QUIVER GRASSMANNIANS OF TYPE \(\tilde{D}_n\),

PART 2: SCHUBERT DECOMPOSITIONS AND \(F\)-POLYNOMIALS

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ABSTRACT. Extending the main result of [14], in the first part of this paper we show that every quiver Grassmannian of a representation of a quiver of type \(\tilde{D}_n\) has a decomposition into affine spaces. In the case of real root representations of small defect, the non-empty cells are in one-to-one correspondence to certain, so called non-contradictory, subsets of the vertex set of a fixed tree-shaped coefficient quiver. In the second part, we use this characterization to determine the generating functions of the Euler characteristics of the quiver Grassmannians (resp. \(F\)-polynomials). Along these lines we obtain explicit formulae for the cluster variables of type \(\tilde{D}_n\). In the case of indecomposables of large defect these formulae are in line of the well-known multiplication formula in the theory of cluster algebras.

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INTRODUCTION

In this paper we continue the consideration of quiver Grassmannians of type \(\tilde{D}_n\) initiated in [14]. Denoting the unique imaginary Schur root by \(\delta\), there it is shown that every quiver Grassmannian of a real root representation of dimension \(\alpha\) with \(\langle \delta, \alpha \rangle = -1\) has a cell decomposition into affine spaces. It is also shown that this is true for every indecomposable representation lying in an exceptional tube and, moreover, for every Schur representation of dimension \(\delta\).

Passing to dual representations, this result can be easily extended to all indecomposable real root representations of dimension \(\alpha\) of small defect, i.e. \(|\langle \delta, \alpha \rangle| \leq 1\). We use this result to obtain the first main result of this paper, which says that this statement is in fact true for every representation of type \(\tilde{D}_n\).

The focus of the second part of the paper is on the generating functions of the Euler characteristics of quiver Grassmannians (resp. \(F\)-polynomials) of indecomposable representations of \(\tilde{D}_n\), i.e.
for a fixed representation $M$ of $Q$, we consider

$$F_M(x) = \sum_{e \in M_{Q_0}} \chi(\text{Gr}_e(X))x^e.$$ 

Initially, we use the combinatorial description of the non-empty cells in [14] to obtain explicit formulae for the $F$-polynomials of representations of small defect. Subsequently, the results of the first part can be used to obtain explicit formulae for the $F$-polynomials of all indecomposable representations of dimension $\alpha$ of large defect, i.e. $|\langle \delta, \alpha \rangle| = 2$. This representation-theoretical approach is in line with the multiplication formula of [6].

Thanks to the Caldero-Chapoton-formula, see [4] and [5], these results can be used to obtain an explicit description of the cluster variables of the mutation finite cluster algebras of type $\tilde{D}_n$. As far as mutation finite cluster algebras are concerned, results of this nature were only known for cluster algebras of type $A$, which include those of type $\tilde{A}_n$, see [8] and [12]. Moreover, since the shape of the obtained formulae is indeed easy, it seems that there are generalizations to other mutation finite cluster algebras.

**Schubert decomposition.** In order to obtain the Schubert decompositions of quiver Grassmannians of indecomposable real root representations $M$ of small defect, it is necessary to consider the coefficient quivers $\Gamma_M$ listed in [14, Appendix B]. Recall that every subset $\beta \subset (\Gamma_M)_0$ of cardinality $e$ defines a possibly empty Schubert cell $C^M_\beta \subset \text{Gr}_e(M)$ induced by the Schubert decompositions of the product of usual Grassmannians $\prod_{q \in Q_0} \text{Gr}_e(M_q)$. The first aim of this paper is to generalize Theorem 4.4 of [14] to all indecomposable representations of $\tilde{D}_n$, i.e.:

**Theorem A.** Let $M$ be an indecomposable representation of $\tilde{D}_n$. Then there exists a coefficient quiver $\Gamma_M$ of $M$ such that the Schubert decomposition $\text{Gr}_e(M) = \bigsqcup C^M_\beta$ is a decomposition into affine spaces and empty cells. Here $\beta$ runs through all subsets of $(\Gamma_M)_0$ of cardinality $e$.

The generalization of Theorem 4.4 of [14] to representations of large defect and to representations of the homogeneous tubes is subject of section 1. Since the quiver Grassmannians of representations lying in the homogeneous tubes behave similar to those of large defect, throughout the paper, we exclude them when referring to representations of small defect. While the construction of the cell decomposition of quiver Grassmannians of representations of small defect is highly combinatorial, in the cases of large defect the main idea is to consider exact sequences which are close to being almost split. It turns out that every indecomposable representation $B$ of large defect can be written as the middle term of such a sequence between indecomposables $M$ and $N$ of small defect. In particular, there exists a coefficient quiver of $B$ with vertex set $(\Gamma_M)_0 \cup (\Gamma_N)_0$ where $\Gamma_M$ and $\Gamma_N$ are those considered in [14]. Generalizations of results of [4] can be used to show that this setup preserves cell decompositions in such a way that every pair of subsets $(\beta, \beta')$ of $(\Gamma_M)_0 \times (\Gamma_N)_0$ determines a (possibly empty) cell $C^B_{\beta, \beta'}$ of a certain quiver Grassmannian of the middle term which turns out to be an affine space. Since all cells can be obtained using this construction, this already proves that every quiver Grassmannian of indecomposables of large defect has a cell decomposition into affine spaces, see Theorem 1.15:
Theorem B. Let $B$ be a real root representation of defect $-2$. Then there exist indecomposable representations $M$ and $N$ of defect $-1$ and respective coefficient quivers $\Gamma_M$ and $\Gamma_N$ such that the (induced) Schubert decomposition

$$\text{Gr}_e(B) = \bigsqcup_{(\beta, \beta')} C^B_{(\beta, \beta')}$$

is a decomposition into affine spaces and empty cells. Here $(\beta, \beta')$ runs through all non-contradictory subsets of $(\Gamma_M)_0 \times (\Gamma_N)_0$ such that the cardinalities of $\beta$ and $\beta'$ sum up to $e$.

There is also a very explicit description in terms of the Auslander-Reiten quiver of those pairs corresponding to an empty cell. As shown in [4] in the case of almost split sequences there is only one such pair, consisting of the cokernel and the trivial subrepresentation.

With similar methods we can also show that every quiver Grassmannian of an indecomposable representation lying in one of the homogeneous tubes has a cell decomposition into affine spaces, see Theorem 1.22.

In this paper and also in [14] we consider preprojective representations rather than preinjective ones. In section 1.8, we prove that passing to the opposite quiver and to dual representations Schubert decompositions are preserved. Thus all results can be transferred to the case of preinjective representations in the natural way.

Theorems A and B have strong implications on the geometry of $\text{Gr}_e(M)$. In particular, the closures of the non-empty Schubert cells form an additive basis for the singular cohomology ring of $\text{Gr}_e(M)$ and they show that the cohomology is concentrated in even degree. Therefore, we can compute the Euler characteristic of $\text{Gr}_e(M)$ as

$$\chi(\text{Gr}_e(M)) = \#\{\beta \subset (\Gamma_M)_0 \text{ of type } e \text{ such that } C^M_{\beta} \text{ is not empty}\}.$$

The construction of the cell decomposition in terms of those of the representations of small defect yields a description of the $F$-polynomial of the indecomposables of large defect, see Theorem 1.17. As already mentioned, in terms of cluster algebras this result translates to the well-known multiplication formula of [6]. As far as cluster variables are concerned, we are thus left with the determination of $F$-polynomials of indecomposables of small defect. The investigation of $F$-polynomials and the derivation of explicit formulae is the main topic of sections 2, 3 and 4.

Calculation of $F$-polynomials. The $F$-polynomials of representations of the homogeneous tubes play an important role in the formulae for $F$-polynomials of indecomposable representations. Since they only depend on the dimension vector and are independent of the chosen tube, we may denote them by $F_{r\delta}$. Considering the cell decompositions into affine spaces, we first obtain a recursive formula for $F_{r\delta}$ which can be used to obtain an explicit formula in terms of $F_{\delta}$ in Corollary 4.12. More detailed, we have

$$F_{r\delta} = \frac{1}{2z}(\lambda^*_+ - \lambda^*_-)$$

where

$$z = \frac{1}{2} \sqrt{F_{\delta}^2 - 4x\delta}, \quad \lambda_\pm = \frac{F_{\delta}}{2} \pm z.$$
Besides the combinatorial description of the non-empty cells, there are two other main ingredients which are used to obtain explicit formulae for the \( F \)-polynomials of indecomposable representations. The first one is studied in section 2. The main idea is to reduce the determination of the \( F \)-polynomials to smaller quivers, i.e. \( \tilde{D}_n \) for \( n \leq 6 \). Here we use that most linear maps of indecomposable representations of \( \tilde{D}_n \) for large \( n \) are isomorphisms. In combination with the reflection functor introduced in [2], it turns out that this is a powerful tool. In section 3, we review the reflection functor and its consequences for quiver Grassmannians, which were also studied in [17] and [10].

If \( \tilde{D}_n \) is in subspace orientation, we are left with counting admissible subsets as defined in section 4.7. Since the coefficient quivers under consideration follow a certain recursion and since the description of these subsets is very easy (and again easier for \( n \leq 6 \)), we get recursive formulae for the \( F \)-polynomials. It turns out that these recursive formulae can be used to obtain explicit formulae for all \( F \)-polynomials of real root representations of small defect in section 4. All these explicit formulae are in terms of the \( \delta \)-polynomials of representations whose dimension is smaller than \( \delta \). Since there also exists an explicit formula for \( F_{r\delta} \) in terms of \( F_{\delta} \), we are left with the easy task of calculating \( \delta \)-polynomials of representations of dimension \( \alpha \leq \delta \). While these \( \delta \)-polynomials do depend on the orientation of \( \tilde{D}_n \), the upshot is that the formulae for the remaining \( F \)-polynomials turn out to be independent of the orientation.

In order to state the main result of the second part, we need some notation. If \( \alpha \) is a real root we denote by \( M_\alpha \) the unique indecomposable representation of this dimension and by \( F_\alpha \) the corresponding \( F \)-polynomial. In the tube of rank \( t \) there exist \( t \) chains of irreducible morphisms

\[
M_{0,1} \hookrightarrow M_{0,2} \hookrightarrow \ldots \hookrightarrow M_{0,t-1} \hookrightarrow M_{1,0} := M_\delta \hookrightarrow M_{1,1} \hookrightarrow \ldots
\]

where the \( m_l(r) := \dim M_{r,l} \) are real roots and the imaginary root representations \( M_{r,0} := M_{r\delta} \) are uniquely determined by this chain. In particular, for every real root \( \alpha \) in the tube of rank \( t \) there exists an exceptional root \( m_l(0) \) such that \( \alpha = r\delta + m_l(0) \). Under the convention that \( F_\delta > 0 \) if \( \alpha \in \mathbb{N} \mathbb{Q}_0 \) has at least one negative coefficient and setting \( m_r(0) := \delta \), we obtain the second main result of this paper, see Theorems 1.17, 4.14, 4.18 and 4.25:

**Theorem C.**

(i) For the representations \( M_{m_l(r)} \) where \( l = 0, \ldots, t-1 \) (lying in the exceptional tube of rank \( t \)), we have

\[
F_{m_l(r)} = F_{m_l(0)} F_{r\delta} + x^{m_l+1(0)} F_{m_{r-l}(0) - m_{l+1}(0)} F_{(r-1)\delta}.
\]

(ii) Let \( M \) be preprojective of defect \(-1\) such that \( t_M = \dim M - r\delta \leq \delta \). If \( \delta - t_M \) is injective we have

\[
F_M = F_{t_M} F_{r\delta} - x^\delta F_{(r-1)\delta}.
\]

If \( \delta - t_M \) is not injective we have

\[
F_M = F_{t_M} F_{r\delta} - x^{\tau^{-1} t_M} F_{\delta - \tau^{-1} t_M} F_{(r-1)\delta}.
\]

Here \( \tau \) is the Auslander-Reiten-translation.
Let $B$ be an indecomposable representation of defect $-2$. Then there exist indecomposable representations $M$ and $N$ of defect $-1$ such that

$$F_B = F_N F_M - x \dim \tau^{-1}M F_{N/\tau^{-1}M}.$$  

Passing to the dual, we obtain analogous formulae for indecomposable representations of positive defect.

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1. Schubert decomposition of quiver Grassmannians

One main result of this section is that every quiver Grassmannian of a representation of a quiver of type $\tilde{D}_n$ of large defect has a cell decomposition into affine spaces, see section 1.6. With similar methods we can show that this is also the case for representations of the homogeneous tubes, see section 1.7. This can be used later to obtain formulae for the $F$-polynomials.

In order to prove this, we first introduce some notation in sections 1.1 and 1.2. In section 1.3, we recall some results from the theory of cluster algebras which are linked to our considerations. In sections 1.4 and 1.5 we state some lemmas that are important for the proof of the main results. Finally, we show in section 1.8 how the results can be used to pass from representations of negative defect to representations of positive defect (resp. from preprojectives to preinjectives).

1.1. Quiver representations. We fix $k = \mathbb{C}$ as our ground field. This suffices for the application of the results to cluster algebras. Actually, all results concerning the representation theory of quivers and quiver Grassmannians of representations remain true when passing to any algebraically closed field $k$.

