}_\mathcal{E}(m)$ decomposes into the Schubert cells $C_{\beta}(m)$ where $\beta$ ranges through all subsets of $\mathcal{B}$ of type $e$. The pull-back of this decomposition yields the decomposition

$$\varphi : \prod_{\beta \subset \mathcal{B} \text{ of type } e} C^M_{\beta} \longrightarrow \text{Gr}_\mathcal{E}(M),$$

i.e. a morphism of $k$-schemes such that the restriction of $\varphi$ to one cell $C^M_{\beta}$ is a locally closed embedding and such that $\varphi$ induces a bijection between $K$-rational points for every field extension $K$ of $k$. We call this decomposition the Schubert decomposition of the quiver Grassmannian $\text{Gr}_\mathcal{E}(M)$ (w.r.t. $\mathcal{B}$). In agreement with [11], we also write

$$\text{Gr}_\mathcal{E}(M) = \bigsqcup_{\beta \subset \mathcal{B} \text{ of type } e} C^M_{\beta},$$

for the Schubert decomposition. We use the modified symbol “$\bigsqcup^\circ$” in order to avoid a confusion with the disjoint union of $k$-schemes.
Note that
\[
\text{Gr}_\varepsilon(M) = \bigsqcup_{\beta \subset \mathcal{B}} \text{C}^M_{\beta, \text{red}}
\]
is also a decomposition of \(\text{Gr}_\varepsilon(M)\), which we call the reduced Schubert decomposition of \(\text{Gr}_\varepsilon(M)\).

**Remark 2.1.** The Schubert decomposition of \(\text{Gr}_\varepsilon(M)\) depends only on the ordering of the subsets \(\mathcal{B}_p\) of \(\mathcal{B}\) and not on the ordering of elements \(b \in \mathcal{B}_p\) and \(b' \in \mathcal{B}_{p'}\) for different \(p \neq p'\). However, we choose to endow \(\mathcal{B}\) with a linear order (and thus superfluous information at this point) since this is needed for the Schubert decomposition of a push-forward of \(M\), cf. Section 4.

**Remark 2.2.** Note that the Schubert cells \(C_{\beta}(m)\) of \(\text{Gr}_\varepsilon(m)\) are affine spaces as products of Schubert cells of usual Grassmannians, but that the Schubert cells \(C^M_{\beta}\) are in general not affine spaces. Since every projective \(k\)-scheme can be realized as a quiver Grassmannian, it is clear that this cannot be the case. Even if there exists a decomposition of \(\text{Gr}_\varepsilon(M)\) into affine spaces for some choice of an ordered basis, a different choice of ordered basis might yield Schubert cells of a different shape (see Example 2.7). In view towards Theorem 1.1, I expect that every affine \(k\)-scheme of finite type can appear as a Schubert cell of a quiver Grassmannian for appropriate \(Q, M, \varepsilon\) and \(\beta \subset \mathcal{B}\).

### 2.1. Schubert decompositions for acyclic quiver.

In case the quiver \(Q\) is acyclic, i.e. without oriented cycles, we find the following alternative description of the Schubert decomposition of \(\text{Gr}_\varepsilon(M)\), cf. Section 6 in [2]. Let \(H = k[Q]\) be the path algebra of \(Q\) and \(H^*\) be the unit subgroup. We can embed \(H^*\) as a subgroup of \(\text{GL}(M, k)\). Since \(Q\) is acyclic, \(H^*\) is contained in a Borel subgroup \(B\) of \(\text{GL}(M, k)\).

The choice of a Borel subgroup \(B\) of \(\text{GL}(M, k)\) is equivalent to the choice of an ordered basis \(\mathcal{B}\) for \(M\) with the property that \(B\) is the subgroup of upper triangular matrices in this basis. Note that in general, the basis \(\mathcal{B}\) does not satisfy Hypothesis (H).

This choice defines a Schubert decomposition
\[
\text{Gr}_{|\varepsilon|}(M) = \bigsqcup_j X_j
\]
of the usual Grassmannian \(\text{Gr}_{|\varepsilon|}(M)\) of submodules of rank \(|\varepsilon|\) of \(M\). As explained in [2], the subscheme \(\text{Gr}_{|\varepsilon|}(M)^{H^*}\) of fixed points equals the disjoint union of all quiver Grassmannians \(\text{Gr}_{\varepsilon'}(M)\) with \(|\varepsilon'| = |\varepsilon|\). Therefore, we yield the decomposition
\[
\bigsqcup_{|\varepsilon'| = |\varepsilon|} \text{Gr}_{\varepsilon}(M) = \bigsqcup_j X_j^{H^*},
\]
which restricts to a decomposition \(\text{Gr}_\varepsilon(M) = \bigsqcup_j (X_j^{H^*} \cap \text{Gr}_\varepsilon(M))\) into reduced subschemes.
This decomposition coincides with the (reduced) Schubert decomposition that we have defined in the previous section. In particular, if $\mathcal{B}$ satisfies Hypothesis (H), the cells $X^H \cap \text{Gr}_e(M)$ coincide with the reduced cells $C^M_{\beta, \text{red}}$. This means that the decomposition $\text{Gr}_e(M) = \coprod (X^H \cap \text{Gr}_e(M))$ is the same as $\text{Gr}_e(M) = \coprod C^M_{\beta}$.

2.2. **K-rational points of Schubert cells.** Let $K$ be a ring extension of $k$. Using the canonical covering $\{U_\beta\}$ of $\text{Gr}_e(m)$, we can describe the $K$-rational points of a Schubert cell $C^M_{\beta}$ as follows.

A $K$-rational point of $C^M_{\beta}$ defines a subrepresentation $V$ of $M_K = M \otimes_k K$. This subrepresentation satisfies that for every $p \in Q_0$, the submodule $V_p$ of $M_p$ is generated by a set of vectors $v^p = (v_b)_{b \in \beta_p}$ where each $v_b$ is of the form

$$v_b = 1 \cdot b + \sum_{b' \in \mathcal{B}_p \setminus \beta_p} v_{b', b} \cdot b'$$

for some $v_{b', b} \in K$. Note that the coefficients $v_{b', b}$ are uniquely determined by $V$. This means that a $K$-rational point $V$ corresponds to a $|\mathcal{B}| \times |\beta|$-matrix $\nu$ with coefficients $v_{b', b} \in K$, which satisfy $v_{b, b} = 1$ and $v_{b', b} = 0$ whenever $b' > b$ or $b' \in \beta$, but $b' \neq b$. In other words, $\nu$ is in row echelon form and all coefficients of a row containing a pivot 1 are zero, except for the pivot itself.

Conversely, a choice of $v_{b', b} \in K$ yields a $K$-rational point $V$ of $C^M_{\beta}(m)$, which, however, does not have to lie in $C^M_{\beta}(K)$. For certain cases of $M$, we will work out the conditions on the coefficients $v_{b', b}$ to come from a $K$-rational point $V$ of $C^M_{\beta}$ (see Section 4.1).

2.3. **Examples.**

**Example 2.3** (One point quiver and usual Grassmannians). Let $Q$ be a quiver that consists of a single point and $M = k^m$ the $Q$-module with basis $\mathcal{B} = (b_1, \ldots, b_m)$. We consider the usual Schubert decomposition

$$\text{Gr}(e, m) = \coprod_{1 \leq i_1 < \cdots < i_e \leq m} C_{i_1, \ldots, i_e}$$

where $C_{i_1, \ldots, i_e}$ is the reduced subscheme of $\text{Gr}(e, m)$ with $K$-rational points

$$\{ V \subset M_K \mid \text{for all } l = 1, \ldots, n \text{ and } k \text{ such that } i_k \leq l < i_{k+1}, \dim(V_l \cap N) = k \}$$

for any field extension $K$ of $k$ where $V_l = \text{span}\{b_1, \ldots, b_l\}$. Then $C_{i_1, \ldots, i_e}$ can be identified with $C^M_{\beta}$ for $\beta = \{b_{i_1}, \ldots, b_{i_e}\}$ if $\mathcal{B}$ is ordered by $b_1 < \cdots < b_e$. This shows that we recover the Schubert decomposition of usual Grassmannians as a special case.

**Example 2.4** (Flag varieties). The same is true for flag variety if we realize them as follows. Let $e = (e_1, \ldots, e_r)$ be the type of the flag variety $X = X(e_1, \ldots, e_r)$ of subspaces of $k^m$. Let $Q$ be the quiver

$$1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{r-1}}{\longrightarrow} r$$
and $M$ the $Q$-module $k^m \xrightarrow{id} \cdots \xrightarrow{id} k^m$. Then $X$ is isomorphic to $\text{Gr}_e(M)$. If we order the standard basis $B = \{ b_{k,p} | k = 1, \ldots, m; p = 1, \ldots, r \}$ of $M$ lexicographically, i.e. $b_{k,p} < b_{l,q}$ if $p < q$ or if $p = q$ and $k < l$, then the decomposition

$$\text{Gr}_e(M) = \coprod_{\beta \subset B \text{ of type } e} C^M_{\beta}$$

coinsides with the usual decomposition of $X$ into Schubert cells.

**Example 2.5** (One loop quiver). Let $Q$ be the quiver with one vertex $p$ and one arrow $\alpha : p \to p$. Let $M$ be the $Q$-module given by $M_p = k^m$ with the standard basis $B = (b_1, \ldots, b_m)$ and by $M_\alpha = J(\lambda)$ where $J(\lambda)$ is a maximal Jordan block with $\lambda$ on the diagonal and $1$ on the upper side-diagonal. Let $e \leq m$. Considering $K$-rational points for a field extension $K$ of $k$, one sees easily that $\text{Gr}_e(M, K) = \mathbb{A}^0(K) = C^M_\beta(K)$ with $\beta = \{ b_1, \ldots, b_e \}$, and $C^M_\beta = \emptyset$ for other subsets $\gamma \subset B$ of cardinality $e$.

This means that the reduced cell $C^{m,\text{red}}_\beta$ is isomorphic to $\mathbb{A}^0$. However, the non-reduced structure of $C^M_\beta$ is more involved. For our choice of ordering, it turns out that $C^M_\beta$ is indeed reduced, while $\text{Gr}_e(M)$ is not. For another choice of ordering $C^M_\beta$ might be isomorphic to the non-reduced scheme $\text{Gr}_e(M)$.

We explain this in the example $m = 2$ and $e = 1$. The quiver Grassmannian $\text{Gr}_e(M)$ is given as the vanishing set of the homogeneous equation

$$\det \left( \begin{array}{cc} X & \lambda X + Y \\ Y & \lambda Y \end{array} \right) = \lambda XY - \lambda XY - Y^2 = Y^2$$

(also cf. Example 2 in [3]). This means that $\text{Gr}_e(M) = \text{Spec}(k[\varepsilon]/(\varepsilon^2))$ is non-reduced. However, the cell $C^M_\beta$ is defined by the open condition $X$ invertible and the closed condition $Y = 0$.

The latter equation forces $C^M_\beta$ to be $\text{Spec} k = \mathbb{A}^0$.

If we reverse the order of $B$, i.e. $b_2 > b_1$, then the unique $K$-rational point of $\text{Gr}_e(M)$ is still contained in the cell $C^{M,\text{red}}_\beta$, but $C^M_\beta$ is only defined by the open condition $X$ invertible. This means that $C^M_\beta \simeq \text{Gr}_e(M)$ is a non-reduced scheme.

**Example 2.6** (Kronecker quiver). Let $Q$ be the Kronecker quiver with two vertices $1$ and $2$ and two arrows $\alpha, \beta : 1 \to 2$. A regular representations of $Q$ is a $Q$-module $M$ with $M_1 = M_2 = k^n$, $M_\alpha = \text{id}$ and $M_\beta = J(\lambda)$ for some positive integer $n$ and some $\lambda \in k$. By Theorem 2.2 in [4], the quiver Grassmannian $\text{Gr}_e(M)$ decomposes into affine spaces $X_L$, which coincide with the reduced Schubert cells $C^{M,\text{red}}_\beta$ of $\text{Gr}_e(M)$ w.r.t. the standard ordered basis of $M$.

**Example 2.7.** Consider the quiver $Q = 1 \xrightarrow{\alpha} 2$ and the module $M$ that is described as follows. Let $M_1 = M_2 = k^2$, and let $B = \{ b_1, \ldots, b_4 \}$ be the standard basis, i.e. $b_1 = (1, 0)$ and $b_2 = (0, 1)$ in $M_1$ and $b_3 = (1, 0)$ and $b_4 = (0, 1)$ in $M_2$. Let $M_\alpha : M_1 \to M_2$ be the linear map that is described by the matrix $M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the bases $B_1$ and $B_2$. For the dimension vector $e = (1, 1)$
and the ordering $b_1 < b_2 < b_3 < b_4$ of $\mathcal{B}$, we have the decomposition of $\text{Gr}_\xi(M)$ into the four cells $C^M_{\{b_1,b_3\}}, C^M_{\{b_2,b_3\}}, C^M_{\{b_1,b_4\}}$ and $C^M_{\{b_2,b_4\}}$.