We shortly review some basics on quiver representations, see [1] and [9] for more details. Let $Q = (Q_0, Q_1)$ be a quiver with vertices $Q_0$ and arrows $Q_1$ denoted by $p \overset{v}{\rightarrow} q$ or $v : p \rightarrow q$ for $p, q \in Q_0$. We assume that $Q$ has no oriented cycles. In most parts of this paper we consider quivers $Q$ of extended Dynkin type $\tilde{D}_n$, i.e. the underlying graph of $Q$ is

A vertex $p \in Q_0$ is called sink if there does not exist an arrow $p \overset{v}{\rightarrow} q \in Q_1$. A vertex $q \in Q_0$ is called source if there does not exist an arrow $q \overset{v}{\rightarrow} p \in Q_1$. For an arrow $p \overset{v}{\rightarrow} q$, let $s(v) = p$ and $t(v) = q$. We denote by $a(p, q)$ the number of arrows from $p$ to $q$. For a vertex $p \in Q_0$, let

$$N_p := \{ q \in Q_0 \mid \exists p \overset{v}{\rightarrow} q \in Q_1 \lor \exists q \overset{v}{\rightarrow} p \in Q_1 \}$$

be the set of neighbors of $p$. Consider the abelian group $\mathbb{Z}Q_0 = \bigoplus_{q \in Q_0} \mathbb{Z}q$ and its monoid of dimension vectors $\mathbb{N}Q_0$. A finite-dimensional complex representation $M$ of $Q$ is given by a tuple

$$M = ((M_q)_{q \in Q_0}, (M_v)_{v \in Q_1} : M_p \rightarrow M_q)$$
of finite-dimensional complex vector spaces and $k$-linear maps between them.

Let $\text{Rep}(Q)$ denote the category of finite-dimensional representations of $Q$. The dimension vector $\text{dim}M \in \mathbb{N}Q_0$ of $M$ is defined by $\text{dim}M = \sum_{q \in Q_0} \text{dim}_k M_q q$. Let $R_{\alpha}(Q)$ denote the affine space of representations of dimension $\alpha$. Moreover, we denote by $Q^{\text{op}}$ the quiver obtained from $Q$ when turning around all arrows. Taking dual vector spaces and adjoint linear maps for each arrow, we obtain the dual representation $M^{\ast}$ of $Q^{\text{op}}$ for every representation $M$ of $Q$.

A dimension vector $\alpha$ is called a root if there exists an indecomposable representation of this dimension. It is called Schur root if there exists a representation with trivial endomorphism ring with this root as dimension vector. A representation $M$ with $\alpha = \text{dim}M$ is called exceptional if we have $\text{Ext}(M, M) = 0$. In the case of real roots, there only exists one indecomposable representation $M$ up to isomorphism having $\alpha$ as dimension vector. We denote this representation by $M_{\alpha}$. We denote by $S_q$ the simple representation corresponding to the vertex $q$ and by $s_q$ its dimension vector.

On $\mathbb{Z}Q_0$ we have a non-symmetric bilinear form, the Euler form, which is defined by

$$\langle \alpha, \beta \rangle = \sum_{q \in Q_0} \alpha_q \beta_q - \sum_{p \twoheadrightarrow q \in Q_1} \alpha_p \beta_q$$

for $\alpha, \beta \in \mathbb{Z}Q_0$.

Recall that for two representations $M, N$ of $Q$ we have

$$(1.1) \quad \langle \text{dim}M, \text{dim}N \rangle = \text{dim}_k \text{Hom}(M, N) - \text{dim}_k \text{Ext}(M, N)$$

and $\text{Ext}^{i}(M, N) = 0$ for $i \geq 2$. For two representation $M$ and $N$, define $[M, N] := \text{dimHom}(M, N)$. As usual let $M^\perp = \{ N \in \text{Rep}(Q) \mid \text{Hom}(M, N) = \text{Ext}(M, N) = 0 \}$.

If $Q$ is of extended Dynkin type, we denote by $\delta$ the unique imaginary Schur root which is actually independent of the orientation. Following [9, section 7], the defect of a representation $M$ is defined as $\delta(M) := \langle \delta, \text{dim}M \rangle$. Clearly the defect is additive on dimension vectors. For indecomposables, we have $|\delta(M)| \leq 2$. We say that an indecomposable representation $M$ has small defect if $|\delta(M)| \leq 1$ and large defect if $|\delta(M)| = 2$. As already mentioned we exclude the representations from the homogeneous tubes when referring to representations of small defect.

1.2. Coefficient quivers. We introduce coefficient quivers and tree modules following the presentation given in [15]. Let $Q$ be a quiver, $\alpha \in \mathbb{N}Q_0$ a dimension vector and $M$ with $\text{dim}M = \alpha$ a representation of $Q$. A basis of $M$ is a subset $\mathcal{B}$ of $\bigoplus_{q \in Q_0} M_q$ such that

$$\mathcal{B}_q := \mathcal{B} \cap M_q$$

is a basis of $M_q$ for all vertices $q \in Q_0$. For every arrow $p \xrightarrow{\nu} q$, we may write $M_{\nu}$ as a $(\alpha_q \times \alpha_p)$-matrix $M_{\nu, \mathcal{B}}$ with coefficients in $k$ such that the rows and columns are indexed by $\mathcal{B}_q$ and $\mathcal{B}_p$ respectively. If

$$M_{\nu}(b) = \sum_{b' \in \mathcal{B}_q} \lambda_{b', b} b'$$

with $\lambda_{b', b} \in k$ and $b \in \mathcal{B}_p$, we obviously have $(M_{\nu, \mathcal{B}})_{b', b} = \lambda_{b', b}$.

Definition 1.1. The coefficient quiver $\Gamma(M, \mathcal{B})$ of a representation $M$ with a fixed basis $\mathcal{B}$ has vertex set $\mathcal{B}$ and arrows between vertices are defined by the condition: if $(M_{\nu, \mathcal{B}})_{b', b} \neq 0$, there
exists an arrow \((v, b, b') : b \to b'\). If \(\mathcal{B}\) is ordered linearly, we say that \(\Gamma(M, \mathcal{B})\) is an ordered coefficient quiver.

A representation \(M\) is called a tree module if there exists a basis \(\mathcal{B}\) for \(M\) such that the corresponding coefficient quiver is a tree.

In order to shorten notation, we sometimes denote an arrow \((v, b, b')\) by \(v\) where \(p \xrightarrow{v} q\) is the corresponding arrow of the original quiver.

1.3. **Quiver Grassmannians, Cluster algebras and \(F\)-polynomial.** For a representation \(M\) with \(m = \dim M\), the quiver Grassmannian \(\text{Gr}_e(M)\) is the set of subrepresentations \(U\) of \(M\) with \(\dim U = e\). It is a closed subvariety of the product \(\prod_{q \in Q_0} \text{Gr}(e_q, m_q)\) of the usual Grassmannians \(\text{Gr}(e_q, m_q)\).

Let \(\mathbb{Q}[x_q^{\pm 1} \mid q \in Q_0]\) be the \(\mathbb{Q}\)-algebra of Laurent polynomials in the variables \(x_q\) for \(q \in Q_0\). Denoting by \(\chi\) the Euler characteristic in singular cohomology, as in [4], we set

\[
X_M = \sum_{e \in \mathbb{N}Q_0} \chi(\text{Gr}_e(M)) \prod_{q \in Q_0} x_q^{-(e, s_q) - (s_q, m - e)}.
\]

With \(Q\) we can associate a cluster algebra \(\mathcal{A}(Q)\), see [11] for more details, and its cluster category \(\mathcal{C}_Q\) introduced in [3]. We cite [5, Theorem 4]:

**Theorem 1.2.** The correspondence \(M \mapsto X_M\) provides a bijection between the set of indecomposable objects of \(\mathcal{C}_Q\) without self-extensions and the set of cluster-variables of \(\mathcal{A}(Q)\).

Actually, this bijection restricts to a bijection between indecomposable exceptional representations of \(Q\) and cluster variables of \(\mathcal{A}(Q)\) excluding the initial variables. In [6, Theorem 2], which generalizes [4, Proposition 3.10], the following multiplication formula is shown:

**Theorem 1.3.** Let \(M\) and \(N\) be indecomposable objects of \(\mathcal{C}_Q\) such that \(\dim \text{Ext}_{\mathcal{C}_Q}(M, N) = 1\). Then we have

\[
X_M X_N = X_B + X'_B
\]

where \(B\) and \(B'\) are up to isomorphism the unique middle terms of the non-split triangles

\[
N \to B \to M \to SN, \quad M \to B' \to N \to SM.
\]

Note that we have

\[
\dim \text{Ext}_{\mathcal{C}_Q}(M, N) = \dim \text{Ext}(M, N) + \dim \text{Ext}(N, M),
\]

see [3]. Moreover, if \(\text{Ext}(M, N) = k\), the middle term \(B\) is the one induced by the non-splitting sequence in the module category. But since \(\text{Ext}(N, M) = 0\) in this case, using the terminology of [5], the middle term \(B'\) is just an object of \(\mathcal{C}_Q\). But it actually has a corresponding representation in the module category which can be determined explicitly.

In this paper, we mostly consider the generating function \(F_M\) of the Euler characteristics of the corresponding quiver Grassmannians of \(M\), also called \(F\)-polynomial, i.e.

\[
F_M(x) = \sum_{e \in \mathbb{N}Q_0} \chi(\text{Gr}_e(M)) x^e
\]
where \( x^e = \prod_{q \in Q_0} x_q^{e_q} \) for \( e \in \mathbb{N}Q_0 \), see also [10]. It is closely related to the cluster variables \( X_M \). Indeed, setting

\[
m'_q = \sum_{p \in Q_0} a(p, q)m_p - m_q
\]

and considering the variable transformation \( x_q \mapsto x'_q \) with

\[
x'_q = \prod_{p \in Q_0} x_p^{a(q, p) - a(p, q)},
\]

it is straightforward to check that we have

\[
X_M = x^{m'_f} F_M(x').
\]

1.4. Short exact sequences and quiver Grassmannians. As already mentioned, the first aim of this paper is to prove that every quiver Grassmannian of an indecomposable representation of large defect has a cell decomposition into affine spaces. To do so we write representations of large defect as the middle term of certain short exact sequences between indecomposables of small defect. Then we can combine Theorem A with the following observations relating the quiver Grassmannians of the middle term to those of the outer terms. In general, given two representations \( M, N \) and an exact sequence

\[
0 \rightarrow M \xrightarrow{i} B \xrightarrow{\pi} N \rightarrow 0,
\]

following [4, section 3], this yields a morphism of algebraic varieties

\[
\Psi_e : \text{Gr}_e(B) \rightarrow \coprod_{f + g = e} \text{Gr}_f(M) \times \text{Gr}_g(N), \quad U \mapsto (i^{-1}(U), \pi(U)).
\]

Note that we have \( i^{-1}(U) \cong U \cap M \) and \( \pi(U) \cong (U + M)/M \cong U/U \cap M \).

The following is shown in the course of the proof of [4, Lemma 3.11] in the case of almost split sequences. Actually, the same proof applies in our situation:

**Lemma 1.4.** If \( \Psi_e^{-1}(A, V) \) is not empty, we have \( \Psi_e^{-1}(A, V) = \mathbb{A}^{[V, M/A]} \).

**Remark 1.5.** Let \( M \) and \( N \) be representations of \( Q \) with ordered coefficient quivers \( \Gamma_M \) and \( \Gamma_N \). Assume that there exists an exact sequence

\[
0 \rightarrow M \xrightarrow{i} B \xrightarrow{\pi} N \rightarrow 0
\]

such that the induced coefficient quiver \( \Gamma_B \) of \( B \) is obtained by adding exactly one extra arrow that connects \( \Gamma_N \) to \( \Gamma_M \). This induces an ordering on \( \Gamma_B \). Clearly we have \( B_q = M_q \oplus N_q \) and \( (\Gamma_B)_0 = (\Gamma_M)_0 \cup (\Gamma_N)_0 \). Every subrepresentation \( U \) of \( N \) lies in a Schubert cell \( C^N_B \) for some subset \( \beta \subset (\Gamma_N)_0 \) of type \( \dim U \). We can consider \( \beta \) also as a subset of \( (\Gamma_B)_0 \). If \( U \) lifts to a subrepresentation of \( B \), i.e. \( \oplus_{q \in Q_0} U_q \subset \oplus_{q \in Q_0} B_q \) defines a subrepresentation of \( B \), the corresponding Schubert cell \( C^B_\beta \) is also not empty. This is for instance the case if \( \text{Ext}(U, M) = 0 \).

If, additionally, \( N \) and \( B \) are indecomposable real root representations of \( \bar{D}_n \) such that we have \( |\delta(N)|, |\delta(B)| \leq 1 \) and the coefficient quivers are those fixed in [14, Appendix B], the corresponding \( \beta \)-state \( \Sigma_\beta \) of the Schubert system \( \Sigma \) is non-empty if and only if it is non-contradictory of the first
and second kind. In this case, we also say that the $\beta$-state lifts. Note that the dimensions of the corresponding Schubert cells $C^N_\beta$ and $C^B_\beta$ may differ. This is because, the fibre $\Psi^{-1}_{\dim U}(0, U)$ is isomorphic to $\mathbb{A}^{\dim [U,M]}$ by Lemma 1.4.

1.5. **Quiver Grassmannians of exceptional regular representations.** For the remaining part of this section, $Q$ is assumed to be of extended Dynkin type $\tilde{D}_n$. In order to prove the main result of section 1, we need some properties concerning the quiver Grassmannians of exceptional regular representations. More detailed, we need that the cell decomposition of [14, Theorem 4.4] is compatible with the decomposition of subrepresentations into direct sums of regular and preprojective representations. To do so we consider the basis and coefficient quivers respectively of the exceptional regular representations lying in the tubes of rank 2 and $n - 2$ respectively treated in [14, Appendix B].

**Proposition 1.6.** Let $M$ be an exceptional regular representation of $\tilde{D}_n$. Considering the coefficient quivers of [14, Appendix B], we have: if there exists $U \in C^M_\beta$ such that $U \cong R \oplus T$ with $R$ regular and $T$ preprojective, this is true for all $U \in C^M_\beta$.

**Proof.** First recall the shape of the exceptional tubes listed in [14, Appendix B]. If $M$ lies in a tube of rank 2, this is clearly true because $M$ has no regular subrepresentation. In the tube of rank $n - 2$ there exist $n - 2$ chains of irreducible inclusions

$$M^j_1 \subset M^j_2 \subset \ldots \subset M^j_{n-3}$$

of exceptional regular representations. Thus we have $M \cong M^j_i$ for some $i \in \{1, \ldots, n - 3\}$ and $j \in \{1, \ldots, n - 2\}$. We proceed by induction on $i$. If $i = 1$, we have that $M^j_1$ has only preprojective subrepresentations and the claim follows.

If $e$ is the dimension vector of a regular subrepresentation of $M$, we have that $e = \dim M^j_l$ with $l \leq i$. Moreover, $e$ is an exceptional root and thus by [7, Corollary 4] we have

$$\dim \text{Gr}_e(M) = \langle e, \dim M - e \rangle = \dim \text{Hom}(M_e, M) - 1 = 0.$$  

In particular, there exists a unique subset of the vertex set of the coefficient quiver of $M$ corresponding to $\text{Gr}_e(M)$. Moreover, there exists a short exact sequence

$$0 \to M^j_i \to M \to M^j_k \to 0$$

with $k + l = i$ where we set $M^j_0 = 0$. By construction of the coefficient quivers in [14, Appendix B], we obtain the coefficient quiver of $M$ by gluing the one of $M^j_k$ by an outgoing arrow to the one of $M^j_i$. 
Assume that $U \cong M^j_i \oplus T$ is a subrepresentation of $M$ such that $T$ is preprojective. Since $\text{Ext}(T, M^j_i) = 0$, we obtain a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M^j_i \\
& \downarrow & \downarrow \\
0 & \longrightarrow & M^j_i \oplus T \\
& & {}^\text{i}_T \uparrow \\
& & 0 \\
\end{array}
$$

In particular, $T$ lies in a cell $C^{M^j_i}_{\beta}$ defined by a subset of the vertex set of the coefficient quiver of $M^j_i$. Moreover, the non-contradictory $\beta$-state defined by $T$ lifts to a non-contradictory $\beta$-state of $\Gamma_M$, see Remark 1.5. By induction hypothesis, we have that all representations in the cell $C^{M^j_i}_{\beta}$ of $T$ decompose into preprojective representations. Moreover, we have $\Psi^{-1}_{\dim M^j_i + \dim T}(M^j_i, T) = \mathbb{A}^0$. This shows that the lifted cell has the same dimension. Thus every representation in the cell of $U$ decomposes into a direct sum of $M^j_i$ and a preprojective representation. \hfill $\square$

1.6. **Representations of large defect.** The main aim of this section is to show that the Schubert decomposition of indecomposable representations of small defect obtained in [14, Theorem 4.4] extends to a Schubert decomposition of indecomposable representation of large defect. It turns out that similar methods can be applied to show that every quiver Grassmannian coming along with a representation lying in one of the homogeneous tubes has a cell decomposition into affine spaces.

As already mentioned we can restrict to the case of preprojective roots. Recall that the preprojectives of defect $-1$ are precisely the Auslander-Reiten translates of the projectives corresponding to the outer vertices, i.e. of $P_qa$, $P_qb$, $P_qc$, and $P_qd$. The preprojectives of defect $-2$ are Auslander-Reiten translates of projectives corresponding to the inner vertices, i.e. of $P_{q_0}, \ldots, P_{q_{n-4}}$.

**Remark 1.7.**

(i) An indecomposable preprojective representation $M$ has no proper factor $N$ such that $\delta(N) \leq \delta(M)$. This follows because the defect is additive on exact sequences and preprojective representations have only preprojective subrepresentations.

(ii) If $M$ is preprojective with $\delta(M) = -1$ and $N$ is preprojective, then every non-zero morphism $f : M \rightarrow N$ is injective. Indeed, since $\text{Im}(f)$ is a subrepresentation of $N$, we have $\delta(\text{Im}(f)) \leq -1 \leq \delta(M)$. Since $\text{Im}(f)$ is a factor of $M$, the first part yields $\text{Im}(f) = M$.

(iii) If $M$ and $N$ are preprojective and there is no path from $M$ to $N$ in the Auslander-Reiten quiver we have $\text{Hom}(M, N) = \text{Ext}(N, M) = 0$. The first statement is clear, the second follows by the Auslander-Reiten formula $\dim \text{Ext}(N, M) = \dim \text{Hom}(M, \tau N)$. Indeed there is a path from $\tau N$ to $N$ and thus no path from $M$ to $\tau N$ by assumption.

For $\tilde{D}_n$ in subspace orientation, there exists almost split sequences of the form

$$0 \rightarrow \tau^{-l} P_p \rightarrow \tau^{-l} P_{q_0} \rightarrow \tau^{-l} P_p \rightarrow 0, \quad 0 \rightarrow \tau^{-l} P_{p'} \rightarrow \tau^{-l} P_{q_{n-4}} \rightarrow \tau^{-l} P_{p'} \rightarrow 0$$

for $p \in \{q_a, q_b\}$, $p' \in \{q_c, q_d\}$ and $l \geq 1$. In this case the initial part of the preprojective component of the Auslander-Reiten quiver looks as follows
where we use the abbreviations $P_i := P_{q_i}$. If $n$ is even, the remaining part of the preprojective component is obtained from this and looks for every orientation as follows:

where the usual arrows indicate monomorphisms and the $\tilde{P}_i$ are Auslander-Reiten translates of $P_i$. In subspace orientation we have

$$\tilde{P}_0 = \tau^{-\frac{n+3}{2}}P_0, \quad \tilde{P}_2 = \tau^{-\frac{n+1}{2}}P_2, \quad \ldots \quad \tilde{P}_{n-6} = \tau^{-2}P_{n-6}, \quad \tilde{P}_{n-4} = \tau^{-1}P_{n-4}.$$
If $n$ is odd, the remaining part of the preprojective component is obtained from this and looks for every orientation as follows:

\[
\begin{align*}
\tilde{P}_0 & \quad \cdots \quad \tilde{P}_2 \\
\vdots & \quad \cdots \quad \vdots \\
\tilde{P}_{n-5} & \quad \cdots \quad \tilde{P}_{n-3} \\
\tilde{P}_c & \quad \cdots \quad \tilde{P}_d \\
\end{align*}
\]

where the usual arrows indicate monomorphisms and the $\tilde{P}_i$ are Auslander-Reiten translates of $P_i$. In subspace orientation we have

\[
\tilde{P}_0 = \tau^{-\frac{n+3}{2}} P_0, \quad \tilde{P}_2 = \tau^{-\frac{n+1}{2}} P_2, \ldots \quad \tilde{P}_{n-5} = \tau^{-2} P_{n-5}, \quad \tilde{P}_{n-3} = \tau^{-1} P_{n-3}, \quad \tilde{P}_c = P_c, \quad \tilde{P}_d = P_d.
\]

If $N$ is preprojective with $\delta(N) = -1$, we denote by $\hat{N}$ its neighbor in the Auslander-Reiten quiver satisfying $\delta(\hat{N}) = -1$. In subspace orientation, for $\tau^{-s} P_d$ we have

\[
\tau^{-s} P_a = \tau^{-s} P_b \quad \text{and} \quad \tau^{-s} P_c = \tau^{-s} P_d
\]

and the corresponding relations when permuting $a, b$ and $c, d$ respectively.

For a representation $C$ with $\delta(C) = -1$, we define $\rho C := \tau^{-1} C$ and $\kappa C := \rho C$. Then we get a chain of irreducible inclusions

\[
C \subset \rho C \subset \rho^2 C \subset \ldots \subset \rho^k C \subset \ldots
\]

Let $\mathcal{C}C$ denote the full subcategory of $\text{Rep}(Q)$ which contains the objects $\rho^l C / \rho^k C$ for $l > k \geq 1$ and which is closed under exact sequences and images.

**Lemma 1.8.**

(i) The category $\mathcal{C}C$ is equivalent to the full subcategory of $\text{Rep}(Q)$ whose objects are the representations of the tube of rank $n-2$.

(ii) For $1 \leq l \leq n-3$ and $m \geq 0$, we have

\[
\dim \text{Ext}(\kappa^{m(n-2)+l} C, C) = m + 1, \quad \dim \text{Hom}(C, \kappa^{m(n-2)+l} C) = m.
\]
Proof. For two representations $M, N \in \text{Rep}(Q)$, we have $\text{Hom}(M, N) = \text{Hom}(\tau^{-1}M, \tau^{-1}N)$ and $\text{Ext}(M, N) = \text{Ext}(\tau^{-1}M, \tau^{-1}N)$ by the Auslander-Reiten formulae, see [1, Theorem IV.2.13]. Thus we can assume that $C = P_a$ and the first part of the lemma is straightforward.

For the second part, one observes that $\dim \kappa_m(n - 2)C = \dim C + m\delta$ for $m \geq 0$. Since we have $\text{Ext}(M, N) = 0$ or $\text{Hom}(M, N) = 0$ for two preprojective representations and since $\delta(C) = -1$, the claim follows by formula (1.1).

Given two representations $M, N$ and an exact sequence

$$0 \to M \xrightarrow{i} B \xrightarrow{\pi} N \to 0,$$

we consider the morphism of algebraic varieties

$$\Psi_e : \text{Gr}_e(B) \to \prod_{f + g = e} \text{Gr}_f(M) \times \text{Gr}_g(N), \; U \mapsto (i^{-1}(U), \pi(U))$$

defined in section 1.4. If non-empty, the dimensions of the fibres depend on the direct sum decomposition of the subrepresentations of $M$ and $N$. In general, it is already difficult to say in which cases the fibre is empty. But in the case of representations of large defect there are sequences which are close to being almost split so that the fibres can be determined in any case. This extends the following result, see [4, Lemma 3.11] and [7, Proposition 2].

**Theorem 1.9.** Let $M$ be a representation of $Q$ and $\tau M$ be its Auslander-Reiten translate. Consider the almost split sequence

$$0 \to \tau M \to B \to M \to 0.$$

Then we have

$$\Psi_e^{-1}(A, V) = \begin{cases} \emptyset & \text{if } (A, V) = (0, M) \\ \mathbb{A}^{[V, \tau M/A]} & \text{otherwise.} \end{cases}$$

In particular, we have

$$\chi(\text{Gr}_e(B)) = \begin{cases} \sum_{f + g = e} \chi(\text{Gr}_f(M)) \chi(\text{Gr}_g(\tau M)) & \text{if } e \neq \dim M \\ \sum_{f + g = e} \chi(\text{Gr}_f(M)) \chi(\text{Gr}_g(\tau M)) - 1 & \text{if } e = \dim M. \end{cases}$$

In general, a representation of large defect cannot be written as the middle term of an almost split sequence. But we can modify the preceding statement to make it applicable for our purposes. If $B$ is indecomposable preprojective of defect $-2$, in the Auslander-Reiten quiver exist the following
Here $M$ and $N$ are two indecomposable preprojective representations of $\hat{D}_n$ of defect $-1$. More precisely we have:

**Lemma 1.10.** Every indecomposable preprojective representation $B$ with $\delta(B) = -2$ is obtained as the middle term of an exact sequence

$$0 \rightarrow M \rightarrow B \rightarrow N \rightarrow 0$$

such that $N = \kappa^l M \in M^\perp$ with $l \leq n - 3$, $\text{Ext}(N, M) = k$ and $\text{Hom}(N, M) = 0$.

**Proof.** The representation $B$ is an Auslander-Reiten translate of a projective representation corresponding to an inner vertex $q_i$. The claim follows by Lemma 1.8 together with the Auslander-Reiten-formulae. Indeed, we can assume that $M = P_a$. 

In the following, we refer to the indecomposable representations lying properly in the above triangle or corresponding to a point $T \neq B$ on the path from $B$ to $N$ as $(M, N)$-inner representation. The remaining ones, i.e. those which are outside the triangle or on the path from $M$ to $B$, as $(M, N)$-outer representations. We drop $(M, N)$ if it is clear which representations are considered. Finally, if a $(M, N)$-inner (resp. $(M, N)$-outer) representation is also a subrepresentation of $N$, we call it inner (resp. outer) subrepresentation of $N$ if we fixed a triangle. In order to investigate such a triangle, the Auslander-Reiten formulae assure that we can mostly without loss of generality assume that $M = P_a$ and $N = \kappa^l M$.

We state some observations which are obtained from considering the Auslander-Reiten quiver in the case of quivers of type $\hat{D}_n$ or general Auslander-Reiten theory, see for instance [1, sections IV, VIII.2]:

**Remark 1.11.**

(i) If $C$ and $Z$ are indecomposable preprojective representations, every morphism $f \in \text{Hom}(C, Z)$ is a finite composition of irreducible morphisms. Thus we have $\text{Hom}(C, Z) = 0$ if there is no path from $C$ to $Z$. In particular, we see that $\text{Hom}(C, B) = \text{Hom}(C, M) = 0$ for every inner representation $C$.

(ii) By a dimension consideration, we see that $N$ has no proper subrepresentation which is isomorphic to an outer representation lying on the border of the triangle. Furthermore, by induction on $l$, we see that $N$ has no inner subrepresentation of defect $-2$ because all representations on the border except $M$ have defect $-2$. 

\[ \begin{array}{cccccc}
M & \rightarrow & \kappa M & \rightarrow & \ldots & \rightarrow \kappa^{l-1} M \\
\downarrow & & \downarrow & & \downarrow & \\
\rho M & \rightarrow & \ldots & \rightarrow & \rho^{l-1} M \\
\downarrow & & \downarrow & & \downarrow & \\
B & & \ldots & \ldots & \ldots & \end{array} \]
Lemma 1.12. 

(i) The inner subrepresentations $C$ of $N$ are precisely the representations $C = \kappa^i M$ for $i = 1, \ldots, l$.

(ii) If $A$ is a non-zero subrepresentation of $M$ and $V$ is any subrepresentation of $N$ we have $\text{Ext}(V, M/A) = 0$.

(iii) If $0 \neq V \subseteq N$, the corresponding injection either factors through $B$ or $V \cong C \oplus L$ where $C$ is an inner subrepresentation of $N$ with $\text{Hom}(C, B) = \text{Hom}(C, M) = 0$.