We use the notation $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for the submodule $V$ of $M_K$ with dimension vector $\xi = (1,1)$ and $V_1 = \langle \left( \begin{array}{c} a \\ c \end{array} \right) \rangle$ and $V_2 = \langle \left( \begin{array}{c} b \\ d \end{array} \right) \rangle$ where the coefficients lie in a field extension $K$ of $k$. Bearing the condition $M_\alpha(V_1) \subset V_2$ in mind, we find the following description for the $K$-rational points of the four cells:

$$C^M_{\{b_1,b_3\}}(K) = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \simeq \mathbb{A}^0(K)$$

$$C^M_{\{b_2,b_3\}}(K) = \left\{ \begin{bmatrix} v & 1 \\ 0 & 0 \end{bmatrix} \mid v \in K \right\} \quad \simeq \mathbb{A}^1(K)$$

$$C^M_{\{b_1,b_4\}}(K) = \left\{ \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \mid \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \in \text{span}\left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\} \right\} \quad = \emptyset$$

$$C^M_{\{b_2,b_4\}}(K) = \left\{ \begin{bmatrix} v & w \\ 1 & 1 \end{bmatrix} \mid v = 0, w \in K \right\} \quad \simeq \mathbb{A}^1(K).$$

Thus $\text{Gr}_\xi(M)$ is isomorphic to two projective lines that intersect in one point.

A reordering of the $b_1$ and $b_2$ is the same as reordering the rows of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If we calculate the Schubert cell of $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}: k^2 \to k^2$ with the same ordering of the basis elements as above, we find that $C^M_{\{b_1,b_3\}} = \emptyset$, that $C^M_{\{b_2,b_3\}} = C^M_{\{b_1,b_4\}} = \mathbb{A}^0$ and that $C^M_{\{b_2,b_4\}}$ is isomorphic to two affine lines that intersect in one point. This shows that in general, it depends on the ordering of the basis $\mathcal{B}$ whether the Schubert decomposition yields affine spaces as Schubert cells or not.

2.4. Disjoint unions of quivers. A subquiver $S$ of $Q$ is a quiver such that $S_0 \subset Q_0$ and $S_1 \subset Q_1$, and such that the source and target maps of $S$ and $Q$ coincide. Let $M$ be a $Q$-module with basis $\mathcal{B}$. Then the restriction $M_S$ of $M$ to a subquiver $S$ of $Q$ is the $S$-module with $M_{S,p} = M_{p}$ for $p \in S_0$ and $M_{S,\alpha} = M_{\alpha}$ for $\alpha$ in $S$. The set $\mathcal{B}_S = \mathcal{B} \cap M_S$ is a basis for $S$. The following fact is obvious, but useful.

**Lemma 2.8.** Let $Q = S \sqcup T$ be the disjoint union of $T$ and $S$ and let $M$ be a $Q$-module with ordered basis $\mathcal{B}$. Let $M_S$ and $M_T$ be the restrictions of $M$ to $S$ resp. $T$. Then $C^M_\beta = C^M_S \times C^M_T$ for every subset $\beta \subset \mathcal{B}$ where $\beta_S = \beta \cap M_S$ and $\beta_T = \beta \cap M_T$. \hfill $\square$

**Example 2.9.** This yields a generalization of the previous examples. Namely, if $Q$ is a quiver and $M$ a representation such that the restriction of $M$ to each connected component $S$ of $Q$ is isomorphic to one of the $S$-modules of Examples 2.3–2.7 then there is an ordered basis $\mathcal{B}$ of $M$ such that

$$\text{Gr}_\xi(M) = \bigoplus_{\beta \subset \mathcal{B} \text{ of type } \xi} C^M_\beta$$

is a decomposition into affine spaces.
3. TREE EXTENSIONS

In this section, we investigate Schubert cell decompositions for trees. More precisely, we prove a relative theorem for tree extension $T$ of quivers $S$ that puts the Schubert cells of the tree extension into relation to the Schubert cells of $S$.

Let $T$ be a quiver with subquiver $S$. We denote by $T - S$ the subquiver that consists of all arrows of $T$ that are not in $S$ and all vertices that are not in $S$, or that are sources or targets of an arrow in $T - S$. Note that $S$ and $T - S$ can have vertices in common, but no edge. We denote by $T/S$ the quotient quiver, which is obtained from $T$ by removing all edges in $S$ and identifying all vertices of $S$. We say that $T$ is a tree extension of $S$ if $T/S$ is a tree (as a geometric graph).

Let $M$ be a $T$-module with ordered basis $B$. We write $\beta < \beta'$ for two subsets $\beta$ and $\beta'$ of $B$ if $b < b'$ for all $b \in \beta$ and $b' \in \beta'$, and $\beta \leq \beta'$ if $(\beta - \beta') \leq (\beta' - \beta)$. We write $p < q$ for two vertices $p$ and $q$ of $T$ if $B_p < B_q$. Note that the relation $\beta \leq \beta'$ differs from the relation $\beta \leq \beta'$ from Section 2.

Let $S$ be a subquiver of $T$. We denote the restriction of $M$ to $S$ by $M_S$. Let $B_S$ a basis of $M_S$ and assume that $T$ is a tree extension of $S$. An extension of $B_S$ to $M$ is an ordered basis $B$ of $M$ whose intersection with $M_S$ is $B_S$ as ordered sets. An extension $B$ of $B_S$ is ordered above $S$ if $B_S \leq B$, if $p_0 < \cdots < p_r$ for all paths $(p_0, \ldots, p_r)$ with $p_0 \in S_0 \cap T_0$ and $p_1, \ldots, p_r \in T_0 - S_0$ pairwise distinct and if $M_\alpha$ is the identity matrix w.r.t. the ordered bases $B_{S(\alpha)}$ and $B_{S(1(\alpha))}$.

3.1. Results for tree extensions. We will prove all results together at the end of this section.

Lemma 3.1. Let $T$ be a tree extension of $S$. Let $M$ be a $T$-module such that $M_\alpha$ is an isomorphism for all arrows $\alpha$ in $T - S$. Let $M_S$ be the restriction of $M$ to $S$ and $B_S$ an ordered basis of $M_S$. Then there exists an extension $B$ of $B_S$ that is ordered above $S$.

Theorem 3.2. Let $T$ be a tree extension of $S$. Let $M$ be a $T$-module and $M_S$ the restriction of $M$ to $S$. Let $B$ be an ordered basis of $M$ that is ordered above $B_S = B \cap M_S$. Let $\beta$ be a subset of $B$ and $\beta_S = \beta \cap B_S$. Then the following holds true.

(i) The Schubert cell $C_\beta^M$ is empty if and only if $C_{\beta_S}^M$ is empty or if there exists an arrow $\alpha : p \to q$ in $T - S$ such that $M_\alpha(\beta_p) \not\subset \beta_q$.

(ii) If $C_\beta^M$ is not empty, then $C_\beta^M \simeq C_{\beta_S}^{M_S} \times k^{n_\beta}$ for $\beta_S = \beta \cap M_S$ and some $n_\beta \geq 0$.

Theorem 3.3. Let $T$ be a tree extension of $S$. Let $M$ be a $T$-module such that $M_\alpha$ is an isomorphism for all arrows $\alpha$ in $T - S$. Let $M_S$ be the restriction of $M$ to $S$. Let $e_S$ be a dimension vector for $T$ and $e_S$ the restriction of $e$ to $S$. Let $\kappa = \#T_0 - \#S_0$. Then there is a sequence $S = T^{(0)} \subset T^{(1)} \subset \cdots \subset T^{(\kappa)} = T$ of tree extensions of $S$ and a sequence

$$\Phi : \text{Gr}_{\kappa}(M) \xrightarrow{\varphi_\kappa} \text{Gr}_{\kappa-1}(M^{(\kappa-1)}) \xrightarrow{\varphi_{\kappa-1}} \cdots \xrightarrow{\varphi_1} \text{Gr}_{\kappa-1}(M^{(0)}) = \text{Gr}_{\kappa}(M_S)$$

of fibre bundles $\varphi_i$ whose fibres are Grassmannians $\text{Gr}(\tilde{e}_i, \tilde{m}_i)$ for certain integers $\tilde{e}_i \leq \tilde{m}_i$ and $i = 1, \ldots, \kappa$. Here $M^{(i)}$ and $e^{(i)}$ are the restrictions of $M$ resp. $e$ to $T^{(i)}$.

In particular, the morphism $\Phi : \text{Gr}_{\kappa}(M) \to \text{Gr}_{\kappa}(M_S)$ is smooth and projective.
Remark 3.4. Note that the sequence \( S = T^{(0)} \subset T^{(1)} \subset \cdots \subset T^{(\kappa)} = T \) and the corresponding quiver Grassmannians \( \text{Gr}_{\alpha}(M_p) \) are not unique, but depend on a choice of numbering of the vertices in \( T_0 - S_0 \). However, the fibres \( \text{Gr}(\tilde{\alpha}, \tilde{m}_p) \) are uniquely determined up to permutation of indices, and the morphism \( \Phi : \text{Gr}_{\alpha}(M) \to \text{Gr}_{\alpha}(M_S) \) is canonical. In so far, Theorem 3.3 can be seen as a Krull-Schmidt theorem for quiver Grassmannians of tree extensions.

Proof of Lemma 3.1, Theorem 3.2 and Theorem 3.3. All claims will be proven by an induction on \( \kappa = \#(T_0 - S_0) \). If \( \kappa = 0 \), then \( T = S \) and there is nothing to prove. This establishes the base case.

If \( \kappa > 0 \), then we choose an end of \( T \) that does not lie in \( S \), i.e. a vertex in \( T_0 - S_0 \) that is connected to only one arrow \( \alpha \). We consider the case that this vertex is the head of \( \alpha \) separately from the case that it is the tail of \( \alpha \).

Case 1: There is an arrow \( \alpha : p \to q \) such that \( q \) is an end of \( T \) that does not lie in \( S \).

Proof of Lemma 3.1. Define \( T' = T - \{ q, \alpha \} \) and \( M' \) as the restriction of \( M \) to \( T' \). By the induction hypothesis, there exists an ordered basis \( \mathcal{B}' \) of \( T' \) that satisfies Lemma 3.1. We define \( \mathcal{B}_q := M_\alpha(\mathcal{B}'_p) \) as an ordered set. Since \( M_\alpha \) is an isomorphism, \( \mathcal{B}_q \) is a basis of \( M_q \). We define \( \mathcal{B} = \mathcal{B}' \cup \mathcal{B}_q \) where the order of \( \mathcal{B} \) is defined such that \( \mathcal{B}' < \mathcal{B}_q \). Then all claims of Lemma 3.1 follow immediately.

Proof of Theorem 3.2. We argue by considering \( K \)-rational points where \( K \) is a ring extension of \( k \). This will establish the statement \( C^M_\beta \cong C^{M_S}_{\beta_S} \times Z \) for a scheme \( Z \) with \( Z_{\text{red}} = \mathbb{A}^{n_\beta} \). An additional argument will show that \( Z \) is already reduced.

Let \( \beta \subset \mathcal{B} \) and define \( \beta' = \beta \cap M' \). Let \( V \) be a \( K \)-rational point, i.e. a subrepresentation of \( M_K = M \otimes_k K \). As explained in Section 2.2, \( V \) can be identified with a \( |\mathcal{B}| \times |\beta| \)-matrix in row echelon form with coefficients \( v_{b',b} \in K \), pivots \( v_{b,b} = 1 \) for \( b \in \beta \) and \( v_{b',b} = 0 \) if \( b' > b \) or \( b' \in \beta \), but \( b' \neq b \).

If we define \( V_p = V \cap M_p \), then \( M_\alpha(V_p) \subset V_q \) implies that pivots are mapped to pivots. Therefore if \( C^M_\beta \) contains a \( K \)-rational point, then \( M_\alpha(\beta_p) \subset \beta_q \), and the restriction \( V' \) of \( V \) to \( T' \) is a \( K \)-rational point of \( C^M_{\beta'} \). Conversely, if \( C^M_{\beta'} \) contains a \( K \)-rational point \( V' \) and \( M_\alpha(\beta_p) \subset \beta_q \), then the image \( M_\alpha(V_{p'}) \) has generating vectors with pivots in \( \beta_q \), and therefore \( V' \) can be extended to a \( K \)-rational point of \( C^M_{\beta} \). Since a scheme contains a \( K \)-rational point for some ring extension \( K \) of \( k \) if and only if the scheme is non-empty, this proves part (i) of Theorem 3.2.

In case \( C^M_{\beta} \) is non-empty, it contains a \( K \)-rational point \( V \) for some ring extension \( K \) of \( k \). The columns of \( V_q \) whose pivot corresponds to an element \( M_\alpha(b) \in \beta_q \) for \( b \in \beta_p \) are determined by the \( b \)-th column of \( V_p \). All other columns can be chosen freely for \( V \), which have

\[
n'_{\beta} = \sum_{b \in (\beta_q - M_\alpha(\beta_p))} \# \{ b' \in \beta_q | b' < b \text{ and } b' \notin \beta_q \}
\]

free coefficients. Since all equations are already defined over \( k \), this establishes the isomorphism \( C^M_{\beta} \cong C^{M'}_{\beta'} \times Z \) with \( Z_{\text{red}} = \mathbb{A}^{n'_{\beta}} \).
To see that the factor $A_{\alpha}^{\beta}$ is reduced, recall that the defining equations for $V$ are linear in the coordinates of $V_q$. This is also true for the corresponding relations between the Plücker coordinates of $V$, cf. [9 §9.1, Lemma 2]. As a solution space of linear equations, the scheme $C_{\beta}^M$ is reduced. The finishes the proof of Theorem 3.2.

**Proof of Theorem 3.3** Let $\mathbf{m}$ be the dimension vector of $M$, $\mathbf{e} \leq \mathbf{m}$ and $\mathbf{e}'$ the restriction of $M$ to $T'$. We argue by considering $K$-rational points and prove that the natural morphism $\varphi : Gr_{\mathbf{e}}(M) \to Gr_{\mathbf{e}'}(M')$ is a fibre bundle with fibre $Gr(e, \tilde{m})$ for $\tilde{e} = e_q - e_p$ and $\tilde{m} = m_q - e_p$, up to a possible non-reduced structure sheaf of the fibre, which we will exclude by an additional argument. Using the induction hypothesis, this will establish the theorem.

Note that in the case that $e_p > e_q$, we face the trivial case of an empty quiver Grassmannian $Gr_{\mathbf{e}}(M)$ and an empty fibre $Gr(e_q - e_p, m_q - e_p)$. Thus we may assume that $e_q \geq e_p$. If $V$ is a $K$-rational point of $Gr_{\mathbf{e}}(M)$ for some ring extension $K$ of $k$, then $V_p$ determines an $e_p$-dimensional subspace of $V_q$ since $M_0(V_p) \subset V_q$. This means that $V_q/M_0(V_p)$, varies through $M_q/M_0(V_p)$, which can be identified with a $K$-rational point of $Gr(e, \tilde{m})$. Therefore, the fibre $\varphi^{-1}(V)$ of every $K$-rational point $V$ of $Gr_{\mathbf{e}'}(M')$ is isomorphic to $Gr(e, \tilde{m})(K)$.

We will show that $\varphi(K)$ trivializes locally. To do so, we consider a $K$-rational point $V$ of $Gr_{\mathbf{e}}(M)$ and define $V' = \varphi(V)$. We choose a basis $B'$ of $M'$ and order it in such a way that $V'$ can be identified with an $\mathbf{m} \times \mathbf{e}'$-matrix in row echelon form that has pivots in the bottom rows of the rows $B_{\beta'}$ for each vertex $\beta'$ of $T'$, i.e. such that $\beta_{\beta'} \geq B_{\beta'}$. Then the corresponding Schubert cell $C_{\beta'}^M(K)$ for $\beta' = \bigcup_{\beta' \in T'} \beta_{\beta'}$ is an open neighborhood of $V'$ in $Gr_{\mathbf{e}'}(M')(K)$.

Further, we can assume that $B_{\beta'}$ is ordered such that also $\beta_q \geq B_q$ if we extend $B'$ to a basis $B$ of $M$ by the rule $B_q = M_\alpha(B_{\beta})$ and define $\beta_q$ as subset of $B_q$ that corresponds to rows that contain a pivot element of $V$. Then the Schubert cell $C_{\beta}^M(K)$ for $\beta = \beta' \cup \beta_q$ is an open neighborhood of $V$ in $Gr_{\mathbf{e}}(M)(K)$ and $\varphi$ restricts to a morphism $\varphi(K) : C_{\beta}^M(K) \to C_{\beta'}^M(K)$.

Since $B$ is an extension of $B'$ that is ordered above $T'$, we can apply Theorem 3.2 (iii) to obtain an isomorphism $C_{\beta}^M(K) \cong C_{\beta'}^M(K) \times A^n(K)$ for some $n \geq 0$. This shows that $\varphi$ is locally trivial, i.e. a fibre bundle.

The fibre of $\varphi$ is reduced since it is given by a system of linear equations in the Plücker coordinates, cf. the proof of Theorem 3.2. This finishes the proof of Theorem 3.3.

**Case II:** There is an arrow $\alpha : p \to q$ such that $p$ is an end of $T$ that does not lie in $S$.

**Proof of Lemma 3.7** We proceed similar to Case I. We define $T' = T - \{p, \alpha\}$ and $M'$ as the restriction of $M$ to $T'$. By the induction hypothesis, there exists an ordered basis $B'$ of $T'$ that satisfies the lemma. We define $B_q := M_\alpha^{-1}(B_{\beta'})$ as an ordered set. Note that $M_\alpha$ is an isomorphism, thus $B_q$ is a basis of $M_q$. We define $B = B' \cup B_q$ where the order of $B$ is defined such that $B' < B_q$. Then all claims of Lemma 3.1 follow immediately.

**Proof of Theorem 3.2** If $C_{\beta}^M$ is non-empty, it contains a $K$-rational point $V$ for some ring extension $K$ of $k$. Then the restriction $V'$ of $V$ to $T'$ is a $K$-rational point of $C_{\beta'}^M$, which shows
that $C^M_{\beta'}$ is non-empty. If $v$ is the matrix associated to $V$, then the condition $M\alpha(V_p) \subset V_q$ shows that pivots are mapped to pivots, which means that $M\alpha(\beta_p) \subset \beta_q$. Conversely, if $C^M_{\beta'}$ contains a $K$-rational point $V'$ for some ring extension $K$ of $k$ and $M\alpha(\beta_p) \subset \beta_q$, then we can extend $V'$ to a $T$-module $V$ by defining $V_p$ as follows: if $V'$ is the matrix associated to $V'$, then we define $V_p$ as the span of the column vectors of $V'$ that are labelled by those $b \in \beta_q$ that lie in the image $M\alpha(\beta_p)$. This shows part (i) of the theorem.

Assume $C^M_{\beta}$ is non-empty, i.e. it contains a $K$-rational point $V$ with associated matrix $v$. For $b \in \beta_p$, the column vector $v_b$ of the submatrix $v^p$ of $v$ is determined by the column vector $v_{M\alpha(b)}$ of $\beta'$, up to adding a linear combination of the column vectors $v_{b'}$ of $\beta'$ for which $b' \in \beta_q - M\alpha(\beta_p)$ and $b' < M\alpha(b)$. This yields

$$n'_{\beta} = \sum_{b \in \beta_p} \#\{ b' \in \beta_q \mid b' < M\alpha(b) \text{ and } b' \notin M\alpha(\beta_p) \}$$

free coefficients. Therefore, $C^M_{\beta} \simeq C^M_{\beta'} \times A^\beta$. Note that the factor $A^\beta$ is reduced for the same reason as explained in Case I. By the induction hypothesis, this establishes part (ii) of Theorem 3.2.

Proof of Theorem 3.3 The only difference to Case I is that $V_p$ varies while $V_q$ is fixed. Since $M\alpha(V_p) \subset V_q$, this means that the $e_p$-dimensional $V_p$ varies in an $e_q$-dimensional space, i.e. the fibre of $\varphi : \text{Gr}_N(M) \to \text{Gr}_N(M')$ is $\text{Gr}(e_p,e_q)$. The rest of the proof is exactly as in Case I.

This finishes the proof of Lemma 3.1, Theorem 3.2, and Theorem 3.3.

4. Push-forwards

In this section, we generalize the results on Schubert cells for tree extensions to push-forwards along certain morphisms from tree extensions to other quivers.

A morphism $F : T \to Q$ of quivers is a map $F : T_0 \cup T_1 \to Q_0 \cup Q_1$ such that $F(T_i) \subset Q_i$ for $i = 0,1$ and such that for every arrow $\alpha$ in $T$, we have $F(s(\alpha)) = s(F(\alpha))$ and $F(t(\alpha)) = t(F(\alpha))$. We define the push-forward $N = F_*M$ of a $T$-module $M$ as the $Q$-module with $N_{\bar{\beta}} = \bigoplus_{\bar{p} \in F^{-1}(\bar{\beta})} M_{\bar{p}}$ for $\bar{\beta} \in Q_0$ and

$$N_{\tilde{\alpha}} : \left( n_{\bar{p}} \right)_{\bar{p} \in F^{-1}(\bar{\beta})} \mapsto \left( m_q \right)_{q \in F^{-1}(\tilde{\eta})} \text{ with } m_q = \sum_{\alpha \in F^{-1}(\tilde{\alpha})} M_{\alpha}(n_{s(\alpha)})$$

for an arrow $\tilde{\alpha} : \bar{p} \to \tilde{\eta}$ of $Q$. Note that a basis $B$ of $M$ is also a basis of $N = F_*M$.

A morphism $F : T \to Q$ of quivers is a winding if for all arrows $\alpha \neq \alpha'$ of $T$ with $F(\alpha) = F(\alpha')$, also $s(\alpha) \neq s(\alpha')$ and $t(\alpha) \neq t(\alpha')$. Note that every inclusion of quivers is a winding and that windings are closed under compositions. Note further that the push-forward of a $T$-module $M$ along a winding $F : T \to Q$ satisfies that the sums defining the $m_q$ in Equation (1) range over at most 1 element, and that every $n_p$ occurs in at most one of the sums defining the different
of \( F^{-1}(\tilde{q}) \). In other words, \( N_{\tilde{\alpha}} \) can be represented as a monomial block matrix whose non-zero blocks correspond to the \( M_{\alpha} \) for \( \alpha \in F^{-1}(\tilde{\alpha}) \).

4.1. **Defining equations for Schubert cells.** In this section, we describe which matrices correspond to submodules of a push-forward \( N = F_*M \) of a \( T \)-module \( M \) along a winding \( F : T \to Q \).

Let \( \mathcal{B} \) be an ordered basis of \( M \) and \( \beta \subset \mathcal{B} \) a subset. Assume that \( \mathcal{C}^M_{\beta} \neq \emptyset \). For \( q \in Q \), define \( \mathcal{B}_q = \bigcup_{p \in F^{-1}(q)} \mathcal{B}_p \) and \( \beta_q = \beta \cap \mathcal{B}_q \). Let \( K \) be a ring extension of \( k \). By the observations of Section 2.2 every \( K \)-rational point \( W \) of \( C^F_{\gamma} \) can be identified with a matrix \( w = (w_{b,b'})_{b \in \mathcal{B}, b' \in \beta} \) with coefficients \( w_{b,b'} \in K \) satisfying (a) \( w_{b,b'} = \delta_{b,b'} \) for \( b, b' \in \beta \) where \( \delta_{b,b'} \) is the Kronecker delta; (b) \( w_{b,b'} = 0 \) if \( b \in \mathcal{B}_q \) and \( b' \in \beta_{q'} \) for distinct vertices \( q \) and \( q' \) of \( Q \); and (c) \( w_{b,b'} = 1 \) if \( b > b' \).

Conversely, the column vectors of a matrix \( w = (w_{b,b'}) \) span a sub-\( K \)-module of \( F_*M_K \), which is a sub-\( Q \)-module if and only if for every arrow \( \tilde{\alpha} : \tilde{p} \to \tilde{q} \) in \( Q \), we have \( M_{\alpha}(W_{\tilde{p}}) \subset W_{\tilde{q}} \). If we write \( w_{b} \) for the \( b \)-th column vector of \( w \), i.e. \( w_b = (w_{b',b})_{b' \in \mathcal{B}} \), then \( M_{\alpha}(W_{\tilde{p}}) \subset W_{\tilde{q}} \) if and only if for every \( b \in \beta_{\tilde{p}} \), there are \( \lambda_{b',b} \in K \) for \( b' \in \beta_{\tilde{q}} \) such that

\[
M_{\alpha}(w_b) = \sum_{b' \in \beta_{\tilde{q}}} \lambda_{b',b}w_{b'}.
\]

We rewrite Equation (2) as follows. Since \( w_{b',b'} = \delta_{b',b'} \), we conclude that \( \lambda_{b',b} = (M_{\alpha}(w_b))_{b'} \). Let \( F^{-1}(\tilde{\alpha}) = \{\alpha_i : p_i \to q_i\}_{i=1,...,r} \) be the fibre of \( F \) over \( \tilde{\alpha} \). Define \( w_{b',b} \) as the submatrix \((w_{b,b'})_{b \in \mathcal{B}_{\tilde{q}}, b' \in \beta_{\tilde{q}}} \) of \( w \) where \( p' \) and \( q' \) are vertices of \( T \). Since \( F \) is a winding, \( M_{\alpha} \) decomposes into a direct sum of the \( M_{\alpha_i} \) for \( i = 1, \ldots, r \) and possibly a trivial morphism. Thus if \( b \in \beta_{p_j} \) and \( b' \in \beta_{q_i} \), then \( \lambda_{b',b} = (M_{\alpha_i}(w_{b}))_{b'} = (M_{\alpha_i}(w_{b',b}))_{b' \in \beta_{p_j}} \). If we define \( M_{\alpha_i}|_{\beta_{p_j}} \) as the submatrix of \( M_{\alpha_i} \) that contains only the \( b \)-th rows where \( b \) is in \( \beta_{q_i} \) (but all columns), then Equation (2) (for varying \( b \in \beta_{\tilde{p}} \)) can be expressed as

\[
M_{\alpha_i}|_{\beta_{p_j}} \cdot w_{p_i,p_j} = \sum_{i=1}^{r} w_{q_i,q_i} \cdot M_{\alpha_i}|_{\beta_{q_i}} \cdot w_{p_i,p_j}
\]

for varying \( i \) and \( j \).

Note that the equations of this system that correspond to rows \( b \in \beta_{q_i} \) reduce to \( M_{\alpha_i}(w_{b}) = M_{\alpha_i}(w_{b}) \), which is trivially satisfied. Therefore, only the equations for rows in \( \mathcal{B}_{q_i} - \beta_{q_i} \) yield proper conditions.

4.2. **Comparison of \( C^M_{\gamma} \) and \( C^F_{\gamma} \).** Let \( F : T \to Q \) be a morphism of quivers and \( M \) a \( T \)-module. Given an ordered basis \( \mathcal{B} \) of \( M \) resp. \( F_*M \) and a subset \( \beta \subset \mathcal{B} \), we like to compare the Schubert cells \( C^M_{\beta} \) of \( \text{Gr}_e(M) \) and \( C^F_{\gamma} \) of \( \text{Gr}_F(Q) \) where \( F(e) \) is the type of \( \beta \) as a subset of \( F_*M \), i.e. \( F(e) = (f_{\tilde{b}})_{b \in Q_0} \) with \( f_{\tilde{b}} = \#(\beta \cap F_*M_{\tilde{b}}) = \sum_{p \in F^{-1}(\tilde{b})} e_p \).