(iv) If $C \oplus L \subseteq N$ such that $C$ is an inner subrepresentation, we have that $L \subseteq N/C$ is preprojective, and, moreover, using the notation of [14, Theorem 4.4], we have

$$\{V' \in \text{Gr}_{\dim C + \dim L}(N) \mid V' \cong C \oplus L', L' \text{ preprojective}\} \cong \{L' \in \text{Gr}_{\dim L}(N/C) \mid L' \text{ preprojective}\}$$

is a union of cells $C^N_{\beta}$ of the cell decomposition of $\text{Gr}_{\dim C + \dim L}(N)$ into affine spaces.

Proof. Consider the chain of inner representations

$$\kappa M \subset \kappa^2 M \subset \ldots \subset \kappa^l M = N.$$ 

Apart from these representations the only representations of defect $-1$ in the triangle are of the form $\rho^i M$ for $i = 1, \ldots, l$. But since $M \subset \rho^i M$, they are no subrepresentations of $N$. Thus we get (i).

Since $N$ is preprojective, $V$ is also preprojective. By the preceding remark, $M/A$ has no preprojective direct summand. Thus we get (ii).

The third part is just a reformulation of parts of Remarks 1.7 and 1.11. Note that we have $\text{Hom}(C, B) = 0$ and thus $B$ has no subrepresentation isomorphic to $C \oplus L$ where $C$ is an inner subrepresentation.

By Lemma 1.8, we have that $N/C$ is an exceptional regular representation. In addition, it follows that $\text{Hom}(C, N/C) = \text{Ext}(C, N/C) = 0$. Now [7, Corollary 4] yields $\text{Gr}_{\dim C}(N) = \{\text{pt}\}$. Moreover, since we have $N = \rho^m C$ for some $m \leq n - 4$, by construction of the coefficient quiver $\Gamma_N$ of $N$ in [14, Appendix B], this subrepresentation corresponds to the full subquiver $\Gamma_C$ of $\Gamma_N$ which consists of the first $\dim C$ vertices. Since $C$ is a subrepresentation of $N$ and $N/C$ a factor, the full subquiver $\Gamma_{N/C}$ consisting of the vertices $(\Gamma_N)_0 \setminus (\Gamma_C)_0$ is connected to $\Gamma_C$ by an outgoing arrow.

If $L \subset N/C$ has a regular direct summand, we have $\text{Hom}(L, N) = 0$. In particular $L \oplus C$ is no subrepresentation of $N$. Note that, if $L$ is regular indecomposable, we can consider the inclusion

(iii) If $Z$ is an outer representation, an injective morphism $f \in \text{Hom}(Z, N)$ with $f \neq 0$ factors through $B$. To see this, first one observes that every such mono factors through an outer representation $N'$ which lies on the path from $M$ to $B$ and we thus have $M \subset N' \subset B$. Since $\text{Hom}(M, N) = 0$, we have $N' \neq M$ and thus $\delta(N') = -2$. By the Auslander Reiten formulae, it follows $M \in N'^\perp$ which yields $\text{Hom}(N', B) \cong \text{Hom}(N', N)$. Thus if an injection factors through $N'$ it already factors through $B$.

(iv) Actually, we can assume that $N = \kappa^l M$ with $l \leq n/2 - 1$. This is because the two lower rows of the Auslander-Reiten quiver behave dual to the upper ones. In particular, if $n = 4, 5$ we can assume that $N = \tau^{-1} M$ and, moreover, we are in the situation of Theorem 1.9.

Using the notation of Lemma 1.10, we have

\begin{align*}
\text{(i)} & \quad \text{The inner subrepresentations } C \text{ of } N \text{ are precisely the representations } C = \kappa^i M \text{ for } i = 1, \ldots, l. \\
\text{(ii)} & \quad \text{If } A \text{ is a non-zero subrepresentation of } M \text{ and } V \text{ is any subrepresentation of } N \text{ we have } \text{Ext}(V, M/A) = 0. \\
\text{(iii)} & \quad \text{If } 0 \neq V \subseteq N, \text{ the corresponding injection either factors through } B \text{ or } V \cong C \oplus L \text{ where } C \text{ is an inner subrepresentation of } N \text{ with } \text{Hom}(C, B) = \text{Hom}(C, M) = 0. \\
\text{(iv)} & \quad \text{If } C \oplus L \subseteq N \text{ such that } C \text{ is an inner subrepresentation, we have that } L \subseteq N/C \text{ is preprojective, and, moreover, using the notation of [14, Theorem 4.4], we have } \\
\{V' \in \text{Gr}_{\dim C + \dim L}(N) \mid V' \cong C \oplus L', L' \text{ preprojective}\} \cong \{L' \in \text{Gr}_{\dim L}(N/C) \mid L' \text{ preprojective}\} \\
\text{is a union of cells } C^N_{\beta} \text{ of the cell decomposition of } \text{Gr}_{\dim C + \dim L}(N) \text{ into affine spaces.}
\end{align*}
\[ \text{Hom}(L, N/C) \hookrightarrow \text{Ext}(L, C). \] Then we even have that the middle term \( C' \) of the corresponding sequence is also an inner representation, see also Remark 1.13.

Thus assume that \( L \subseteq N/C \) is preprojective. By Proposition 1.6, we have that there exist cells \( C_{\beta_1}^{N/C}, \ldots, C_{\beta_n}^{N/C} \) such that the subsets \( \beta_i \) have cardinality \( e \) and such that all representations in these cells are preprojective. Therefore we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C & \longrightarrow & N & \longrightarrow & N/C & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & V & \longrightarrow & L & \longrightarrow & 0
\end{array}
\]

where \( V \) is a preprojective representation. If \( \text{Ext}(L, C) = 0 \), we have \( V \cong C \oplus L \). Thus the subrepresentation \( L \) lifts to a subrepresentation of \( N \). Moreover, we have \( \Psi_{\dim C + \dim L}(C, L) = A^0 \) so that the \( \beta \)-states corresponding to \( L \) in \( N/C \) are in bijection with the one of \( C \oplus L \) in \( N \), see also Remark 1.5.

We claim that \( \text{Ext}(L, C) = 0 \) for all preprojective subrepresentations \( L \subseteq N/C \). We can without loss of generality assume that \( L \) is indecomposable, \( N \in \{ \tau^{-1}P_a, \tau^{-1}P_b \} \) and that \( C = \tau^{-r}P_a \) where \( 1 \leq r \leq l \leq n - 3 \). In particular, we have \( \delta > \dim N \geq \dim C \). If \( L = \kappa'C \) for some \( n - 3 \geq l \geq 1 \), the representation \( V \) would be an indecomposable inner representation with \( \delta(V) = -2 \). Indeed, in this case we have \( C \in \perp V \). But since \( N \) has no subrepresentations of defect \(-2 \), this is not possible. Also \( l \geq n - 2 \) is not possible because then we had \( \dim L > \delta > \dim N/C \).

We have \( \dim \text{Ext}(L, C) = \dim \text{Hom}(C, \tau L) \) by the Auslander–Reiten formulae. If \( \text{Ext}(L, C) \neq 0 \), since \( C \) is of defect \(-1 \), by the second part of Remark 1.7, it follows that \( C \subseteq \tau L \). If \( \delta(L) = -2 \), there exists a chain of inclusions

\[ \tau L \subseteq L \subseteq \tau^{-1}L \subseteq \ldots \]

In particular, we have \( \dim L \geq \dim \tau L \geq \dim C \). But for \( C = \tau^{-r}P_a \) we either have \( \dim C_{q_a} \geq 1 \) or \( \dim C_{q_b} \geq 1 \) and thus \( \dim V_{q_c} \geq 2 > \delta_{q_a} \geq \dim N_{q_a} \) or \( \dim V_{q_b} \geq 2 > \delta_{q_b} \geq \dim N_{q_b} \). But this is not possible because \( V \) is a subrepresentation of \( N \).

Thus it remains to deal with the case if \( L \in \{ \tau^{-1}P_a, \tau^{-1}P_b \} \) for some \( l \geq 0 \). There exists a \( r \geq 1 \) such that \( C \subseteq \tau^{-s}P_{n-4} \) and \( C \not\subseteq \tau^{-t}P_{n-4} \) for \( s \geq r \) and \( r - 1 \geq t \geq 0 \). Moreover, there exists almost-split sequences

\[
0 \to \tau^{-k}P_c \to \tau^{-r}P_{n-4} \to \tau^{-(k+1)}P_c \to 0, \quad 0 \to \tau^{-k'}P_d \to \tau^{-r}P_{n-4} \to \tau^{-(k'+1)}P_d \to 0
\]

for some \( k, k' \geq 0 \). By the choice of \( r \), we have \( \text{Hom}(C, \tau^{-k}P_c) = \text{Hom}(C, \tau^{-k'}P_d) = 0 \) because otherwise there were a path of irreducible morphism from \( C \) to \( \tau^{-k}P_c \) which were forced to factor through \( \tau^{-r+1}P_{n-4} \). Thus, keeping in mind the second part of Remark 1.7, it follows that \( C \subseteq \tau^{-l}P_c \), \( \tau^{-l}P_d \) for all \( l > k, k' \) and the claim follows as in the case of \( \delta(L) = -2 \).

\[ \square \]

**Remark 1.13.** By the results of this section, it follows that every subrepresentation \( C \oplus L \subseteq N \) such that the fibre of \( (0, C \oplus L) \) is empty are in bijection with the subrepresentations of \( N/\tau^{-1}M \). To check this it suffices to keep in mind that every inner representation is obtained as the middle term of an exact sequence between a regular subrepresentation of \( N/\tau^{-1}M \) and \( \tau^{-1}M \).
This lemma enables us to prove several properties concerning the morphism between the involved quiver Grassmannians induced by the exact sequence under consideration:

**Proposition 1.14.**  
(i) The fibre $\Psi^{-1}_e(A,V)$ is empty if and only if $A = 0$ and $V \cong C \oplus L$ where $C$ is an inner subrepresentation and $L = 0$ or $L \subseteq N/C$ is preprojective.

(ii) If $\Psi^{-1}_e(A,V)$ is not empty, we have $\Psi^{-1}_e(A,V) = \mathbb{A}^{[V,M/A]}$.

(iii) For all subrepresentations $V \subseteq N$ and $A \subseteq M$ of a fixed dimension, we have $[V,M/A] = (\dim V, \dim M/A)$. In particular, the dimensions of the non-empty fibres of $\Psi_e$ only depend on the dimension vectors of $V$ and $A$ respectively.

**Proof.** Most parts of the proof are adopted from [4, Lemma 3.11]. Let $U \subseteq B$. If $i^{-1}(U) = 0$, we have $\pi(U) \cong V$ and thus the first part is just a reformulation of the last two parts of Lemma 1.12.

If $V$ is direct sum of outer representations, by the third part of Lemma 1.12, every injection $V \hookrightarrow N$ factors through $B$. In this case the second part of the claim follows by Lemma 1.4.

Thus let $C$ be an inner subrepresentation of $N$ and $A \neq 0$. Then we have $C = \kappa^m M$ with $1 \leq m \leq l \leq n - 3$. Since $\text{Hom}(N/C, M) = 0$, because $N/C$ is exceptional regular by Lemma 1.8, considering the appropriate long exact sequence, we obtain $\text{Ext}(N/C, M) = 0$ and thus $\text{Ext}(C, M) \cong \text{Ext}(N, M) = k$. Since the representation $C$ is preprojective and $M/A$ has no preprojective direct summand, we have $\text{Ext}(C,M/A) = 0$ and thus we get a surjection

$$\text{Ext}(C,A) \twoheadrightarrow \text{Ext}(C,M) \cong \text{Ext}(N,M).$$

In particular, we get a commutative diagram

\[
\begin{array}{ccccc}
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & N & \longrightarrow & 0 \\
& & i_A & & & \uparrow i_C & & & \\
0 & \longrightarrow & A & \longrightarrow & U & \longrightarrow & C & \longrightarrow & 0
\end{array}
\]

showing that $\Psi^{-1}_e(A,C) \neq \emptyset$ where $e = \dim U$. Furthermore, by Lemma 1.4 we have $\Psi^{-1}_e(A,C) = \mathbb{A}^{[C,M/A]}$. Finally, if $V = C \oplus L$ is a direct sum of an inner subrepresentation and outer subrepresentations and $A \neq 0$, we can combine both cases in order to show that the fibre is not empty. Moreover, we can conclude again that $\Psi^{-1}_e(A,V) = \mathbb{A}^{[V,M/A]}$. Thus the second part follows.

If $A \neq 0$, we have $\text{Ext}(V,M/A) = 0$ because $M/A$ has no preprojective direct summand by the preceding considerations. If $A = 0$ and if the fibre is not empty, by the first part, $V$ is forced to be a direct sum of outer subrepresentation. Let $V \cong V_1 \oplus \ldots V_r$ be its direct sum decomposition. Since $V_i$ does also not lie on the border, by the second part of Remark 1.11, there is no path from $M$ to $V_i$ in the Auslander-Reiten quiver for all $1 \leq i \leq r$. Thus we have $\text{Ext}(V_i, M) = 0$ and, thus, $\dim \text{Hom}(V_i, M) = \langle V_i, M \rangle$.

The considerations of this section together with [14, Theorem 4.4] now yield that there exists a cell decomposition for every quiver Grassmannian attached to preprojective representations (resp. preinjective representations).