There is a canonical closed embedding

\[
\iota^M_{F,\beta} : C^M_{\beta} \to C^F_{\beta}
\]
by sending a submodule $V$ of $M$ to the submodule $F_*V$ of $F_*M$. If $V$ is represented by the matrix $v$, then $F_*V$ is represented by the same matrix $v$. This defines a canonical closed embedding

$$\iota^M_{\underline{\alpha}} : \text{Gr}_{\underline{\alpha}}(M) \longrightarrow \text{Gr}_{F(\underline{\alpha})}(F_*M).$$

Under a certain assumption on $F : T \to Q$, there exists a retraction to $\iota^M_{F,\beta}$. Namely, a morphism $F : T \to Q$ is called strictly ordered (w.r.t. $\mathcal{B}$) if for each pair of distinct arrows $\alpha : p \to q$ and $\alpha' : p' \to q'$ of $T$ with $F(\alpha) = F(\alpha')$, we have that either $p < p'$ and $q < q'$ or $p > p'$ and $q > q'$. In other words, the ordering of $\mathcal{B}$ defines a natural ordering of the arrows in the fibre $F^{-1}(\tilde{\alpha})$ for every arrow $\tilde{\alpha}$ of $Q$. Note that every strictly ordered morphism is a winding.

For a strictly ordered winding $F : T \to Q$, we can define a morphism $\pi^M_{F,\beta} : C^F_{\beta, M} \to C^M_\beta$ as follows. Let $W$ be a $K$-rational point where $K$ is a ring extension of $k$ and let $w$ be the associated $|\mathcal{B}| \times |\beta|$-matrix with coefficients in $K$. We regard $w$ as a block matrix $(w_{p,p'})_{p,p' \in T_0}$ where $w_{p,p'}$ is the submatrix of $w$ whose rows are labelled by elements of $\mathcal{B}_p$ and whose columns are labelled by elements of $\beta_{p'}$. Then $w_{p,p'}$ is the zero matrix if $F(p) \neq F(p')$ or if $p' < p$. If $\alpha : \tilde{p} \to \tilde{q}$ is an arrow in $Q$ and $F^{-1}(\tilde{\alpha}) = \{\alpha_i : p_i \to q_i \}_{i=1,...,r}$, then the submatrices $w_{p_i,p_j}$ satisfy Equation\footnote{Equation for matrix submatrices.} Since $F$ is strictly ordered, this reduces to

$$M_{\alpha_i} \cdot w_{p_i,p_j} = w_{q_i,q_j} \cdot M_{\alpha_i|_{\beta_{q_i}}} \cdot w_{p_i,p_j}$$

in case that $i = j$. This means that also the block matrix $v = (v_{p,p'})_{p,p' \in T_0}$ with $v_{p,p} = w_{p,p}$ and $v_{p,p'} = 0$ if $p \neq p'$ satisfies the Equation\footnote{Equation for matrix submatrices.} for all choices of $\tilde{\alpha}$, $i$ and $j$. Therefore $v$ is associated to a $K$-rational point $V'$ of $C^F_{\beta,M}$, which is the image $\iota^M_{F,\beta}(V)$ of the $K$-rational $V$ of $C^M_\beta$. This defines the morphism

$$\pi^M_{F,\beta} : C^F_{\beta,M} \longrightarrow C^M_\beta,$$

which is a retract to the embedding $\iota^M_{F,\beta} : C^M_\beta \to C^F_{\beta,M}$. These morphisms for various $\beta$ patch to a morphism

$$\pi^M_{\underline{\alpha}} : \text{Gr}_{F(\underline{\alpha})}(F_*M) \longrightarrow \text{Gr}_{\underline{\alpha}}(M),$$

which is a retract to the embedding $\iota^M_{\underline{\alpha}} : \text{Gr}_{\underline{\alpha}}(M) \to \text{Gr}_{F(\underline{\alpha})}(F_*M)$.

We summarize the facts of this section in the following proposition.

**Proposition 4.1.** Let $F : T \to Q$ be a morphism of quivers and $M$ a $T$-module with ordered basis $\mathcal{B}$. For every subset $\beta \subseteq \mathcal{B}$ of type $\underline{\alpha}$, there is a closed embedding $\iota^M_{F,\beta} : C^M_\beta \to C^F_{\beta,M}$, which is the restriction of a closed embedding $\iota^M_{\underline{\alpha}} : \text{Gr}_{\underline{\alpha}}(M) \to \text{Gr}_{F(\underline{\alpha})}(F_*M)$.

If $F : T \to Q$ is a strictly ordered winding, then $\iota^M_{F,\beta}$ has a retract $\pi^M_{F,\beta} : C^F_{\beta,M} \to C^M_\beta$, which is the restriction of a retract $\pi^M_{\underline{\alpha}} : \text{Gr}_{F(\underline{\alpha})}(F_*M) \to \text{Gr}_{\underline{\alpha}}(M)$ to $\iota^M_{\underline{\alpha}}$. \hfill $\square$

**4.3. The main theorem.** Let $S \subseteq T$ be a subquiver and $M$ be a $T$-module with ordered basis $\mathcal{B}$. Let $\mathcal{B}_S$ be the restriction of $\mathcal{B}$ to $S$. We say that a morphism $F : T \to Q$ of quivers is ordered above $S$ (w.r.t. $\mathcal{B}$) if $\mathcal{B}$ is ordered above $\mathcal{B}_S$ and if for all arrows $\alpha : p \to q$ and $\alpha' : p' \to q'$ in $T - S$ with $F(\alpha) = F(\alpha')$, the relations $q > q'$ and $q > q'$ imply that $p > p'$. For instance, every strictly ordered morphism $F : T \to Q$ is ordered above $S$. 
Theorem 4.2. Let $T$ be a tree extension of $S$ and $M$ a $T$-module with ordered basis $\mathcal{B}$. Let $M_S$ be the restriction of $M$ to $S$ and $\mathcal{B}_S = \mathcal{B} \cap M_S$. Let $F : T \to Q$ be a winding that is ordered above $S$. Consider a subset $\beta \subset \mathcal{B}$ such that $C^M_\beta \neq \emptyset$, and let $n_\beta \geq 0$ be the integer such that $C^M_\beta \simeq C^{M_S}_{\beta_S} \times \mathbb{A}^{n_\beta}$ where $\beta_S = \beta \cap \mathcal{B}_S$ (cf. Theorem 3.2).

Then there is an integer $n_{F,\beta} \geq 0$ and an isomorphism $C^{F, M}_\beta \simeq C^{F, M_S}_{\beta_S} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F, \beta}}$ such that

\[
\begin{array}{ccc}
C^M_\beta & \sim & C^{M_S}_{\beta_S} \times \mathbb{A}^{n_\beta} \\
\downarrow_{\iota_{F,\beta}} & & \downarrow \left(\begin{array}{c} M_\beta \\ \iota_{F,\beta} \cdot \id \end{array} \right) \\
C^{F, M}_\beta & \sim & C^{F, M_S}_{\beta_S} \times \mathbb{A}^{n_{F,\beta}} \times \mathbb{A}^{n_\beta}
\end{array}
\]

commutes.

Proof. We prove the theorem by induction on $\kappa = \#(T_0 - S_0)$. If $\kappa = 0$, then $T = S$ and there is nothing to prove. If $\kappa > 0$, then we consider again two cases.

Case I: There is an arrow $\alpha : p \to q$ such that $q$ is an end of $T$ that does not lie in $S$.

Define $T' = T - \{q, \alpha\}$. Let $M'$ be the restriction of $M$ to $T'$ and $\mathcal{B}' = \mathcal{B} \cap M$. We can assume that $q$ is the largest vertex of $T$ w.r.t. the ordering of $\mathcal{B}$. By the induction hypothesis, there are non-negative numbers $n_\beta'$ and $n_{F,\beta'}$ that satisfy the assertions of the theorem for $T'$, $M'$ and $\mathcal{B}'$. We consider $K$-rational points to conclude that there are $n_\beta \geq n_\beta'$ and $n_{F,\beta} \geq n_{F,\beta'}$ that satisfy the assertions of the theorem for $T$, $M$ and $\mathcal{B}$.

Let $W$ be a $K$-rational point and $w$ the associated $|\mathcal{B}| \times |\beta|$-matrix. We assume that the restriction $W_{T'}$ of $W$ to $T'$ is fixed, and we will show that the possible extensions of $W'$ to $T$ define an affine space. This means we consider the submatrices $w^{i, q}$ as undetermined and the rest of $w$ as fixed. Note here that $w^{q, q'} = 0$ for $q' \neq q$ since we chose $q$ as the largest vertex of $T$.

The submatrices $w^{i, q}$ are object to Equation (3) for those $\tilde{\alpha}$, $i$ and $j$ for which $q$ appears as an upper index. Since the only arrow connecting to $q$ is $\alpha : p \to q$, we obtain a condition for every arrow in $F^{-1}(\tilde{\alpha} : \tilde{p} \to \tilde{q}) = \{\alpha_i : p_i \to q_i\}_{i=1, \ldots, r}$ where $\alpha_r = \alpha$ and $\tilde{\alpha} = F(\alpha)$. For each choice of $i$ and $j$, the vertex $q = q_r$ appears as an upper index in only one term of Equation (3), namely in

$$w^{q_i, q} \cdot M_{\alpha_i | \beta_q} \cdot w^{p_i, p_j} = w^{q_i, q} \cdot w^{p_i, p_j} |_{\beta_q}$$

(note that $M_{\alpha_i}$ is the identity matrix because $F : T \to Q$ is ordered above $S$).

If $p > p_j$, this term equals 0. Since $F : T \to Q$ is ordered above $S$, the inequalities $q \geq q_j$ and $q > p$ imply $p \geq p_j$. Therefore $w^{q_i, q} \cdot M_{\alpha_i | \beta_q} \cdot w^{p_i, p_j}$ equals 0 unless $p_j = p$. In this case, we face only one equation

$$w^{q_i, q} \cdot w^{p_i, p_j} |_{\beta_q} = w^{p_i, p} - \sum_{l=1}^{r-1} w^{q_i, q_l} \cdot w^{p_i, p_j} |_{\beta_q}$$

for every $i \in \{1, \ldots, r\}$ (note again that all $M_{\alpha_i}$ equal the identity matrix).
This means that the coefficients of $w^{q', q}$ have to satisfy a system of linear equations that are equal to constants that depend only on $w^{p', q'}$ with $p', q' < q$. Since $w^{p, p}$ is of full rank, this system of linear equations has indeed a solution, and therefore the coefficients of $w^{q', q}$ vary in an affine space of some dimension $\tilde{n}_{q, i}$. A change of coordinates establishes an isomorphism $C_{\beta}^{F, M} \simeq C_{\beta'}^{F, M'} \times \mathbb{A}^{n'_q}$ for $n'_q = \sum_{i=1}^r \tilde{n}_{q, i}$. Note that the factor $\mathbb{A}^{n'_q}$ is reduced for the same reason as explained in the proof of Theorem 3.2: it is the solution space of linear equations in the Plücker coordinates.

The solutions in matrices $v$ that correspond to $K$-rational points $V$ of $C_{\beta}^{M}$ yield an isomorphism $C_{\beta}^{M} \simeq C_{\beta'}^{M'} \times \mathbb{A}^{n'_q}$ for some $n'_q \geq 0$. The closed embedding $\iota_{F, \beta} : C_{\beta}^{M} \rightarrow C_{\beta}^{F, M}$ yields a commutative diagram

$$
\begin{array}{ccc}
C_{\beta}^{M} & \sim & C_{\beta'}^{M'} \times \mathbb{A}^{n'_q} \\
\cap & & \cap \\
C_{\beta}^{F, M} & \sim & C_{\beta'}^{F, M'} \times \mathbb{A}^{n'_{F, q}} \times \mathbb{A}^{n'_{q}}
\end{array}
$$

where $n'_{F, q} = n'_q - n'_q$ is a non-negative integer. Using the inductive hypothesis, the theorem holds for the numbers $n_{\beta} = n_{\beta'} + n'_q$ and $n_{F, \beta} = n_{F, \beta'} + n'_{F, q}$ in Case I.

**Case II:** There is an arrow $\alpha : p \rightarrow q$ such that $p$ is not in $S$ and an end of $T$.

We proceed similar to Case I. We define $T' = T - \{p, \alpha\}$. Let $M'$ be the restriction of $M$ to $T'$ and $B' = B \cap M$. We can assume that $p$ is the largest vertex of $T$ w.r.t. the ordering of $B$. By the induction hypothesis, there are numbers $n_{\beta'}$ and $n_{F, \beta'}$ that satisfy the assertions of the theorem for $T', M'$ and $B'$.

Let $W$ be a $K$-rational point of $C_{\beta}^{F, M}$ and $w$ the associated matrix. As in Case I, we investigate how the coefficients of $w$ that lie in one of the submatrices $w^{p', p}$ depend on the other coefficients of $w$ (note that $w^{p, p'} = 0$ if $p \neq p'$). These dependencies are expressed by Equation (3) for $\tilde{\alpha} = F(\alpha)$ and those $i$ and $j$ such that $p$ appears as an upper index. Let $F^{-1}(\tilde{\alpha} : \tilde{p} \rightarrow \tilde{q}) = \{\alpha_i : p_i \rightarrow q_i\}_{i=1, \ldots, r}$ be the fibre of $\tilde{\alpha} = F(\alpha)$ with $\alpha_r = \alpha$. Again, since $w^{p, p'} = 0$ if $p' \neq p$, the only non-trivial equations are

$$
(5) \quad w^{p_k, p} = \sum_{i=1}^r w^{q_k, q_i} \cdot w^{p_i, p} |_{\beta_{q_i}}
$$

for $k = 1, \ldots, r$ where we used that all $M_{\alpha_i}$ equal the identity matrix since $F : T \rightarrow Q$ is ordered above $S$.

Since this system of linear equations yields a trivial equation for rows $b \in \beta_{q_k}$, we are only concerned with rows $b'$ with $b' \in \beta_{q_k} - \beta_{q_k}$. On the other hand, $w^{p_i, p} |_{\beta_{q_i}}$ depends only on rows $b \in \beta_{q_k}$. Define

$$
M_{p, 0} = \text{span} \left( M_{\alpha}^{-1}(\beta_{q}) \right) \quad \text{and} \quad M_{p, 1} = \text{span} \left( M_{\alpha}^{-1}(\beta_{q} - \beta_{q}) \right)
$$
and let $\pi_i : M_p \to M_{p,i}$ be the orthogonal projections for $i = 0, 1$. Then $w^{p_i \cdot p_i | \beta, \gamma}$ depends only on $\pi_0(W_p) \subset M_{p,0}$ and the relevant rows of $w^{p_k \cdot p_k}$ depend only on $\pi_1(W_p) \subset M_{p,1}$. Since $M_{\alpha_k}$ is an isomorphism for all $k \in \{q, \ldots, r\}$, we find for any choice of $\pi_0(W_p)$ a solution in $\pi_1(W_p)$ such that Equation (5) holds for the submatrices $w^{p_k \cdot p_k}$ of $w$ for $k = 1, \ldots, r$. This means that there is an affine space of solutions in $w^{p_k \cdot p_k}$ for every $k = 1, \ldots, r$ and that $C_{\beta, M} \simeq C_{\beta', M'} \times A_n^r$ for some $n_k' \geq 0$.

For the same reasons as in Case I, there are non-negative integers $n_k'$ and $n_{F, q}'$ with $n_k' = n_q'+ n_{F, q}'$ and a commutative diagram

$$
\begin{array}{c}
C^M_{\beta} \sim \to C^M_{\beta'} \times A_n^{n_q'} \\
\downarrow \quad \downarrow \\
C_{\beta, F, M} \sim \to C_{\beta', F, M'} \times A_n^{n_q'} \times A_n^{n_{F, q}'}.
\end{array}
$$

Using the inductive hypothesis, the theorem holds for the numbers $n_{\beta} = n_{\beta'} + n_q'$ and $n_{F, \beta} = n_{F, \beta'} + n_{F, q}'$ also in Case II. This finishes the proof of the theorem. \hfill \square

**Remark 4.3.** Note that there are examples of windings $F : T \to Q$ such that there is no choice of ordered basis $B$ that satisfies the assumption that for all arrows $\alpha : p \to q$ and $\alpha' : p' \to q'$ in $T - S$ with $F(\alpha) = F(\alpha')$, the relations $q > q'$ and $q > p$ imply that $p > p'$. It is not clear to me whether this assumption is necessary or might be completely removed. Theorem 1.2 (a) and its consequences (e.g. Cor. 3.1) in [10] work completely without this hypothesis and indicate that it is superfluous.

The following is an example of a winding that cannot be ordered above $S$. Let $S$ be the quiver with one vertex $s$,

$$
T = \bullet \circ \circ \circ \circ \circ \bullet \text{ and } Q = \bullet \circ \circ \circ \circ \circ \bullet.
$$

The winding $F : T \to Q$ is defined by $F(\alpha_1) = \tilde{\alpha}$ and $F(\beta_1) = \tilde{\beta}$ for $i = 1, 2$. Then one of the bullets next to $s$ in $T$ has to be the smallest vertex of $T_0 - \{s\}$. By the symmetry of interchanging the roles of $\alpha$ and $\beta$, we may assume that the vertex to the right of $s$, i.e. $s(\alpha_2)$, is the smallest one. Independent of the order of the other vertices, we have that $t(\alpha_2) > s(\alpha_2)$ and $t(\alpha_2) > s = t(\alpha_1)$. But since $s(\alpha_2)$ is the smallest vertex in $T_0 - \{s\}$, we have $s(\alpha_1) > s(\alpha_2)$, which shows that $F : T \to Q$ cannot be ordered above $S$.

The following winding $F : T \to Q$, defined by $F(\alpha_1) = \tilde{\alpha}$, et cetera, cannot be ordered above any subquiver $S$ of $T$ that consists of a single vertex:

$$
T = \bullet \circ \circ \circ \circ \circ \bullet \text{ and } Q = \bullet \circ \circ \circ \circ \circ \bullet.
$$
Remark 4.4. Let \( Q \) be an arbitrary quiver. In [10, Thm. 12 (b)], Haupt provides a formula for the Euler characteristic of \( \text{Gr}_e(M) \) where \( M \) is a band module of \( Q \). To explain, \( M \) is called a band module if it is an irreducible \( Q \)-module that is the push-forward of a \( T \)-module \( N \) along a winding \( F : T \to Q \) where \( T \) is a quiver of type \( \tilde{A}_n \) (i.e. connected with \( n \) vertices of valency 2 and \( n \) arrows) and \( N_\alpha \) is an isomorphism for all \( \alpha \in \Gamma_1 \). The result in [10] hints that quiver Grassmannians of band modules have a reduced Schubert decomposition into affine spaces.

Note that a positive result for band modules combined with Theorem 4.2 establishes a reduced Schubert decomposition of \( \text{Gr}_e(F_*M) \) into affine cells when \( F_*M \) is the push-forward of a representation \( M \) of a quiver \( T \) of geometric genus 1 along a winding \( F : T \to Q \), with the technical restriction that the winding \( F \) has to be ordered above the unique subquiver \( S \) of \( T \) that is of type \( \tilde{A}_n \).

5. Consequences of the push-forward theorem

In this section, we will describe a series of consequences of Theorem 4.2. Whenever we have a Schubert decomposition of some quiver Grassmanian \( \text{Gr}_e(M) \) into affine spaces where \( M \) is an \( S \)-module for some quiver \( S \), we can use the Theorem 4.2 to extend this result to a larger class of quiver Grassmannians. We formulate this method in following statement.

Corollary 5.1. Let \( T \) be a tree extension of \( S \) and \( M \) a \( T \)-module with ordered basis \( \mathcal{B} \). Let \( F : T \to Q \) be a winding that is ordered above \( S \). Denote by \( M_S \) the restriction of \( M \) to \( S \). If \( C_{F_*M}^\beta_S \) is an affine space or empty for every subset \( \beta_S \subset \mathcal{B} \cap M_S \), then

\[
\text{Gr}_e(M) = \bigsqcup_{\beta \subset \mathcal{B} \text{ of type } e} C_{\beta}^{F_*M}
\]

is a decomposition into affine spaces for any dimension vector \( e \).

Proof. This follows immediately from Theorem 4.2. \( \square \)

Example 5.2. Corollary 5.1 allows us to expand Example 2.9. Let \( T \) be a tree extension of \( S \) and \( M \) a \( T \)-module such that the restriction of \( M \) to each connected component of \( S \) is one of the quiver representations as considered in Examples 2.3–2.7. Let \( \mathcal{B} \) be an ordered basis of \( M \) and \( F : T \to Q \) a winding that is ordered above \( S \) such that \( F|_S : S \to Q \) is injective on vertices and arrows. Then

\[
\text{Gr}_e(M) = \bigsqcup_{\beta \subset \mathcal{B} \text{ of type } e} C_{\beta}^{F_*M}
\]

is a decomposition into affine spaces for any dimension vector \( e \).

5.1. Direct sums of modules.

Theorem 5.3. Let \( T \) be a tree extension of \( S \) and let \( M^{(i)} \) be \( T \)-modules for \( i = 1, 2 \). Assume that \( M^{(i)}_\alpha \) is an isomorphism for all arrows \( \alpha \) of \( T - S \) and for \( i = 1, 2 \). Define \( M = M^{(1)} \oplus M^{(2)} \), and let \( M_S \) and \( M^{(i)}_S \) be the respective restrictions to \( S \). Let \( \mathcal{B}_S \) be an ordered basis of \( M_S \) such...
that \( \mathcal{B}_S = \mathcal{B}_S^{(1)} \cup \mathcal{B}_S^{(2)} \) where \( \mathcal{B}_S^{(i)} = \mathcal{B}_S \cap M_i \) for \( i = 1, 2 \). Let \( \beta_S \subset \mathcal{B}_S \) be a subset such that there is an \( n_S \geq 0 \) with

\[
C_{\beta_S}^{M_S} \simeq C_{\beta_S}^{M_S^{(1)}} \times C_{\beta_S}^{M_S^{(2)}} \times \mathbb{A}^{n_S}.
\]

Then there is an extension \( \mathcal{B} \) of \( \mathcal{B}_S \) to \( M \) that is ordered above \( S \) such that for every subset \( \beta \subset \mathcal{B} \) with \( \beta_S = \beta \cap \mathcal{B}_S \), there is some \( n \geq n_S \) such that

\[
C_{\beta}^{M} \simeq C_{\beta}^{M^{(1)}} \times C_{\beta}^{M^{(2)}} \times \mathbb{A}^{n}
\]

if \( C_{\beta}^{M} \) is not empty.

**Proof.** Define \( T' = T \amalg T \) and \( S' = S \amalg S \subset T' \). Then \( T' \) is a tree extension of \( S' \). Let \( \iota_1 : T \to T' \) and \( \iota_2 : T \to T' \) be the inclusions into the first resp. the second summand of \( T' = T \amalg T \). Define the \( T' \)-module \( M' = M^{(1)} \amalg M^{(2)} \) whose restriction to \( \iota_1(T) \) is \( M^{(1)} \) and whose restriction to \( \iota_2(T) \) is \( M^{(2)} \). By Lemma 3.1, we can extend \( \mathcal{B}_S \) to an ordered basis \( \mathcal{B} \) of \( M' \) that is ordered above \( S' \) and that satisfies \( M_\alpha(\mathcal{B}_p) \subset \mathcal{B}_q \) for every arrow \( \alpha : p \to q \) in \( T' - S' \).

Define \( F : T' \to T \) by \( F|_{\iota_1(T)} = \text{id}_T = F|_{\iota_2(T)} \). Then \( F \) is a winding that is ordered above \( S' \) and, by the very definition of push-forwards, \( F_*M' = M \) and \( F_*M_S = M_S \). Further, we have

\[
C_{\beta}^{M'} = C_{\beta}^{M^{(1)}} \times C_{\beta}^{M^{(2)}}, \quad C_{\beta}^{M_S} = C_{\beta}^{M^{(1)}} \times C_{\beta}^{M^{(2)}} \quad \text{(by Lemma 2.8)}
\]

This allows us to apply Theorems 3.2 and 4.2 and the hypothesis of this theorem (to which we refer to by (H)) to conclude

\[
C_{\beta}^{M} = C_{\beta}^{M'} = C_{\beta}^{F_*M'} = C_{\beta}^{F_*M_S} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F,\beta}} = C_{\beta}^{M_S} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F,\beta}}
\]

\[
(\text{H})
\]

\[
C_{\beta}^{M^{(1)}} \times C_{\beta}^{M^{(2)}} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F,\beta}} = C_{\beta}^{M'} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F,\beta}}
\]

\[
3.2
\]

\[
C_{\beta}^{M^{(1)}} \times C_{\beta}^{M^{(1)}} \times \mathbb{A}^{n_{\beta}} \times \mathbb{A}^{n_{F,\beta}};
\]

thus the claim of the theorem is satisfied for \( n = n_S + n_{F,\beta} \). \( \square \)

5.2. **Monomial representations of forests.** Let \( Q \) be a quiver and \( M \) a \( Q \)-module over \( k \). The support of \( M \) is the full subquiver \( Q_M \) of \( Q \) with vertices \( Q_M,0 = \{ p \in Q | M_p \neq 0 \} \). A \( Q \)-module \( M \) is **thin** if \( \text{rk}M_p \leq 1 \) for every vertex \( p \) of \( Q \), and \( M \) is **sincere** if \( \text{rk}M_p \geq 1 \) for every vertex \( p \) of \( Q \). A \( Q \)-module \( M \) is **monomial** if it admits a basis \( \mathcal{B} \) such that \( M_\alpha \) is a monomial matrix for all \( \alpha \in Q_1 \). A forest is a quiver that is a union of trees.

**Theorem 5.4.** Let \( Q \) be a forest, \( M \) a monomial \( Q \)-module and \( \underline{e} \) a dimension vector for \( Q \). Then there is an ordered basis \( \mathcal{B} \) of \( M \) such that

\[
\text{Gr}_{\underline{e}}(M) = \prod_{\beta \subset \mathcal{B} \text{ of type } \underline{e}} C_{\beta}^{M}
\]

is a decomposition into affine spaces.
Proof. This theorem follows essentially by the same arguments as used in the proofs of Theorems 4.2 and 5.3. For completeness, we show how to reduce the claim to Theorem 4.2.