**Theorem 1.15.** Let $B \in \text{Rep}(Q)$ be an indecomposable preprojective representation with $\delta(B) = -2$. Then there exist two preprojective representations $M$ and $N = \kappa^l M$ with $\delta(M) = \delta(N) = -1$.
and a short exact sequence

\[ 0 \to M \to B \to N \to 0 \]

such that for \( \Psi_e \) we have

\[ \Psi_e^{-1}(A, V) = \begin{cases} 0 & \text{if } A = 0, V \cong C \oplus L, C \text{ an } (M,N)\text{-inner subrepresentation} \\ A_{(\dim V, \dim M/A)} & \text{otherwise} \end{cases} \]

Moreover, \( \Psi_e^{-1}(A, V) \) is constant over \( C^M_{\beta} \times C^N_{\beta'} \subseteq \text{Gr}_{\dim A}(M) \times \text{Gr}_{\dim V}(N) \) for each pair \((\beta, \beta')\) of type \((\dim A, \dim V)\) and \( \text{Gr}_e(B) \) has a cell decomposition into affine spaces.

Using Theorem 1.24, we obtain:

**Corollary 1.16.** Let \( B \in \text{Rep}(Q) \) be a preinjective representation with \( \delta(B) = 2 \). Then every quiver Grassmannian \( \text{Gr}_e(B) \) has a cell decomposition into affine spaces.

The results of this section can now be used to obtain the \( F \)-polynomials of indecomposable representation \( B \) of large defect. It is straightforward to check that, in terms of cluster variables, this corresponds to the multiplication formula, see Theorem 1.3:

**Theorem 1.17.** Let \( B \) be an indecomposable representation with \( \delta(B) = -2 \). If \( 0 \to M \to B \to N \to 0 \) is a short exact sequence as in Theorem 1.15 we have

\[ F_B = F_N F_M - \chi_{\dim \tau^{-1} M} F_{N/\tau^{-1} M}. \]

**Proof.** First recall that \( \text{Gr}_{\dim C}(N) = \{\text{pt}\} \) for every inner representation \( C \in C \). Every regular subrepresentation \( V \) of \( N/C \) gives rise to a subrepresentation \( C' \) where \( C' \) is also an inner subrepresentation of \( N \) such that fibre of \((0, C')\) is empty. Moreover, every preprojective subrepresentation \( V \) gives rise to a subrepresentation \( C \oplus V \) of \( N \) such that the fibre of \((0, C \oplus V)\) is empty. We can also combine both cases in the natural way. Choosing \( C = \tau^{-1} M \) as in Remark 1.13, these observations can be summarized to

\[ \chi(\text{Gr}_e(B)) = \sum_{f + g = e} \chi(\text{Gr}_f(M)) \chi(\text{Gr}_g(N)) - \sum_{f = e - \dim \tau^{-1} M} \chi(\text{Gr}_f(N/\tau^{-1} M)). \]

Now it straightforward that, in terms of \( F \)-polynomials, this translates to the claim. \( \square \)

Clearly, the analogous statement holds for preinjective representations \( B \) with \( \delta(B) = 2 \).

1.7. **Representations of the homogeneous tubes.** In this section, we consider quiver Grassmannians of indecomposable representations lying in one of the homogeneous tubes. It turns out that they are independent of the chosen tube because it is straightforward to see that quiver Grassmannians of indecomposable representations of dimension \( \delta \) are independent of the chosen homogeneous tube. We fix a homogeneous tube and denote by \( M_{r \delta} \) the indecomposable representation of dimension \( r \delta \) which lies in this tube where \( r \geq 1 \). There exists a chain of irreducible inclusions

\[ \{0\} \hookrightarrow M_{\delta} \hookrightarrow M_{2\delta} \hookrightarrow \ldots \hookrightarrow M_{r\delta} \hookrightarrow \ldots \]
Actually, we can recursively construct all representations $M_{r\delta}$ by considering non-splitting short exact sequences

$$0 \to M_{(r-1)\delta} \xrightarrow{i_r} M_{r\delta} \xrightarrow{\pi_r} M_{\delta} \to 0.$$ 

The idea is to proceed along the lines of section 1.6. Thus we start with considering the morphism

$$\Psi_e^r : \text{Gr}_e(M_{r\delta}) \to \prod_{f+g=e} \text{Gr}_f(M_{(r-1)\delta}) \times \text{Gr}_g(M_{\delta}), \ U \mapsto (i_r^{-1}(U), \pi_r(U)).$$

Remark 1.18. We have $\text{Gr}_{m\delta}(M_{r\delta}) = \{pt\}$ if $m \leq r$. Indeed, the only subrepresentations of $M_{r\delta}$ are preprojective or contained in the tube of $M_{r\delta}$. Since $\delta(m\delta) = 0$ and since the defect is additive, a subrepresentation of dimension $m\delta$ cannot contain a preprojective direct summand.

Lemma 1.19. Let $A$ be a subrepresentation of $M_{(r-1)\delta}$ such that $0 \neq A' := \pi_{r-1}(A) \subset M_{\delta}$. Then we have $\text{Ext}(M_{\delta}, M_{(r-1)\delta}/A') = 0$.

Proof. Since $A'$ is a proper subrepresentation of $M_{\delta}$, we have that $A'$ is preprojective and $M_{\delta}/A'$ is preinjective. Thus we have $\text{Ext}(M_{\delta}, M_{\delta}/A') = 0$. But since $A'$ is preprojective, the inclusion $A' \hookrightarrow M_{\delta}$ factors through $M_{(r-1)\delta}$. In particular, we get (using the universal property of the cokernel of $A' \hookrightarrow M_{(r-1)\delta}$) a commutative diagram

$$0 \to M_{(r-2)\delta} \xrightarrow{\cdot} M_{(r-1)\delta}/A' \xrightarrow{\cdot} M_{\delta}/A' \to 0$$

$$0 \to M_{(r-2)\delta} \xrightarrow{\cdot} M_{(r-1)\delta} \xrightarrow{\cdot} M_{\delta} \to 0$$

But since the lower sequence does not split and, moreover, since $\text{Ext}(M_{\delta}, M_{(r-2)\delta}) = k$, this shows that $\text{Hom}(M_{\delta}, M_{\delta}/A') \to \text{Ext}(M_{\delta}, M_{(r-2)\delta}/A')$ is surjective. Thus we have

$$\text{Ext}(M_{\delta}, M_{(r-1)\delta}/A') \cong \text{Ext}(M_{\delta}, M_{\delta}/A') = 0,$$

which completes the proof.

Lemma 1.20. The fibre $(\Psi_e^r)^{-1}(A, V)$ is empty if and only if $V = M_{\delta}$ and $i_{r-1}(A) \cong A$, i.e. $A$ is already a subrepresentation of $M_{(r-2)\delta}$.

Proof. If $V = 0$, the fibre of $(A, V)$ is clearly not empty because every subrepresentation of $M_{(r-1)\delta}$ is already a subrepresentation of $M_{r\delta}$.

If $0 \neq V \subsetneq M_{\delta}$, we have that $V$ is preprojective and thus the canonical inclusion factors through $M_{r\delta}$. In particular, the fibre of $(A, V)$ is not empty.

Thus assume that $V = M_{\delta}$. If $A = 0$, the fibre is empty because the sequence does not split. For general $A \subset M_{(r-1)\delta}$, we consider the long exact sequence

$$0 \to \text{Hom}(M_{\delta}, A) \to \text{Hom}(M_{\delta}, M_{(r-1)\delta}) \to \text{Hom}(M_{\delta}, M_{(r-1)\delta}/A) \to$$

$$\text{Ext}(M_{\delta}, A) \xrightarrow{\cdot} \text{Ext}(M_{\delta}, M_{(r-1)\delta}) \to \text{Ext}(M_{\delta}, M_{(r-1)\delta}/A) \to 0$$
Thus also in this case the fibre is empty. If \( V\) is not empty, the statement is clearly true for \( V\) and only if \( f_r^A = 0\). This is obviously the case if and only if \( \Ext^1(M_\delta,M_{(r-1)\delta}/A) = k\). In turn the fibre is not empty if and only if \( f_r^A\) is surjective. First assume that \( i_{r-1}^{-1}(A) \cong A\). If \( A = M_{(r-2)\delta}\), we have \( M_{(r-1)\delta}/A \cong M_\delta\) by Remark 1.18. Then every vector space in the above sequence is isomorphic to \( k\). In particular, we have \( f_r^A = 0\).

Furthermore, if \( A \subset M_{(r-2)\delta}\), we have that \( f_r^A\) is the composition

\[
\Ext(M_\delta,A) \to \Ext(M_\delta,M_{(r-2)\delta}) \xrightarrow{f_r^{M_{(r-2)\delta}}} \Ext(M_\delta,M_{(r-1)\delta}).
\]

Thus also in this case the fibre is empty.

If \( A'' := i_{r-1}^{-1}(A) \not\cong A\), we have \( A' = \pi_{r-1}(A) \neq 0\). Then we have the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M_{(r-2)\delta} & \longrightarrow & M_{(r-1)\delta} & \longrightarrow & M_\delta & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0
\end{array}
\]

inducing a diagram

\[
\begin{array}{ccccccc}
\Ext(M_\delta,A'') & \xrightarrow{f_r^{A''}} & \Ext(M_\delta,M_{(r-2)\delta}) & \longrightarrow & \Ext(M_\delta,M_{(r-2)\delta}/A'') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Ext(M_\delta,A) & \xrightarrow{f_r^A} & \Ext(M_\delta,M_{(r-1)\delta}) & \longrightarrow & \Ext(M_\delta,M_{(r-1)\delta}/A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Ext(M_\delta,A') & \longrightarrow & \Ext(M_\delta,M_\delta) & \longrightarrow & \Ext(M_\delta,M_\delta/A') & \longrightarrow & 0
\end{array}
\]

If \( A' = M_\delta\), we clearly have \( \Ext(M_\delta,M_\delta/A') = 0\). By induction, we have that \( f_r^{A''}\) is surjective and thus it follows that \( \Ext(M_\delta,M_{(r-1)\delta}/A) = 0\). Thus the fibre is not empty.

Finally, assume that \( 0 \neq A' = \pi_{r-1}(A) \neq M_\delta\). By Lemma 1.19, we have \( \Ext(M_\delta,M_{(r-1)\delta}/A') = 0\). Thus \( f_r^{A'}\) is surjective. But since \( A'\) is a factor of \( A\), we have \( \Ext(M_\delta,A) \to \Ext(M_\delta,A')\). Thus the fibre of \( (A,M_\delta)\) is not empty. \(\square\)

**Lemma 1.21.** If \( (\Psi_e^r)^{-1}(A,V) \neq \emptyset\), we have \( (\Psi_e^r)^{-1}(A,V) = A^{(\dim V,\dim M_{(r-1)\delta}/A)}\).

**Proof.** By Lemma 1.4, we have

\[
(\Psi_e^r)^{-1}(A,V) = A^{[\dim V,\dim M_{(r-1)\delta}/A]}
\]

if it is not empty. The statement is clearly true for \( V = 0\). If \( 0 \neq V \neq M_\delta\), it is preprojective. Since \( M_{(r-1)\delta}/A\) has no preprojective direct summand, we have \( \Ext(V,M_{(r-1)\delta}/A) = 0\) in this case. If \( V = M_\delta\) and the fibre is not empty, by the considerations in the proof of Lemma 1.20, we have that \( f_r^A\) is surjective. Thus we have \( \Ext(M_\delta,M_{(r-1)\delta}/A) = 0\). \(\square\)

The preceding lemmas can now be used to prove the main result of this section:
Theorem 1.22. Every quiver Grassmannian \( \text{Gr}_e(M_{r\delta}) \) has a cell decomposition into affine spaces. Moreover, this decomposition is compatible with the decomposition

\[
(1.2) \quad \text{Gr}_e(M_{r\delta}) = \{ U \in \text{Gr}_e(M_{r\delta}) \mid \pi_r(U) = 0 \} \cup \{ U \in \text{Gr}_e(M_{r\delta}) \mid \pi_r(U) \neq 0 \}.
\]

Proof. We proceed by induction on \( r \). If \( r = 1 \), the claim follows by [14, Theorem 4.4]. Alternatively, it is straightforward to check by hand that every quiver Grassmannian has a cell decomposition. Since we clearly have \( \pi_1(U) \neq 0 \) for every subrepresentation \( U \subset M_\delta \), also the compatibility follows.

Thus let \( r \geq 2 \). By Lemma 1.20, the fibre of \((A, V)\) is empty if and only if \( A \) is a subrepresentation of \( M_{(r-2)\delta} \) and \( V = M_\delta \). Since \( \text{Gr}_f(M_{(r-1)\delta}) \) and \( \text{Gr}_g(M_\delta) \) have cell decompositions, by Lemma 1.21, it follows that

\[
(\Psi_e')^{-1}(\text{Gr}_f(M_{(r-1)\delta}) \times \text{Gr}_g(M_\delta))
\]

has a cell decomposition if \( g \neq \delta \). If \( g = \delta \), the fibre is empty if \( \pi_{r-1}(A) = 0 \). Since the cell decompositions of the quiver Grassmannians \( \text{Gr}_f(M_{(r-1)\delta}) \) are compatible with the decomposition (1.2) by induction hypothesis, the claim follows in this case in the same way.

Since we have \( \pi_r((\Psi_e')^{-1}(A, V)) = 0 \) if and only if \( V = 0 \), it follows that

\[
\{ U \in \text{Gr}_e(M_{r\delta}) \mid \pi_r(U) = 0 \} = (\Psi_e')^{-1}(\text{Gr}_e(M_{(r-1)\delta}) \times \{ 0 \})
\]

and

\[
\{ U \in \text{Gr}_e(M_{r\delta}) \mid \pi_r(U) \neq 0 \} = (\Psi_e')^{-1}(\bigcup_{f+g=e, g \neq 0} \text{Gr}_f(M_{(r-1)\delta}) \times \text{Gr}_g(M_\delta)).
\]

This already shows that the cell decompositions of the quiver Grassmannians \( \text{Gr}_e(M_{r\delta}) \) are also compatible with decomposition (1.2).

We define \( F_{r\delta} := F_{M_{r\delta}} \). Now the following Corollary is straightforward:

Corollary 1.23. We have

\[
F_{r\delta} = F_{\delta}F_{(r-1)\delta} - x^\delta F_{(r-2)\delta}
\]

for \( r \geq 1 \) where \( F_0 = 1 \) and \( F_{-\delta} := 0 \).