Since $M$ is monomial, there is a basis $B$ such that $M_\alpha(B_p) \subset \{ab|a \in k, b \in B_q\}$ for each arrow $\alpha : p \to q$ in $T$. Since $Q$ is a forest, we can rescale the basis elements, emerging from one vertex of a component of $Q$ to its ends, such that $M_\alpha(B_p) \subset B_q \cup \{0\}$ for every arrow $\alpha : p \to q$ in $Q$. We assume that $B$ satisfies this property.

Since $M$ is monomial and $Q$ a tree, it follows that $M$ decomposes into a direct sum $M \simeq \bigoplus_{i=1}^{r} Q_i$ of thin and indecomposable $Q$-modules $M_i$. Let $Q_i$ be the support of $M_i$ for $i = 1, \ldots, r$ and $\iota_i : Q_i \to Q$ the inclusion. Then $Q_i$ is a tree and the restriction $M_{i,Q_i}$ of $M_i$ to $Q_i$ is a thin sincere $Q_i$-module with $M_\alpha(B_p) = b_q$ for every arrow $\alpha : p \to q$ of $Q_i$ and $B_p = \{b_p\}$ and $B_q = \{b_q\}$.

We can further assume that every $Q_i$ has a vertex $p$ that connects to only one arrow $\alpha : p \to q$ and such that $e_{\iota_i(p)} = 0$. If this is not the case, we add an arrow $\alpha : p \to q$ to $Q_i$ where $q$ is an arbitrary vertex of $Q_i$, which defines a tree $Q_i'$. We extend $Q$ to a forest $Q'$ that contains an arrow $\alpha' : p' \to \iota(q)$, which allows us to extend $\iota_i$ to an inclusion $\iota' : Q_i' \to Q'$ that maps $\alpha$ to $\alpha'$. We extend $M_{i,Q_i}$ to the $Q_i'$-module $M'_{i,Q_i}$ with basis $B' = B \cup \{b_p\}$ by $M'_{i,p} = k$ and $M'_{i,\alpha}(b_p) = b_q$ where $B_q = \{b_q\}$. Extend $M_i$ to the $Q'$-module $M'_i$ whose restriction to $Q_i'$ is $M'_{i,Q_i}$. Extend all other direct summands $M_j$ of $M$ to the $Q'$-module $M'_j$ with $M'_{j,\alpha'} : 0 \to M'_{j,i,\iota(q)}$ and define $M' = \bigoplus_{j=1}^{r} M'_j$. Define the dimension vector $\epsilon'$ for $Q'$ by $e'_p = 0$ and $e'_{p'} = e_{p'}$ for $p' \in Q_0$. Then $Q'$, $M'$ and $\epsilon'$ satisfy the hypothesis of the theorem and $\text{Gr}_\epsilon(M')$ is the same as $\text{Gr}_\epsilon(M)$. Therefore we can assume the existence of the vertex $p$ in $Q_i$.

Define $T = \bigsqcup_{i=1}^{r} Q_i$, which is a forest, and $S = \{p_1, \ldots, p_r\}$ where $p_i$ is a vertex of $Q_i$ with the properties from the last paragraph. Define $N = \bigsqcup_{i=1}^{r} M_{i,Q_i}$. Let $F : T \to Q$ be the morphism of quivers that restricts to $\iota_i : Q_i \to Q$ for each connected component $Q_i$ of $T$. Then $F$ is a winding, $F_*N = M$ and $B$ is a basis for $N$.

By Lemma 3.1, we can choose an arbitrary order for $B_S = B \cap N_S$ and can extend it to an ordering of $B$ (which is uniquely determined by the property $N_\alpha(B_p) \subset B_p$) such that $T$ is ordered above $S$. We can further assume that this ordering of $B$ satisfies that $B_S < B_1 < \cdots < B_r$ where $B_i = B \cap M_{i,Q_i}$. Then $F : T \to Q$ is a winding that is ordered above $S$.

Therefore we can apply Theorem 4.2 to obtain for every subset $\beta$ of $B$ and $\beta_S = \beta \cap N_S$ that $C^M_\beta \simeq C^M_{\beta_S} \times \mathbb{A}^{n_{\beta}}$ for some $n_{\beta} \geq 0$, provided $C^M_{\beta_S}$ is not empty. Since $e_p = 0$ for all $p \in S$, the set $\beta_S$ is empty if $\beta$ is of type $\epsilon$, which means that $C^M_{\beta_S} = \text{Spec} k$ is a point. Therefore $C^M_\beta$ is an affine space for every $\beta \subset B$ of type $\epsilon$. This completes the proof of the theorem. \qed

5.3. Push-forwards from forests and application to string and tree modules. We like to push the statement of Theorem 5.3 a bit further by considering push-forwards of monomial representations along windings $F : T \to Q$ from a forest $T$ to an arbitrary quiver $Q$. Since it is not necessary to consider a fixed subquiver $S$ of $T$ in this situation, we introduce the notion of an ordered winding.

Let $T$ be a forest and $M$ a $T$-module with ordered basis $B$. Then $T$ is ordered (w.r.t. $B$) if for every connected component $T'$ of $T$, there is a vertex $p_0$ such that $p_0 < \cdots < p_r$ for
every (unoriented) path \((p_0, \ldots, p_n)\) of pairwise different vertices \(p_i\) of \(T'\). A winding \(T \to Q\) is ordered (w.r.t. \(B\)) if \(T\) is ordered and if for all \(\alpha : p \to q\) and \(\alpha' : p' \to q'\) with \(F(\alpha) = F(\alpha')\), the conditions \(q > q'\) and \(q > p\) imply \(p > p'\).

**Theorem 5.5.** Let \(T\) be a forest and \(M\) a monomial \(T\)-module.

(i) Choose a vertex \(p_i\) in each connected component \(T_i\) of \(T = \bigcup_{i=1}^n T_i\). Then there is an ordered basis \(B\) of \(M\) such that \(T\) is ordered, such that \(p_i < q\) for all other vertices \(q\) in \(T_i\), such that \(M_\alpha(B_p) \subseteq B_q \cup \{0\}\) for all arrows \(\alpha : p \to q\) of \(T\) and such that \(b'_p > b_p\) implies \(b'_q > b_q\) whenever \(M_\alpha(b_p) = b_q\) and \(M_\alpha(b'_p) = b'_q\) for an arrow \(\alpha : p \to q\) of \(T\).

(ii) Let \(F : T \to Q\) be a winding that is ordered w.r.t. \(B\) and \(\varepsilon\) a dimension vector for \(Q\). Then

\[
\text{Gr}_\varepsilon(F_*M) = \bigoplus_{\beta \in B \text{ of type } \varepsilon} C^F_{\beta,M}
\]

is a decomposition into affine spaces.

**Proof.** As explained in the proof of Theorem 5.4, there is a basis \(B\) of \(M\) that satisfies \(M_\alpha(B_p) \subseteq B_q \cup \{0\}\) for every arrow \(\alpha : p \to q\) in \(T\). This basis can be ordered such that it satisfies (i) of the theorem. If \(T = \bigcup_{i=1}^n T_i\) is a decomposition into its connected components, \(M_i\) the restriction of \(M\) to \(T_i\) and \(B_i = B \cap M_i\), then it is enough to order each \(B_i\) for \(i = 1, \ldots, r\) and to impose \(B_1 < \cdots < B_r\).

We describe the ordering of \(B_i\) for every given a component \(T_i\) informally: define the elements of \(B_{p_i}\) as the smallest elements of \(B_i\), in an arbitrary order. Then define successively the elements of \(B_q\) as greater than the previous elements, where \(q\) varies through the vertices of \(T_i\) moving from \(p_i\) towards the ends of the tree \(T_i\). The elements of each \(B_q\) can be ordered such that the last condition of (i) is satisfied. This yields an ordered basis \(B\) of \(T\) that satisfies (i).

Assume \(F : T \to Q\) is an ordered winding and \(\varepsilon\) a dimension vector for \(Q\). As explained in the proof of Theorem 5.4, \(M\) is isomorphic to the push-forward of a thin sincere \(T'\)-module \(N\) along an ordered winding \(G : T' \to T\) where \(T'\) is also a forest. We identify \(F_*N\) with \(M\) and \(B\) with an ordered basis of \(F_*N\) resp. \(N\). This defines an ordering of \(T'\). In particular, if \(G(p) \neq G(q)\), then \(p > q\) if and only if \(G(p) > G(q)\).

We show that the composition \(H = F \circ G : T' \to Q\) is an ordered winding. Assume that \(H(\alpha) = H(\alpha')\) for two arrows \(\alpha : p \to q\) and \(\alpha' : p' \to q'\) of \(T'\) and that \(q > q'\) and \(q > p\). We have to show that \(p > p'\). If \(G(\alpha) = G(\alpha')\), this is clear since \(G\) is ordered. If \(\beta = G(\alpha) : r \to s\) is distinct from \(\beta' = G(\alpha') : r' \to s'\), then \(q > q'\) means that \(s > s'\) and \(q > p\) means that \(r > s\). Since \(F\) is ordered, we have \(s > s'\), which implies in turn \(p > p'\) as desired.

Finally, we can apply the same trick as used in the proof of Theorem 5.4 to apply Theorem 4.2 without loss of generality, we can assume that \(e_{p_i} = 0\). Therefore the definition \(S = \{p_1, \ldots, p_n\}\) yields a trivial factor \(C^F_{e,S,M} = \text{Spec} k\) in the formula of Theorem 4.2. This shows that all locally closed subschemes \(C^F_{\beta,M}\) in the decomposition of \(\text{Gr}_\varepsilon(F_*M)\) are affine spaces or empty. This finishes the proof of the theorem. \(\square\)
A quiver of type $A_n$ is a connected oriented graph of the shape

$$
1 \rightarrow 2 \leftarrow \cdots \rightarrow n
$$

with $n$ vertices of valency $\leq 2$ and $n - 1$ arrows that can be oriented arbitrarily. Let $Q$ be any quiver and $M$ a $Q$-module. Then $M$ is a string module if there is a winding $F : T \rightarrow Q$ from a quiver $T$ whose connected components $T_i$ are of type $A_{n_i}$ for some $n_i \geq 1$ and if there is a monomial $T$-module $N$ such that $M \cong F^* N$. Note that this is equivalent to the definition in [3].

In particular, $T$ can be recovered from $Q$ and $M$ as the coefficient quiver of $M$.

**Remark 5.6.** Note that every $T$-module $M$ is monomial if $T$ is locally of type $A_n$. This can be seen as follows. Since the disjoint union of monomial representations is monomial, we can assume that $T$ is connected and of type $A_n$. Since every indecomposable module of a quiver of type $A_n$ is thin, $M$ decomposes into a direct sum of thin $T$-modules $M_i$. This allows to pick a basis $B_i$ for each $M_i$, which yields a basis $B = \bigcup B_i$ for $M$ that witnesses that $M$ is monomial.

**Corollary 5.7.** Let $Q$ be a quiver, $M$ be a string module of $Q$ and $e$ a dimension vector for $Q$. Then there is an ordered basis $B$ of $M$ such that

$$
\text{Gr}_{\mathbb{C}}(M) = \bigcap_{\beta \subset B \text{ of type } e} C^M_{\beta}
$$

is a decomposition into affine spaces.

**Proof.** Since $M$ is a string module, there is a winding $F : T \rightarrow Q$ from a quiver $T$ whose connected components $T_i$ are of type $A_{n_i}$ and a monomial $T$-module $N$ such that $M \cong F^* N$.

Choose in each connected component $T_i$ of $T = \bigcup_{i=1}^n T_i$ a vertex $p_i$ of valency 1 as its smallest vertex. By Theorem 5.5 (i), we can choose an ordered basis $B$ of $N$ such that we have $p_i^{(0)} < \cdots < p_i^{(n_i)}$ where $(p_i^{(0)}, \ldots, p_i^{(n_i)})$ is the unique path in $T_i$ from one $p_i^{(0)}$ to the other end. It is clear that w.r.t. this ordering, every winding $F : T \rightarrow Q$ is ordered.

This allows us to apply Theorem 5.5 (ii) to establish the corollary. □

**Example 5.8 (Degenerate flag varieties).** As a particular class of string modules, we re-obtain the result [5, Thm. 7.11], which says that degenerate flag varieties have a decomposition into affine spaces. Indeed, the results of [5] are stronger since the decomposition is given by a group action. We inspect degenerate flag varieties in more detail in Example 6.13.

Let $Q$ be a quiver. A tree module over $Q$ is a $Q$-module $M$ that is isomorphic to the push-forward of a thin sincere and indecomposable $T$-module along a winding $F : T \rightarrow Q$ where $T$ is a tree, cf. Section 2.3 in [10]. A tree module $M$ with ordered basis $B$ is ordered w.r.t. $B$ if $F : T \rightarrow Q$ and $N$ as above can be chosen such that the isomorphism $F_* M \cong N$ identifies $B$ with an basis $B'$ of $N$ and such that $F$ is ordered w.r.t. $B'$. A tree module $M$ with ordered basis $B$ is strictly ordered if it is ordered for some $F : T \rightarrow Q$ and $N$ such that $F$ is a strictly ordered morphism (cf. Section 4.2).
Corollary 5.9. Let $Q$ be a quiver, $M$ an ordered tree module w.r.t. an ordered basis $B$ and $\epsilon$ a dimension vector for $Q$. Then

$$\text{Gr}_\epsilon(M) = \coprod_{\beta \subset B \text{ of type } \epsilon} C^M_\beta$$

is a decomposition into affine spaces.