In section, we use this recursive formula to obtain an explicit formula for the \( F \)-polynomial.

1.8. Representations of positive defect. For indecomposable representations \( M \) of positive defect, we can deduce that \( \text{Gr}_e(M) \) decomposes into affine spaces from the corresponding fact for indecomposable representations of negative defect when passing to dual representations.

Let \( M \) be a representation of \( Q \) with ordered basis \( \mathcal{B} \) and \( d := \dim M \). It defines the dual basis \( \mathcal{B}^* \) of \( M^* \), which consists of the linear maps \( f_b : M \to \mathbb{C} \) with \( f_b(b') = \delta_b,b' \). We endow \( \mathcal{B}^* \) with the inverse order of \( \mathcal{B} \). Note that the coefficient quiver \( \Gamma(M^*, \mathcal{B}^*) \) is obtained from the coefficient quiver \( \Gamma(M, \mathcal{B}) \) by inverting the arrows.

If \( M = s_{p_1} \cdots s_{p_s} S_p \) is preprojective, then \( M^* = s_{p_1} \cdots s_{p_s} S_p^* \) is preinjective, and vice versa. Therefore the dual \((-)^* \) establishes a correspondence between the preprojective representations of \( Q \) and the preinjective representations of \( Q^{op} \). If \( Q \) is of extended Dynkin type \( \tilde{D}_n \), the absolute value of
the defect depends on whether \( p \in \{q, q_c, q_d\} \). Thus this correspondence restricts to a correspondence between defect \(-1\) (or defect \(-2\)) preprojectives and defect \(1\) (or defect \(2\)) preinjectives.

For a subrepresentation \( U \) of \( M \) with dimension vector \( e \), we define \( U^* = (U_q^*) \) as the collection of subspaces
\[
U_q^* = \{ f \in M_q^* \mid f(n) = 0 \text{ for all } n \in U_q \}
\]
of \( M_q^* \). For a subset \( \beta \) of \( \mathcal{B} \), we define its dual as
\[
\beta^* = \{ f \in \mathcal{B} \mid f(b) = 0 \text{ for all } b \in \beta \},
\]
which is of type \( e^* = d - e \). Note that \( \beta^* \) is the complement of the set of dual elements \( f_b \in \mathcal{B} \) of basis vectors \( b \in \beta \).

**Theorem 1.24.** The association \( U \mapsto U^* \) defines an isomorphism \( \text{Gr}_e(M) \xrightarrow{\sim} \text{Gr}_{e^*}(M^*) \), which restricts to an isomorphism \( C^M_{\beta} \xrightarrow{\sim} C_{\beta^*} \) between Schubert cells for every subset \( \beta \) of \( \mathcal{B} \). Moreover, \( \beta \) is contradictory of the first (second) kind if and only if \( \beta^* \) is contradictory of the first (second) kind.

**Proof.** Let \( \bar{d} = \sum d_p \) be the total dimension of \( d = (d_p) \), \( \bar{e} \) the total dimension of \( e \) and \( \bar{e}^* \) the total dimension of \( e^* \). Then \( M \) has dimension \( \bar{d} \) as a \( \mathbb{C} \)-vector space, a subrepresentation \( U \) with dimension vector \( e \) has dimension \( \bar{e} \) over \( \mathbb{C} \) and a subrepresentation \( U^* \) of \( M^* \) with dimension vector \( e \) has dimension \( \bar{e}^* \) over \( \mathbb{C} \).

The canonical isomorphism \( \Lambda^{\bar{e}}M \xrightarrow{\sim} \Lambda^{\bar{e}^*}M^* \) that sends \( n_1 \wedge \cdots \wedge n_{\bar{e}} \) to the unique element \( f_1 \wedge \cdots \wedge f_{\bar{e}^*} \) with \( f_j(n_i) = 0 \) for all \( i = 1, \ldots, \bar{e} \) and \( j = 1, \ldots, \bar{e}^* \) induces an isomorphism \( \Phi : \mathbb{P}(\Lambda^{\bar{e}}M) \xrightarrow{\sim} \mathbb{P}(\Lambda^{\bar{e}^*}M^*) \) between the corresponding projective spaces.

A subrepresentation \( U \) of \( M \) of type \( e \) corresponds to a point \( \iota(U) \) of \( \mathbb{P}(\Lambda^{\bar{e}}M) \) and \( U^* \) corresponds to a point \( \iota^*(U^*) \) of \( \mathbb{P}(\Lambda^{\bar{e}^*}M^*) \) where \( \iota \) and \( \iota^* \) denote the respective Plücker embeddings. It is clear from the definitions that \( \iota^*(U^*) = \Phi(\iota(U)) \).

The following calculation shows that \( U^* \) is a subrepresentation of \( M^* \), i.e. that \( M^* \) of \( \mathbb{C} \)-vector space, a subrepresentation \( U \) with dimension vector \( e \) has dimension \( \bar{e} \) over \( \mathbb{C} \) and a subrepresentation \( U^* \) of \( M^* \) with dimension vector \( e \) has dimension \( \bar{e}^* \) over \( \mathbb{C} \).

A subrepresentation \( U \) of \( M \) of type \( e \) corresponds to a point \( \iota(U) \) of \( \mathbb{P}(\Lambda^{\bar{e}}M) \) and \( U^* \) corresponds to a point \( \iota^*(U^*) \) of \( \mathbb{P}(\Lambda^{\bar{e}^*}M^*) \) where \( \iota \) and \( \iota^* \) denote the respective Plücker embeddings. It is clear from the definitions that \( \iota^*(U^*) = \Phi(\iota(U)) \).

The following calculation shows that \( U^* \) is a subrepresentation of \( M^* \), i.e. that \( M^* \) of \( \mathbb{C} \)-vector space, a subrepresentation \( U \) with dimension vector \( e \) has dimension \( \bar{e} \) over \( \mathbb{C} \) and a subrepresentation \( U^* \) of \( M^* \) with dimension vector \( e \) has dimension \( \bar{e}^* \) over \( \mathbb{C} \).
we make use of the notation from the Introduction of [14]. This shows the second claim of the theorem.

Let $\Gamma$ be the coefficient quiver of $M$ w.r.t. $\mathcal{B}$ and $\Gamma^*$ the coefficient quiver of $M^*$ w.r.t. $\beta^*$. Then the underlying graphs of $\Gamma$ and $\Gamma^*$ are the same, but all arrows are inverted and, according to our convention of drawing $b \in \mathcal{B}_p$ on top of $b' \in \mathcal{B}_p$ if $b < b'$, we have to turn the illustration of $\Gamma$ as defined in [14, section 1.1] up side down to obtain the illustration of $\Gamma^*$. The extremal arrows occur in an illustration as the maximal slanted down arrows with a fixed label $v \in Q_1$ (or $v^*$) and a fixed end vertex. From this, it is clear that the same edges of the common underlying graph correspond to extremal arrows of $\Gamma$ and $\Gamma^*$. Since we invert the direction of the arrows of $\Gamma$ and $\beta^*$ is the complement of the set of dual elements of $\beta$, we see that $\beta$ is extremal successor closed, i.e. not contradictory of the first kind, if and only if $\beta^*$ is so.

It is easily verified that also the conditions for $\beta$ to be contradictory of the second kind behave well with dualizing. We forgo to spell out the elementary, but somewhat lengthy details, and conclude the proof of the theorem. □

2. Reductions of Quiver Grassmannians

In this section, we show how we can simplify the determination of quiver Grassmannians by passing to smaller quivers and smaller roots respectively. Together with BGP-reflections reviewed in section 3, it turns out that these methods are very useful when calculating generating functions of representations of $\tilde{D}_n$ in section 4.

2.1. Reduction of type one. Let $\alpha$ be a dimension vector of $Q$ and let $M$ be a representation of dimension $\alpha$ such that all linear maps $M_v$ for $v \in Q_1$ have maximal rank. This is true for real root representation. This follows because they are even of maximal rank type, see [16].

Let $e$ be a second dimension vector such that $e \leq \alpha$ and such that $\text{Gr}_e(M) \neq 0$. Assume that $Q$ has a full subquiver of the form

$$p_0 \to p_1 \to p_2.$$  

In terms of quiver Grassmannians, we consider a commutative diagram

$$
\begin{array}{ccc}
k^{\alpha_0} & \rightarrow & k^{\alpha_1} \\
\downarrow & & \downarrow \\
k^{\tilde{\alpha}_0} & \rightarrow & k^{\tilde{\alpha}_1}
\end{array}
\begin{array}{ccc}
k^{\alpha_2} & \rightarrow & k^{\alpha_2} \\
\downarrow & & \downarrow \\
k^{\tilde{\alpha}_2} & \rightarrow & k^{\tilde{\alpha}_2}
\end{array}
$$

where $\alpha_i := \alpha_{p_i}$ and $e_i := e_{p_i}$.

Let $Q(p_1)$ be the quiver of type $\tilde{D}_{n-1}$ resulting from $Q$ when deleting the vertex $p_1$ and the two corresponding arrows and, moreover, when adding an extra arrow $p_0 \to p_2$. Moreover, let $\hat{\alpha}_i, \hat{\alpha}$ be the corresponding dimension vectors and $\hat{M}$ the induced representation.

If $\alpha_0 \leq \alpha_1 = \alpha_2$, all maps in the diagram are injective and we have $e_0 \leq e_1 \leq e_2$. It is easy to check that $\hat{M}$ is indecomposable if and only if $M$ is indecomposable. For a vector space $V$, we denote by $\text{Gr}(l, V)$ the usual Grassmannian (resp. $\text{Gr}(l, r)$ in the case $V = k^r$). Recall that, for a fixed vector space $V$ and a subspace $U$ with $\dim U \leq l$, we have

$$\text{Gr}(l, U, V) := \{ W \in \text{Gr}(l, V) \mid U \subset W \} \cong \text{Gr}(l - \dim U, \dim V - \dim U).$$
Thus it is straightforward that we have

\[ \text{Gr}_e(M) \cong \text{Gr}(e_1 - e_0, e_2 - e_0) \times \text{Gr}_e(\hat{M}). \]

Note that, since \( \alpha_1 = \alpha_2 \) every subspace of \( M_{p_2} \cong k^{\alpha_2} \) can be identified with a subspace of \( M_{p_1} \cong k^{\alpha_1} \) and vice versa.

Note that, in case of a subquiver

\[ p_0 \leftarrow p_1 \leftarrow p_2 \]

with dimension vector \( \alpha \) satisfying \( \alpha_0 \leq \alpha_1 = \alpha_2 \), we can turn around all arrows in \( Q \) to obtain the situation treated above.

We want to consider a similar case: using the same notation, we assume that we are faced with the following situation

\[
\begin{array}{ccc}
k^{e_0} & \subseteq & k^{\alpha_0} \\
\downarrow & & \downarrow \\
k^{e_1} & \subseteq & k^{\alpha_1} \\
& k^{e_2} & \leftarrow & k^{e_3} & \leftarrow & k^{\alpha_2} & \leftarrow & k^{\alpha_3}
\end{array}
\]

where \( \alpha_2 + \alpha_3 \leq \alpha_1 = \alpha_0 \) and \( e \leq \alpha \). If \( M \) is of maximal rank type, we have that \( k^{\alpha_2} \cap k^{\alpha_3} = \{0\} \) when understanding these two vector spaces as subspaces of \( k^{\alpha_1} \).

Since all maps in the diagram are injective, similar to the preceding case, we can reduce this situation to the case

\[
\begin{array}{ccc}
k^{e_0} & \leftarrow & k^{\alpha_0} \\
& k^{e_2} & \leftarrow & k^{e_3} & \leftarrow & k^{\alpha_2} & \leftarrow & k^{\alpha_3}
\end{array}
\]

Note that it is again straightforward to check that \( \alpha \) is a root if and only if \( \hat{\alpha} \) is a root (resp. that the corresponding representation \( M \) is indecomposable if and only if \( \hat{M} \) is indecomposable). As before we get a subrepresentation of \( M \) for a fixed subrepresentation of \( \hat{M} \) together with a subspace \( U \in \text{Gr}(e_1 - e_2 - e_3, e_0 - e_2 - e_3) \). Thus we have

\[ \text{Gr}_e(M) \cong \text{Gr}_e(\hat{M}) \times \text{Gr}(e_1 - e_2 - e_3, e_0 - e_2 - e_3). \]

Note that there is again a dual case obtained when turning around all arrows. In the sequel we will refer to these two procedures as reduction of type one.

2.2. **Reduction of type two.** In this section, under certain additional conditions we obtain a similar result to Theorem 1.15. We should mention that the result is not needed for the proof of the main results of the paper, but seems to be important for future considerations and as a result itself.

In more detail, we consider a short exact sequence of quiver representations

\[ 0 \rightarrow S_q \rightarrow B \rightarrow N \rightarrow 0 \]
with \( \dim \Hom(S_q,B) = 1 \) and \( \Ext(B,S_q) = 0 \). The morphism introduced in section 1.4 induces a decomposition \( \Gr_e(B) = \Gr_e^q(B) \cup \Gr_e^0(B) \) with
\[
\Gr_e^q(B) = \{ U \in \Gr_e(B) \mid S_q \subset U \}, \quad \Gr_e^0(B) = \{ U \in \Gr_e(B) \mid U \cap S_q = 0 \}.
\]

While we can classify the empty fibres by an easy condition, the dimensions of the fibres turn out to be non-constant if we fix dimension vectors on each direct product of quiver Grassmannians.