6. The cohomology of quiver Grassmannians

A Schubert decomposition of a quiver Grassmannian into affine spaces yields certain information about the cohomology of the quiver Grassmannian. We concentrate on the singular cohomology of a complex quiver Grassmannian, i.e. the case $k = \mathbb{C}$. The same arguments can also be used to treat the $l$-adic cohomology with proper support of quiver Grassmannians over the integers.

The basic fact that we use is that if a smooth projective $k$-scheme $X$ of complex dimension $m$ has a decomposition $X = \coprod_{i \in I} Z_i$ into affine spaces $Z_i$, then the cohomology classes of the closures $\overline{Z_i}$ form an $\mathbb{Z}$-basis of the cohomology ring $H^\ast(X, \mathbb{Z})$. If $Z_i \cong \mathbb{A}^{d_i}$, then the class $[Z_i]$ is an element of $H^{2m-2d_i}(X, \mathbb{Z})$. In particular, the odd cohomology of $X$ vanishes. For more details, see Appendix B of [9].

Example 6.1 (Smooth quiver Grassmannians). Let $Q$ be a quiver, $M$ a $Q$-module with ordered basis $B$ and $\epsilon$ a dimension for $Q$. Since the Euler characteristic is additive in decompositions into locally closed subschemes, the Schubert decomposition $\text{Gr}_\epsilon(M) = \coprod C^M_\beta$ is a Schubert decomposition into affine spaces, then the cohomology ring $H^\ast(\text{Gr}_\epsilon(M), \mathbb{Z})$ is additively generated by the classes of $\overline{C^M_\beta}$, where $[\overline{C^M_\beta}] \in H^{2m-2d}(\text{Gr}_\epsilon(M), \mathbb{Z})$ where $d$ is the dimension $C^M_\beta$.

By Theorem 1.2, $\text{Gr}_\epsilon(M)$ is smooth if $M$ is rigid. For rigid $Q$-modules such that $\text{Gr}_\epsilon(M)$ has an integral model, the above facts have already been observed by Fan Qin, cf. Thm. 3.2.6 in [13].

If $\text{Gr}_\epsilon(M)$ has several irreducible components, we can still draw conclusions under some further assumptions as explained in Section 6.2.

6.1. Euler characteristics of quiver Grassmannians. In this section, we draw conclusions from the knowledge of Schubert decompositions on the Euler characteristic of a complex quiver Grassmannian. Therefore, we fix $k = \mathbb{C}$ in this section.

Let $Q$ be a quiver, $M$ a $Q$-module with ordered basis $B$ and $\epsilon$ a dimension for $Q$. Since the Euler characteristic is additive in decompositions into locally closed subschemes, the Schubert decomposition $\text{Gr}_\epsilon(M) = \coprod C^M_\beta$ yields the formula

$$\chi(\text{Gr}_\epsilon(M)) = \sum_{\beta \subset B \text{ of type } \epsilon} \chi(C^M_\beta).$$

In the particular case that $\text{Gr}_\epsilon(M)$ decomposes into affine spaces, the Euler characteristic of $\text{Gr}_\epsilon(M)$ equals the number of non-empty Schubert cells $C^M_\beta$.

We describe some examples.
Example 6.2 (String modules). Let $M$ be a string module over a quiver $Q$. By Corollary 5.7 there is a winding $F : T \to Q$ from a quiver $T$ that is locally of type $A_n$ and a $T$-module with ordered basis $\mathcal{B}$ such that $M = F_* N$ and such that

$$\text{Gr}_{\mathcal{B}}(M) = \bigsqcup_{\beta \in \mathcal{B} \text{ of type } \mathcal{E}} C^M_{\beta}$$

is a decomposition into affine spaces. Therefore $\chi(\text{Gr}_{\mathcal{B}}(M))$ equals the number of non-empty cells Schubert cells $C^M_{\beta}$. Since $F$ can be chosen to be strictly ordered, there exists a retract $\pi^M_{F,\beta}$ to the inclusion $i^M_{F,\beta} : C^N_{\beta} \to C^M_{\beta}$, see Proposition 4.1. Therefore the cell is $C^M_{\beta}$ is non-empty if and only if $C^N_{\beta}$ is non-empty. By Theorem 3.2 (i), the latter cell is non-empty if and only if $N_\alpha(\beta_p) \subseteq \beta_q$ for all arrows $\alpha : p \to q$ in $T$. This recovers Theorem 1 of [3].

Example 6.3 (Tree modules). More generally, let $M$ be an ordered tree module over $Q$ (cf. Section 5.3). By Corollary 5.9 there is a winding $F : T \to Q$ from a quiver $T$ that is a tree and a thin sincere and indecomposable $T$-module $N$ with ordered basis $\mathcal{B}$ such that $M = F_* N$, such that $F$ is ordered and such that

$$\text{Gr}_{\mathcal{B}}(M) = \bigsqcup_{\beta \in \mathcal{B} \text{ of type } \mathcal{E}} C^M_{\beta}$$

is a decomposition into affine spaces. We assume further that $M$ is strictly ordered, i.e. that $F$ is a strictly ordered morphism, to assure the existence of the retract $\pi^M_{F,\beta}$, cf. Proposition 4.1. By the same reasoning as in the previous example, the Euler characteristic of $\text{Gr}_{\mathcal{B}}(M)$ equals the number of non-empty Schubert cells $C^N_{\beta}$ of the $T$-module $N$.

By Theorem 3.2 (i), the cell $C^N_{\beta}$ is non-empty if and only if $N_\alpha(\beta_p) \subseteq \beta_q$ for all arrows $\alpha : p \to q$ in $T$. Since $N$ is thin sincere and indecomposable, the sets $\mathcal{B}_p$ are of cardinality 1 for all $p \in q_0$ and the linear maps $N_\alpha$ are isomorphisms for all arrows $\alpha$ of $T$. Therefore the cell $C^M_{\beta}$ is non-empty if and only if the support $T_\beta$ of the linear span of $\beta$ is a successor closed subquiver of $T$, i.e. a subquiver such that there is no arrow $\alpha$ in the complement $T - T_\beta$ with source $s(\alpha)$ in $T_\beta$.

For a subquiver $S$ of $T$, we define its dimension vector $\text{dim} S = (d_p)_{p \in T_0}$ by $d_p = 1$ if $p \in S_0$ and $d_p = 0$ if not. Then the Euler characteristic of $\text{Gr}_{\mathcal{B}}(M)$ equals the number of successor closed subquiver $S$ of $T$ with $F(\text{dim} S) = \mathcal{E}$. This is Corollary 3.1 of [10] for strictly ordered windings $F$.

Example 6.4 (Push-forwards of monomial representations of forests). Let $T$ be a forest and $M$ a monomial $T$-module. Let $F : T \to Q$ be a strictly ordered winding and $\mathcal{E}$ a dimension vector for $Q$. By Theorem 5.5 (i), there exists an ordered basis $\mathcal{B}$ of $M$ such that $M_\alpha(\mathcal{B}_p) \subseteq \mathcal{B}_q \cup \{0\}$ for all arrows $\alpha : p \to q$ of $T$. Then the same arguments of the previous example establish the formula

$$\chi(\text{Gr}_{\mathcal{B}}(M)) = \# \{ \beta \in \mathcal{B} \text{ of type } \mathcal{E} \mid M_\alpha(\mathcal{B}_p) \subseteq \mathcal{B}_q \cup \{0\} \text{ for all } \alpha : p \to q \in T_1 \}. $$
Example 6.5 (Tree extensions of the Kronecker quiver). Let $T$ be a tree extension of the Kronecker quiver $S = 1 \rightarrow 2$ and $M$ a $T$-module with ordered basis $\mathcal{B}$. Let $F : T \rightarrow Q$ be a strictly ordered winding that is ordered above $S$ and whose restriction $F|_S : S \rightarrow Q$ to $S$ is injective. Assume that be the restriction $M_S$ of $M$ to $S$ is a regular representation of the Kronecker quiver, i.e. the two linear maps of $M_S$ are given by the identity matrix resp. a maximal Jordan block $J(\lambda)$ w.r.t. the basis $B_S = B \cap M_S$ of $M_S$.

Then $\text{Gr}_S(F_*M_S) = \text{Gr}_S(M_S)$ decomposes into affine spaces $X_L = C_{M_S}^{\text{red}}$, (cf. Example 2.6).

6.2. Regular decompositions. The following definition makes sense for all rings $k$. We will apply it in the following theorem to the case of the complex numbers $k = \mathbb{C}$. Let

$$\varphi : \coprod_{i \in I} C_i \longrightarrow X$$

be a decomposition of $X$ into locally closed subschemes $C_i$. A locally closed subscheme $Z$ of $X$ decomposes (w.r.t. $\varphi$) if there is a subset $I_Z \subset I$ such that $\varphi$ restricts to a decomposition

$$\coprod_{i \in I_Z} C_i \longrightarrow Z.$$ 

Note that this is a purely topological property of $Z$. Note further that $I_Z$ is uniquely determined if all cells $Z_i$ are non-empty. The decomposition $\varphi \coprod Z_i \rightarrow X$ is regular if the closures of all cells $Z_i$ (for $i \in I$) decompose w.r.t. $\varphi$. This extends the notion of a regular torification from Section 6.2 of [11].

**Theorem 6.6.** Let $X$ be a complex projective scheme of dimension $m$ whose irreducible components $X_1, \ldots, X_n$ are smooth. Let $\varphi : \coprod_{i \in I} \mathbb{A}^{d_i} \rightarrow X$ be a regular decomposition into affine spaces. Then the following is true.

(i) The classes of the closures $\overline{A_i^{d_i}}$ of the cells of dimension $d_i = d$ form an additive basis of $H^{2m-2d}(X, \mathbb{Z})$ for all integers $d$.

(ii) The cohomology in odd degrees vanishes. Thus the classes of the closures $\overline{A_i^{d_i}}$ of all cells form an additive basis for the cohomology ring $H^*(X, \mathbb{Z})$.

(iii) The embeddings $\iota_i : X_i \rightarrow X$ define a graded inclusion

$$(t_1^n, \ldots, t_n^n) : H^*(X, \mathbb{Z}) \longrightarrow \bigoplus_{i=1}^n H^*(X_i, \mathbb{Z})$$

of rings.

**Proof.** We make an induction on the number $n$ of irreducible components of $X$. If $n = 1$, then $X$ is smooth projective. Thus (i) and (ii) follow from what we have explained in the beginning of Section 6 while (iii) is trivial.
Let $n > 1$. Then $X$ is the union of two closed subsets $Y$ and $X_n$ for $Y = X_1 \cup \ldots \cup X_{n-1}$. Let $i_Y : Y \to X$ and $i_X : X_n \to X$ be the embeddings of $Y$ resp. $X_n$ into $X$ and let $j_Y : Z \to Y$ and $j_X : Z \to X_n$ be the embeddings of $Z = Y \cap X_n$ into $Y$ resp. $X_n$. We consider the Mayer-Vietoris sequence

$$\cdots \to H^d(X, \mathbb{Z}) \xrightarrow{(i_Y^*, i_X^*)} H^d(Y, \mathbb{Z}) \oplus H^d(X_n, \mathbb{Z}) \xrightarrow{(j_Y^*, j_X^*)} H^d(Z, \mathbb{Z}) \to \cdots$$

Let $X_i$ be an irreducible component of $X$ and $C_i = \mathbb{A}^{d_i}$ a cell of the decomposition $\varphi$. If $X_i$ intersects $C_i$ non-trivially, then $C_i$ is contained in $X_i$ since $C_i$ is irreducible. This shows that $X_i$ decomposes w.r.t. $\varphi$. Since $X_i$ is closed in $X$, it contains also the closure $\overline{C_i}$ of each of its cell $C_i \subset X_i$. It follows that for all $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$, the intersection $X_{i_1} \cap \cdots \cap X_{i_r}$ decomposes w.r.t. $\varphi$ and contains the closure of each of its cells $C_i$.

If $[\overline{C_i}]$ is the cohomology class of the closure of $C_i$, then the homomorphism $i_Y^* : H(\mathbb{X}, \mathbb{Z}) \to H(Y, \mathbb{Z})$ sends $[\overline{C_i}]$ to the class of $\overline{C_i}$ if $\overline{C_i}$ is a subscheme of $Y$, or to 0 if not. The analog statement is true for $i_X$, $j_Y$ and $j_X$. By the induction hypothesis, $H^d(Y, \mathbb{Z})$, $H^d(X_n, \mathbb{Z})$ and $H^d(Z, \mathbb{Z})$ are freely generated by the classes of the closures of the cells $C_i$ of dimension $d'$ that are contained in $Y$, $X_n$ resp. $Z$ if $d = 2m - 2d'$ is even, and they are 0 if $d$ is odd. Therefore the homomorphism $(j_Y^*, -j_X^*) : H^d(Y, \mathbb{Z}) \oplus H^d(X_n, \mathbb{Z}) \to H^d(Z, \mathbb{Z})$ is surjective for every degree $d$, which means that the Mayer-Vietoris sequence splits into short exact sequences

$$0 \to H^d(X, \mathbb{Z}) \xrightarrow{(i_Y^*, i_X^*)} H^d(Y, \mathbb{Z}) \oplus H^d(X_n, \mathbb{Z}) \xrightarrow{(j_Y^*, j_X^*)} H^d(Z, \mathbb{Z}) \to 0.$$

The image of the span of the classes $[\overline{C_i}]$ of $H^d(X, \mathbb{Z})$ under $(i_Y^*, i_X^*)$ is a free subgroup of $H^d(Y, \mathbb{Z}) \oplus H^d(X_n, \mathbb{Z})$ and it equals the kernel of $(j_Y^*, -j_X^*)$. Therefore, $H^d(X, \mathbb{Z})$ is freely generated by the classes $[\overline{C_i}]$ for which $d = 2m - 2 \dim C_i$, and it is zero if $d$ is odd, by the induction hypothesis. This establishes (i) and (ii).

Since both $i_Y^* : H^*(\mathbb{X}, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ and $i_X^* : H^*(\mathbb{X}, \mathbb{Z}) \to H^*(X_n, \mathbb{Z})$ are ring homomorphisms by the induction hypothesis, (iii) follows. This completes the proof of the theorem. \(\square\)

**Remark 6.7.** In general, the hypothesis that the decomposition is regular cannot be omitted from Theorem 6.6. For instance, the union of two projective lines that intersect in two points can be decomposed into two affine lines. However, the cohomology of this projective scheme is not generated by the classes of the closures of these two affine lines.

However, Schubert decompositions of quiver Grassmannians have particular properties, and it might be that the assertion of regularity can be dropped for Schubert decompositions of quiver Grassmannians.
Remark 6.8. The inclusion in claim (iii) of Theorem 6.6 is the initial part of an exact sequence of \( \mathbb{Z} \)-modules of the form

\[
0 \longrightarrow H^\ast(X, \mathbb{Z}) \xrightarrow{(i_1, \ldots, i_n)} \bigoplus_{l=1}^n H^\ast(X_l, \mathbb{Z}) \longrightarrow \bigoplus_{1 \leq l_1 < l_2 \leq n} H^\ast(X_{l_1, l_2}, \mathbb{Z}) \longrightarrow \]

\[
\vdots \longrightarrow \bigoplus_{1 \leq l_1 < \cdots < l_{n-1} \leq n} H^\ast(X_{l_1, \ldots, l_{n-1}}, \mathbb{Z}) \longrightarrow H^\ast(X_1, \ldots, n, \mathbb{Z}) \longrightarrow 0
\]

where \( X_{l_1, \ldots, l_r} = X_{l_1} \cap \cdots \cap X_{l_r} \) and the homomorphisms are defined as alternating sum of restriction maps \( H^\ast(X_{l_1, \ldots, l_r}, \mathbb{Z}) \rightarrow H^\ast(X_{l_1, \ldots, l_{r+1}}, \mathbb{Z}) \), similar to those that appear in the definition of singular cohomology or Čech cohomology.

6.3. Examples and conjectures. In this section, we will describe some examples (and counterexamples) of Schubert decompositions of quiver Grassmannians that are regular. Everything can be considered over an arbitrary base ring \( k \).

Example 6.9 (Usual Grassmannians and flag varieties). It is well-known that the Schubert decomposition of a usual Grassmannian or, more generally, of a flag variety is regular (cf. Exercise 13 of §9.4 and p. 159 in [9]). In our notation, this fact takes the following shape.

Let \( e = (e_1, \ldots, e_r) \) be the type of the flag variety \( X \) of subspaces in \( k^m \). Let \( Q \) be a quiver of the form \( 1 \rightarrow \cdots \rightarrow r \) and \( M \) the \( Q \)-module \( k^m \xrightarrow{id} \cdots \xrightarrow{id} k^m \). Then \( \text{Gr}_e(M) \) is isomorphic to \( X \). If we order the standard basis \( B = \{ b_{k,p} | k = 1, \ldots, m; p = 1, \ldots, r \} \) of \( M \) lexicographically, then the decomposition

\[
\text{Gr}_e(M) = \bigcap_{\beta \subset B \text{ of type } e} C^M_\beta
\]

coincides with the usual decomposition of \( X \) into Schubert cells, cf. Example 2.4.

There is a natural action of \( \text{GL}_m \) on \( \text{Gr}_{e_p}(M_p) \) for each \( p \in Q_0 \), and thus a diagonal action on the flag variety \( \text{Gr}_e(M) \). The orbits of the upper triangular Borel subgroup \( B \) of \( \text{GL}_m \) coincide with the Schubert cells \( C^M_\beta \). Since the closure of an orbit is decomposes into orbits, the Schubert decomposition of \( \text{Gr}_e(M) \) is regular. More precisely, we have

\[
\overline{C^M_\beta} = \bigcap_{\gamma \leq \beta} C^M_\gamma
\]

(cf. Section 2 for the definition of \( \gamma \leq \beta \)).

This example generalizes to Schubert cells of products of Grassmannians. Since the Schubert cell of a general quiver Grassmannian \( \text{Gr}_e(M) \) is the pull-back of a Schubert cell of the product Grassmannian \( \text{Gr}_e(\dim M) \) (see Section 2), we have the following fact.

Lemma 6.10. Let \( Q \) be a quiver and \( M \) a \( Q \)-module with dimension vector \( m \) and ordered basis \( B \). If \( \beta \) and \( \gamma \) are dimension vectors of type \( e \) and \( \overline{C^M_\beta} \cap \overline{C^M_\gamma} \neq \emptyset \), then \( \gamma \leq \beta \).
Example 6.11 (Representations of forests). Let $Q$ be a forest with $\kappa$ vertices and $M$ a $Q$-module with ordered basis $\mathcal{B}$ such that $M_\alpha$ is the identity matrix for all arrows $\alpha$ of $Q$. By Theorem 3.3 there is a sequence $Q^{(1)} \subset \cdots \subset Q^{(\kappa)} = Q$ of subquivers and a sequence

$$\Phi : \text{Gr}_e(M) \xrightarrow{\varphi_{\kappa}} \cdots \xrightarrow{\varphi_2} \text{Gr}_e(M^{(1)}) \xrightarrow{\varphi_1} \text{Spec } k$$

of fibre bundles $\varphi_i$ whose fibres are Grassmannians $\text{Gr}(\tilde{e}_i, \tilde{m}_i)$ for certain $\tilde{e}_i \leq \tilde{m}_i$ and $i = 1, \ldots, \kappa$. Here $M^{(i)}$ and $e^{(i)}$ are the restrictions of $M$ resp. $e$ to $Q^{(i)}$. Since every fibre has a regular Schubert decomposition such that the closure of each cell decomposes into the cells with smaller index, we obtain

$$\overline{C^M_\beta} = \prod_{\gamma \leq \beta} C^M_\gamma,$$

which generalizes Example 6.9.

Example 6.12 (Monomial representations). Let $Q$ be a forest and $M$ a monomial $Q$-module. Not every choice of ordering of a basis $\mathcal{B}$ for a monomial $Q$-module yields a regular Schubert decomposition. For instance, reconsider Example 2.7, i.e. the quiver $1 \rightarrow 2$ and $Q$-module $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : k^2 \rightarrow k^2$, together with the standard ordered basis $\mathcal{B} = \{b_1, \ldots, b_4\}$ of $M = M_1 \oplus M_2$. We use the shorthand notation $C_{i,j} = C^M_{\{b_i, b_j\}}$.

We saw in Example 2.7 that $C_{1,3} \simeq \mathbb{A}^0$, $C_{2,3} \simeq C_{2,4} \simeq \mathbb{A}^1$ and $C_{1,4} = \emptyset$. Inspecting the parameters of the sets of $K$-rational points, we see that the closure of $C_{2,4}$ contains a single point of $C_{2,3}$, see Figure [I]

As explained in Example 2.7 a reordering of $b_1$ and $b_2$ (resp. $b_3$ and $b_4$) corresponds to a reordering of the rows (resp. columns) of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We illustrate the Schubert decompositions of all permutations of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in Figure [I] An isolated dot means that the corresponding cell is a single point, i.e. $\mathbb{A}^0$. A line stands for an affine line $\mathbb{A}^1$ whose origin is illustrated as the dot connected to one end of the line. The dot close-by to the other end of a line stands for the closure of the affine line.

![Figure 1. Four different Schubert decompositions of $\text{Gr}_e(M)$](image)

As can be seen in the figure, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ yield decompositions into affine spaces, while the decompositions for $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are regular. In particular, there is
exists ordering for $M$ (corresponding to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) that yields a regular Schubert decomposition into affine spaces.

**Example 6.13** (Degenerate flag varieties). Cerulli, Feigin and Reineke identify in [5] degenerate flag varieties of Dynkin type with certain quiver Grassmannians $\text{Gr}_\beta(M)$ and establish a regular decomposition into the finitely orbits of the action of a certain Borel subgroup of the automorphism group of $M$.

In this example, we consider the case of a complete degenerate flag variety $F^a_\xi$ of flags of type $\xi = (1, \ldots, n)$ in $k^{n+1}$, which can be identified with the quiver Grassmannian $\text{Gr}_\beta(P \oplus I)$ where $P$ is the direct sum over all indecomposable projective $Q$-modules and $I$ is the direct sum over all indecomposable injective $Q$-modules for the equioriented quiver $Q$ of type $A_n$. In Section 7.2 of [5], the reader finds a detailed description of the orbits of $B$ for this case. We will see that this decomposition coincides indeed with the Schubert decomposition w.r.t. a certain choice of ordered basis. It seems to be interesting to work out the connection for the general degenerate flag variety of Dynkin type.

Let $Q = 1 \to \cdots \to n$ be the underlying equioriented quiver of type $A_n$. For $i = 1, \ldots, n$, let $P_i$ be the indecomposable projective $Q$-module with support $i \to \cdots \to n$ and let $I_i$ be the indecomposable injective $Q$-module with support $1 \to \cdots \to i$. Then $P = \bigoplus_{i=1}^n P_i$ and $I = \bigoplus_{i=1}^n I_i$.

The $k$-module $P_{i,j}$ is trivial if $j < i$ and of rank one if $j \geq i$, in which case we denote the corresponding basis vector by $b_{i,j}^P$. The $k$-module $I_{i,j}$ is trivial if $j > i$ and of rank one if $j \leq i$, in which case, we denote the corresponding basis vector by $b_{i,j}^I$. The set

$$\mathcal{B} = \{ b_{i,j}^P \mid 1 \leq i \leq j \leq n \} \cup \{ b_{i,j}^I \mid 1 \leq j \leq i \leq n \}$$

is a basis for $M = P \oplus I$.

We order $\mathcal{B}$ as follows. First note that the relative order of the subsets $\mathcal{B}_i$ (where $i \in Q_0$) is irrelevant for the shape of the Schubert cell $C^M_\beta$ of $\text{Gr}_\beta(M)$, cf. Remark 2.1. For a vertex $i$, we order $\mathcal{B}_i$ by $b_{i,i}^P < \cdots < b_{i,r}^P < b_{i,1}^P < \cdots < b_{i,i}^I$. Then the Schubert cell $C^M_\beta$ coincides with the cell $C_{L_i}$ of [5] (as defined in Section 7.2), and $C^M_\beta$ coincides with the intersection of $\text{Gr}_\beta(M)$ with the product $\prod_{i=1}^n C_{L_i}$ in $\text{Gr}_\beta(m)$. By Theorem 7.11 in [5], this cell coincides with an orbit of the action of $B$ on $\text{Gr}_\beta(M)$.

In the notation of this paper, we can identify this Schubert decomposition with the Schubert decomposition of $\text{Gr}_\beta(M')$ w.r.t. $\mathcal{B}$ where $M'$ is the $Q$-module

$$k^{n+1} \xrightarrow{J(0)} k^{n+1} \xrightarrow{J(0)} \cdots \xrightarrow{J(0)} k^{n+1}$$

and $\mathcal{B}$ is the standard ordered basis of $M'$ (recall that $J(0)$ is a maximal Jordan block with 0 on the diagonal and 1 on the upper side diagonal).

Based on the last two examples and further calculations, I expect that the following statements are true.
Conjecture 6.14. Let $Q$ be a forest and $M$ a $Q$-module with ordered basis $\mathcal{B}$ such that for every arrow $\alpha$ of $Q$, the matrix $M_\alpha$ is a block matrix $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ where $I$ is a square identity matrix. Then $\text{Gr}_\varepsilon(M) = \bigsqcup C_\beta$ is a regular decomposition into affine spaces for every dimension vector $\varepsilon$.

Conjecture 6.15. Let $T$ be a tree extension of $S$ and $M$ a $T$-module such that $M_\alpha$ is an isomorphism for all arrows $\alpha$ in $T - S$. Let $M_S$ be the restriction of $M$ to $S$ and $\mathcal{B}$ be an ordered basis of $M$ that is ordered above $\mathcal{B}_S = \mathcal{B} \cap M_S$. Let $\beta \subset \mathcal{B}$ of type $\varepsilon$ and $\beta_S = \beta \cap M_S$ of type $\varepsilon_S$. Then the following holds true.

(i) If $\overline{C_{\beta_S}^{M_S}}$ decomposes into Schubert cells, then also $\overline{C_\beta^M}$ decomposes into Schubert cells.

(ii) If $\overline{C_{\beta_S}^{M_S}}$ decomposes into Schubert cells and if $\overline{C_{\beta_S}^{M_S}} = \bigsqcup_{\gamma_S \leq \beta_S} C_{\gamma_S}^{M_S}$, then $\overline{C_\beta^M} = \bigsqcup_{\beta' \leq \beta} C_{\beta'}^M$.

In particular, if $\text{Gr}_\varepsilon(S)(M_S) = \bigsqcup C_{\beta_S}^{M_S}$ is a regular decomposition, then so is $\text{Gr}_\varepsilon(M) = \bigsqcup C_\beta^M$.

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