**Proposition 2.1.** Let \( 0 \to S_q \to B \to N \to 0 \) be a short exact sequence with \( \dim \Hom(S_q,B) = 1 \) and \( \Ext(B,S_q) = 0 \). Then the morphism \( \Psi_e \) introduced in section 1.6 induces morphisms
\[
\Psi_e : \Gr_e^q(B) \to \Gr_{e-q}(S_q) \times \Gr_{e-s_q}(N) \cong \Gr_{e-s_q}(N),
\]
\[
\Psi_e : \Gr_e^0(B) \to \Gr_0(S_q) \times \Gr_e(N) \cong \Gr_e(N)
\]
such that the first map is an isomorphism and for the second one we have
\[
\Psi_e^{-1}(V) \cong \begin{cases} 0 & \text{if } \Ext(V,S_q) \neq 0 \\ \mathbb{A}^{[V,S_q]} & \text{otherwise} \end{cases}.
\]

**Proof.** Since \( \dim \Hom(S_q,B) = 1 \), we have \( \Hom(S_q,N) = 0 \). Furthermore, for every subrepresentation \( U \) of \( B \) (resp. \( N \)) we have \( \dim \Hom(S_q,U) \leq 1 \) (resp. \( \Hom(S_q,U) = 0 \)). Thus we indeed get a morphism
\[
\Psi_e : \Gr_e^q(B) \to \Gr_{e-s_q}(N), \quad U \mapsto U/S_q.
\]

Note that we have a chain of modules \( S_q \subset U \subset B \) and we thus have \( U + S_q = U \). For a fixed subrepresentation \( V \) of \( N \) of dimension \( e - s_q \), we obtain a subrepresentation \( U \) of dimension \( e \) of \( B \), when considering the following diagram where \( U \) is the pullback of the morphisms \( i_V \) and \( \pi \):
\[
\begin{array}{ccc}
0 & \longrightarrow & S_q \\
\downarrow i & \quad & \downarrow \pi \\
B & \longrightarrow & N \\
\downarrow \varphi & \quad & \downarrow i_V \\
0 & \longrightarrow & U \\
\end{array}
\]

This shows that \( \Psi_e : \Gr_e^q(B) \to \Gr_{e-s_q}(N) \) is surjective. Again by Lemma 1.4, we obtain that \( \Psi_e^{-1}(S_q,V) = \mathbb{A}^{[V,S_q]} \cong \mathbb{A}^0 \). This shows that \( \Psi_e : \Gr_e^q(B) \to \Gr_{e-s_q}(N) \) is an isomorphism.

Moreover, we get a morphism
\[
\Psi_e : \Gr_e^0(B) \to \Gr_e(N), \quad U \mapsto (U + S_q)/S_q.
\]

Here we have \( U \cap S_q = 0 \), \( (U + S_q)/S_q \cong U \) and \( V := \Psi_e(U) \cong U \). Since \( U \) is a subrepresentation of \( B \), we have \( \Ext(B,S_q) \cong \Ext(V,S_q) = 0 \). Now by Lemma 1.4, we have \( \Psi_e^{-1}(V) = \mathbb{A}^{[V,S_q]} \).

If \( V \) is a subrepresentation of \( N \) such that \( \Ext(V,S_q) \neq 0 \), we thus have \( \Psi_e^{-1}(V) = \Psi_e^{-1}(0,V) = \emptyset \) and the statement follows.

**Corollary 2.2.** Using the notation of Proposition 2.1, we have:

(i) If \( \Gr_e(N) = \emptyset \), it follows that \( \Gr_e(B) \cong \Gr_{e-s_q}(N) \).
(ii) We have
\[ \chi(\text{Gr}_e(B)) = \chi(\text{Gr}_{e-s_q}(N)) + \chi(\text{Gr}_e(N)) \]
if \( \text{Ext}(V, S_q) = 0 \) for all subrepresentations \( V \in \text{Gr}_e(N) \).

2.3. Application to real root representations and examples. We first note that the reductions of type one preserve possible cell decompositions into affine spaces while, in general, type two reductions do not. As Corollary 2.2 suggests, the reduction of type two can often be applied to obtain isomorphisms between quiver Grassmannians. When calculating the generating function of the Euler characteristics of quiver Grassmannians, it turns out that the reduction of type one is a very powerful tool when combining it with BGP-reflections. Actually, we can start with \( \tilde{D}_n \) in subspace orientation where we can apply reductions of type one. Then we can show that the obtained formulae are invariant under BGP-reflections.

Considering the Auslander-Reiten quiver of \( \tilde{D}_n \) in subspace orientation, it can be seen easily that the preprojectives of defect \(-1\) can be reduced to

![Diagram of preprojective quiver](diagram1)

and

![Diagram of preprojective quiver](diagram2)

where the numbers indicate the dimension vector.

Note that in the same way we can reduce the calculation of quiver Grassmannians of indecomposables of preinjective roots to the case of \( \tilde{D}_5 \). Alternatively, we can consider the opposite quiver and restrict to preprojective roots.

Next we consider the non-exceptional real roots. It is easy to check that we can actually reduce all the real roots of the tubes of rank two to cases of the form

![Diagram of real roots](diagram3)

The real roots of the exceptional of rank \( n - 2 \) tube can be reduced to the following cases:
In summary, as far as subspace orientation is concerned, by the introduced reductions steps, we can stick to the exceptional roots of $\tilde{D}_5$ and to the non-exceptional roots of $\tilde{D}_6$. Note that this method cannot simply be applied to the imaginary root representations lying in the exceptional tubes because not all linear maps are of maximal rank.

**Example 2.3.** Consider the following real root representations (indicated by the root) and exact sequence:

$$
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 0
\end{array}
$$

Denoting the middle term by $B$, consider the quiver Grassmannians $\text{Gr}_{e_1}(B)$ and $\text{Gr}_{e_2}(B)$ where

$$
e_1 = \begin{array}{ccc}
1 & \rightarrow & 1 \\
1 & \rightarrow & 1 \\
1 & \rightarrow & 0
\end{array}, \quad e_2 = \begin{array}{ccc}
1 & \rightarrow & 2 \\
1 & \rightarrow & 0
\end{array}
$$

In the first case we get

$$
\text{Gr}_{e_1}(B) = \text{Gr}_{e_1}(N) \cup \hat{\text{Gr}}_{e_1}(N) = \{\text{pt}\} \cup \mathbb{P}^1 \setminus \{\text{pt}\}
$$

where we define $\hat{\text{Gr}}_{e_1}(N) = \{U \in \text{Gr}_{e_1}(N) \mid \text{Ext}(U, S_q) = 0\}$. Note that the representation $U$ of dimension $e_1$ which can not be lifted to a subrepresentation of $B$ has a direct sum decomposition into indecomposables with roots

$$
\begin{array}{cccccccc}
1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 \\
1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
$$

Since $\text{Gr}_{e_2}(N) = \emptyset$ we get an isomorphism $\text{Gr}_{e_2}(N) \cong \text{Gr}_{e_2}(B) \cong \mathbb{P}^1$. 

3. BGP-REFLECTIONS AND QUIVER GRASSMANNIANS

Another method to get morphisms and connections between quiver Grassmannians and the corresponding generating functions is to consider the reflection functor introduced by Bernstein, Gelfand and Ponomarev in [2], see [17, section 5] and [10, section 5]. For a quiver $Q$, consider the matrix $A = (a_{p,q})_{p,q \in Q_0}$ with $a_{p,p} = 2$ and $a_{p,q} = a_{q,p}$ for $p \neq q$, in which $a_{p,q} = |\{v \in Q_1 \mid v: p \to q \lor v: q \to p\}|$. Fixed some $q \in Q_0$ define $\sigma_q : \mathbb{Z}Q_0 \to \mathbb{Z}Q_0$ as

$$\sigma_q(p) = p - a_{q,p}q.$$

Let $Q$ be a quiver and $q \in Q_0$ a sink (resp. a source). Then by $\sigma_q Q$ we denote the quiver which is obtained from $Q$ by turning around all arrows with tail (resp. head) $q$. In both cases we denote the reflection functors, which are additive functors, by $\sigma_q : \text{Rep}(Q) \to \text{Rep}(\sigma_q Q)$. If $M$ is a representation of $Q$ and $q$ is a sink (resp. a source) we consider the linear maps

$$\phi^M_q : \bigoplus M_p \xrightarrow{q \mapsto q} M_q \quad (\text{resp.} \quad \phi^M_q : M_q \xrightarrow{q \mapsto q} \bigoplus M_p).$$

Recall that in both cases we have $(\sigma_q M)_p = M_p$ if $p \neq q$. Moreover, we have $(\sigma_q M)_q = \text{Ker}(\phi^M_q)$ (resp. $(\sigma_q M)_q = \text{coker}(\phi^M_q)$). Now the linear maps $(\sigma_q M)_v$ for $v : p \to q$ (resp. $(\sigma_q M)_v$ for $v : q \to p$) are the natural ones. The functors have the following properties:

(i) If $M \cong S_q$, then $\sigma_q(S_q) = 0$.

(ii) If $M \ncong S_q$ is indecomposable, then $\sigma_q(M)$ is indecomposable such that $\sigma^2_q(M) \cong M$ and $\dim \sigma_q(M) = \sigma_q(\dim M)$.

In order to investigate the behavior of quiver Grassmannians and the corresponding generating functions under the reflection functor, we review and re-prove some results of [17] and [10]. Note that in [10] the more general case of mutations is treated. We define

$$\text{Gr}_e(M,q^r) = \{ U \in \text{Gr}_e(M) \mid \dim \text{Hom}(U,S_q) = r \}.$$  

and

$$\text{Gr}_e(q^r, M) = \{ U \in \text{Gr}_e(M) \mid \dim \text{Hom}(S_q,U) = r \}.$$  

In order to simplify notations, we assume that $q$ is a sink and, moreover, that $M$ is an indecomposable representation of $Q$ with $\alpha := \dim M$. The case when $q$ is a source can be obtained analogously or simply by considering the isomorphisms $\text{Gr}_e(M) \cong \text{Gr}_{\alpha-e}(M^*)$. Following [17, section 5], we consider the following map

$$\pi^r_q : \text{Gr}_e(M,q^r) \to \text{Gr}_{e-rs_q}(M,q^0)$$

where $\pi^r_q$ is defined by $\pi^r_q(U)_q = \text{Im}(\phi^U_q)$ and $\pi^r_q(U)_p = U_p$ if $p \neq q$. Note that, indeed, $\pi^r_q(U)$ is a subrepresentation of dimension $e - rs_q$ of $M$ such that $\text{Hom}(U,S_q) = 0$. By [17, Theorem 5.11], we have for sinks $q$:

**Theorem 3.1.** The morphism $\pi^r_q$ is surjective with fibres isomorphic to $\text{Gr}(r,\alpha_q - e_q + r)$. Moreover, there exists an isomorphism of varieties

$$\sigma_q : \text{Gr}_e(M,q^0) \to \text{Gr}_{e-q^0}(q^0, \sigma_q M), \ U \mapsto \sigma_q U.$$
The analogous statement holds if $q$ is a source.

For every dimension vector $e \in \mathbb{N}Q_0$, there exists some $0 \leq t \leq e_q$ such that $\text{Gr}_{e-ts_q}(M, q^r) = \emptyset$ for $r \geq 1$ and $l \geq t$. Then we have $\text{Gr}_{e-ts_q}(M, q^0) = \text{Gr}_{e-ts_q}(M)$. Fix $t \in \mathbb{N}$ minimal with this property. Thus, for $e' = e - ts_q$ with $l \geq t$ we have $\text{Gr}_{e'}(M) \cong \text{Gr}_{\sigma q'}(\sigma_q M)$. Then we have the following statement:

**Proposition 3.2.** Assume that $q$ is a sink and that $\text{Gr}_e(M) = \text{Gr}_e(M, q^0)$. Then we have

$$
\chi(\text{Gr}_{e+ms_q}(M, q^0)) = \sum_{i=0}^{m} (-1)^{m-i} \binom{\alpha_q - e_q - i}{m-i} \chi(\text{Gr}_{e+is_q}(M)).
$$

**Proof.** First recall that $\chi(\text{Gr}(k, n)) = \binom{n}{k}$ for $k \leq n$. We proceed by induction on $m$. The statement is satisfied for $m = 0$. By Theorem 3.1, we have

$$
\chi(\text{Gr}_{e+ms_q}(M)) = \sum_{i=0}^{m} \binom{\alpha_q - e_q - (m-i)}{i} \chi(\text{Gr}_{e+(m-i)s_q}(M), q^0)).
$$

Applying the induction hypothesis, we get

$$
\chi(\text{Gr}_{e+ms_q}(M, q^0)) = \chi(\text{Gr}_{e+ms_q}(M)) - \sum_{i=1}^{m} \binom{\alpha_q - e_q - (m-i)}{i} \chi(\text{Gr}_{e+(m-i)s_q}(M, q^0))
$$

$$
= \chi(\text{Gr}_{e+ms_q}(M)) - \sum_{i=1}^{m} \binom{\alpha_q - e_q - (m-i)}{i}
$$

$$
\cdot \sum_{j=0}^{m-i} (-1)^{m-i-j} \binom{\alpha_q - e_q - j}{m-i-j} \chi(\text{Gr}_{e+js_q}(M))
$$

$$
= \chi(\text{Gr}_{e+ms_q}(M)) - \sum_{j=0}^{m-1} \chi(\text{Gr}_{e+js_q}(M))
$$

$$
\cdot \sum_{i=1}^{m-j} (-1)^{m-i-j} \binom{\alpha_q - e_q - (m-i)}{i} \binom{\alpha_q - e_q - j}{m-i-j}
$$

$$
= \chi(\text{Gr}_{e+ms_q}(M)) - \sum_{j=0}^{m-1} (-1)^{m-j} \binom{\alpha_q - e_q - j}{m-j} \chi(\text{Gr}_{e+js_q}(M))
$$

$$
\cdot \sum_{i=1}^{m-j} (-1)^{i} \binom{m-j}{i}.
$$

Since we have

$$
\sum_{i=1}^{m-j} (-1)^{i} \binom{m-j}{i} = -1
$$

the claim follows. \qed
We need the following identities which can be proved by induction where
\[
\binom{n}{k} := \frac{n(n-1) \ldots (n-k+1)}{k!}
\]
for \( n \in \mathbb{Z} \).

**Lemma 3.3.**

(i) For natural numbers \( m,t,n \) with \( n \geq t \geq m \), we have
\[
\sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{n-m+r}{n-t} = (-1)^m \binom{n-m}{t}.
\]

(ii) For natural numbers \( m,t,n \) with \( m \leq n < t \), we have
\[
\sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{(n-m+r)!}{(t-m+r)!} = \frac{(n-m)!(t-n+m-1)!}{t!(t-n-1)!}.
\]

**Proof.** We proceed by induction where the result is checked easily for \( m = 0 \). Applying the induction hypothesis and the well-known formula \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \) have
\[
\sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} \binom{n-(m+1)+r}{n-t} = \sum_{r=0}^{m+1} (-1)^r \left( \binom{m}{r} + \binom{m}{r-1} \right) \binom{n-(m+1)+r}{n-t}
\]
\[
= \sum_{r=0}^{m} (-1)^r \binom{m}{r} \left( \binom{n-1-m+r}{n-1-(t-1)} \right)
\]
\[
- \sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{n-m+r}{n-t}
\]
\[
= (-1)^{m+1} \left( - \binom{n-1-m}{t-1} + \binom{n-m}{t} \right)
\]
\[
= (-1)^{m+1} \binom{n-(m+1)}{t}.
\]

The second statement can be proved similarly. We proceed by induction where the result is checked easily for \( m = 0 \). Applying the induction hypothesis we have
\[
\sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} \frac{(n-m-1+r)!}{(t-m-1+r)!} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{(n-1-m+r)!}{((t-1)-m+r)!}
\]
\[
- \sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{(n-m+r)!}{(t-m+r)!}
\]
\[
= \frac{(n-1-m)!(t-n+m-1)!}{(t-1)!(t-n-1)!} - \frac{(n-m)!(t-n+m-1)!}{t!(t-n-1)!}
\]
\[
= \frac{(n-m-1)!(t-n+m)!}{t!(t-n-1)!}.
\]

This completes the proof of the lemma. \( \square \)
Applying the preceding statements, we see that the Euler characteristic of a quiver Grassmannian of a representation, which is obtained by reflecting at a source or sink, is already determined by the Euler characteristics of quiver Grassmannians of the original representation:

**Theorem 3.4.** Let $M$ be a representation of dimension $\alpha$. Let $q$ be a sink and $e \in \mathbb{N}Q_0$ such that $\text{Gr}_e(M) = \text{Gr}_e(M,q^0)$. Let $n := (\sigma_q e)_q$ and $t := \alpha_q - e_q$. We have

$$\chi(\text{Gr}_{\sigma_q e - ms_q}(\sigma_q M)) = \sum_{j=0}^{m} \chi(\text{Gr}_{e + js_q}(M)) \binom{n-t}{m-j}.$$  

**Proof.** Let $q$ be a source and assume that $\text{Gr}_e(M) = \text{Gr}_e(q^0, M)$. This is the case if and only if $\text{Gr}_{\alpha - e}(M^*) = \text{Gr}_{\alpha - e}(M^*, q^0)$. Since $\chi(\text{Gr}_e(M)) = \chi(\text{Gr}_{\alpha - e}(M^*))$, by Theorem 3.1 (for a source $q$), we get

$$\chi(\text{Gr}_{e - ms_q}(M)) = \chi(\text{Gr}_{\alpha - e + ms_q}(M^*)) = \sum_{i=0}^{m} \binom{e_q - (m-i)}{i} \chi(\text{Gr}_{\alpha - e + (m-i)s_q}(M^*, q^0)) = \sum_{i=0}^{m} \binom{e_q - (m-i)}{i} \chi(\text{Gr}_{e + (i-m)s_q}(q^0, M)).$$

Thus if $\text{Gr}_e(M) = \text{Gr}_e(M, q^0)$, applying successively this statement, Theorem 3.1 and Proposition 3.2, we have

$$\chi(\text{Gr}_{\sigma_q e - ms_q}(\sigma_q M)) = \sum_{i=0}^{m} \binom{(\sigma_q e)_q - (m-i)}{i} \chi(\text{Gr}_{\sigma_q e + (i-m)s_q}(q^0, \sigma_q M)) = \sum_{i=0}^{m} \binom{e_q - (m-i)}{i} \chi(\text{Gr}_{e + (m-i)s_q}(M^*, q^0)) = \sum_{i=0}^{m} \binom{e_q - (m-i)}{i} \sum_{j=0}^{m-i} (-1)^{m-i-j} \binom{\alpha_q - e_q - j}{m-i-j} \chi(\text{Gr}_{e + js_q}(M)) = \sum_{j=0}^{m} \chi(\text{Gr}_{e + js_q}(M)) \sum_{i=0}^{m-j} (-1)^{m-i-j} \binom{(\sigma_q e)_q - (m-i)}{i} \binom{\alpha_q - e_q - j}{m-i-j}.$$ 

Set $n = (\sigma_q e)_q$ and $t = \alpha_q - e_q$. First assume that $n \geq t$. Applying Lemma 3.3, we obtain:

$$\chi(\text{Gr}_{\sigma_q e - ms_q}(\sigma_q M)) = \sum_{j=0}^{m} \binom{(t-j)!}{(n-t)!} \binom{n-t-j}{m-j}! (1)^{m-j} \chi(\text{Gr}_{e + js_q}(M)) \cdot \sum_{i=0}^{m-j} (-1)^{i} \binom{m-j}{i} \binom{n-j-(m-j-i)}{n-j-(t-j)} = \sum_{j=0}^{m} \chi(\text{Gr}_{e + js_q}(M)) \binom{(t-j)!}{(n-t)!} \binom{n-m-j}{(n-m)!} \binom{t-j}{(n-t-j)!} \binom{n-j}{t-j}! \binom{n-t-m+j}{(n-t-m+j)!}.$$
Theorem 3.5. Let \( q \) be a sink. We have

\[
F_{\sigma_q M}(x) = (1 + x_q^{-1})^{-\dim M_q} \sum_{e \in \mathbb{N}Q_0} \chi(\text{Gr}_e(M)) \sigma_q(x^e) = (1 + x_q^{-1})^{-\dim M_q} F_M(x')
\]

where

\[
x'_i = \begin{cases} 
  x_i^{-1} & \text{if } i = q \\
  x_i & \text{if } i \neq q \end{cases}
\]

If \( n < t \), again applying Lemma 3.3, we obtain

\[
\chi(\text{Gr}_{\sigma_q e - m \sigma_q}(\sigma_q M)) = \sum_{j=0}^{m} \frac{(t-j)!}{(n-m)!(m-j)!} (-1)^{m-j} \chi(\text{Gr}_{e-j \sigma_q}(M))
\]

\[
\cdot m^{-j} \sum_{i=0}^{m-j} (-1)^i \binom{m-j}{i} \frac{(n-j-(m-j-i))!}{(t-(m-j-i))!}
\]

\[
= \sum_{j=0}^{m} \frac{(t-j)!}{(n-m)!(m-j)!} \chi(\text{Gr}_{e-j \sigma_q}(M))
\]

\[
= \sum_{j=0}^{m} \chi(\text{Gr}_{e-j \sigma_q}(M)) \binom{t-n+m-j-1}{m-j}
\]

This completes the proof of the theorem. \( \square \)

We want to investigate how the \( F \)-polynomial of a representation changes when applying BGP-reflections. Assume that \( \text{Gr}_e(M) = \text{Gr}_e(M, q^0) \). By Theorem 3.4, we know that

\[
\chi(\text{Gr}_{e+rs_q}(M)) \binom{n-t}{i}
\]

contributes to the coefficient of \( x^{\sigma_q(e)-(r+i)s_q} \) for \( r = 0, \ldots, t \) and \( i = 0, \ldots, n-t \). In other words for the coefficient of \( x^{\sigma_q(e)-(r+i)s_q} \) in \( F_{\sigma_q M}(x) \) we get

\[
\sum_{i=0}^{n-t} \binom{n-t}{i} \chi(\text{Gr}_{e+rs_q}(M)) x^{\sigma_q(e)-(r+i)s_q} = x^{\sigma_q(e+s_q)} \chi(\text{Gr}_{e+rs_q}(M)) (1 + x_q^{-1})^{n-t}.
\]

Define

\[
\sigma_q(x^e) := x^{\sigma_q(e)} (1 + x_q^{-1})^{\sigma_q(e) + e_q}.
\]

Recall that \( a(p, q) \) was defined as the number of arrows from \( p \) to \( q \). Finally, we re-obtain [10, Lemma 5.2] in the case where \( q \) is a sink:

**Theorem 3.5.**

(i) Let \( q \) be a sink. We have

\[
F_{\sigma_q M}(x) = (1 + x_q^{-1})^{-\dim M_q} \sum_{e \in \mathbb{N}Q_0} \chi(\text{Gr}_e(M)) \sigma_q(x^e) = (1 + x_q^{-1})^{-\dim M_q} F_M(x')
\]

where

\[
x'_i = \begin{cases} 
  x_i^{-1} & \text{if } i = q \\
  x_i & \text{if } i \neq q \end{cases}
\]
(ii) Let $q$ be a source. We have

$$F_{\sigma q M}(x) = (1 + x_q^{-1})^{(\sigma_q \dim M)_q} F_M(x')$$

where

$$x'_i = \begin{cases} 
  x^{-1}_i & \text{if } i = q \\
  x_i x_q a(q, i) (1 + x_q)^{-a(q, i)} & \text{if } i \neq q 
\end{cases}$$

Proof. The first part follows from the results of this section. Moreover, we have

$$F_{\sigma q M} = F_{(\sigma q M)^*}.$$

Since $q$ is a sink of $Q^{\text{op}}$ and keeping in mind that $F_{M^*}(x_p \mid p \in Q_0) = x^{\dim M} F_M(x_p^{-1} \mid p \in Q_0)$, the statement is straightforward consequence of the first part. □

Remark 3.6. If we consider the quiver with one vertex and if $M$ is the semi-simple of dimension vector $n$ we get the generating function of the usual Grassmannian

$$F_M(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$ 

We have $\sigma_q(x^k) = x^{-k}(1 + x^{-1})^{-n}$ and thus

$$F_{\sigma q M}(x) = F_0 = \sum_{k=0}^{n} \binom{n}{k} x^{-k}(1 + x^{-1})^{-n} = 1.$$

4. Generating functions of Euler characteristics of quiver Grassmannians of type $\tilde{D}_n$

The main aim of this section is to develop explicit formulae for the generating functions of Euler characteristics of quiver Grassmannians (resp. $F$-polynomials) of representations of quivers of type $\tilde{D}_n$. This reduces to counting certain subsets of the vertex set of coefficient quivers of the respective representations. We first derive formulae for the generating functions of indecomposable representations of small defect. To do so, we initially restrict to subspace orientation and generalize the obtained formulae by applying BGP-reflections. By Theorem 1.17, this can be used to obtain formulae for all indecomposable representations of a quiver of type $\tilde{D}_n$. Since we have $F_{M \oplus N} = F_M F_N$ for two representations $M$ and $N$, see [4, Corollary 3.7], we obtain formulae for all representations of $Q$. Throughout this section, we frequently use the notation of section 3.

Remark 4.1. The following observation is trivial, but crucial for the considerations of this section: if $F, G \in k[x_q \mid q \in Q_0]$ and $x_q \mapsto x'_q$ such that $x' \in k[x_q \mid q \in Q_0]$ is a variable transformation, we have

$$(FG)(x') = F(x')G(x') \quad \text{and} \quad F(x') + G(x') = (F + G)(x').$$

In many cases, we can use this to transfer a factorization or a formula for the generation function of a representation $M$ to one of $M'$ which is obtained from $M$ by applying the reflection functor or the methods from section 2.1.

Recall that for a sink (resp. source) $q \in Q_0$ we defined $\sigma_q x^d = (x')^d$ where $x'$ is obtained by the variable transformation of Theorem 3.5. Moreover, this extends to $F_{\sigma q M} = (1 + x_q^{-1})^{-\dim M} \sigma_q F_M$. We frequently use the following lemma:
Lemma 4.2. (i) Let \( d \in \mathbb{N}Q_0 \). For a sink \( q \in Q_0 \), we have
\[
\sigma_q x^d = (1 + x_q^{-1}) \sum_{p \in Q_0} a(p, q) d_p x^{(\sigma_q d)_q} \prod_{p \in Q_0, p \neq q} x^{d_p} = (1 + x_q^{-1}) \sum_{p \in Q_0} a(p, q) d_p x^{\sigma_q d}.
\]

(ii) Let \( q \) be sink of \( Q \). Then for every indecomposable representation \( M \), we have
\[
\dim(\tau M)_q = \sum_{p \in Q_0} a(p, q) \dim M_p - \dim M_q.
\]

Proof. The first part is just a reformulation of the definition. The second statement follows because the Auslander-Reiten translate can be obtained by any admissible sequence of BGP-reflections at sinks. \( \square \)

Remark 4.3. A dimension vector \( \alpha \in \mathbb{N}Q_0 \) can be the root of \( Q \) w.r.t. different orientations. Though the \( F \)-polynomial \( F_\alpha \) depends on the chosen orientation of \( Q \), we opt to suppress it from the notation. Note that also the \( F \)-polynomial \( F_\delta \) of a representation from a homogeneous tube depends on the orientation.

Note that there is another recursive description of \( F_{r \delta} \) induced by Theorem 1.9, see Lemma 4.11 for more details.

4.1. Reduction steps and generating functions. In this section, we analyze the behavior of the generating functions under reduction of type one, i.e. we have an indecomposable representation \( M \) of dimension \( \alpha \) such that \( \alpha_i = \alpha_{i+1} = \alpha_{i+2} + 1 \). Let \( M \) be of maximal rank type and \( \hat{M} \) and \( \hat{\varepsilon} \) the induced representation and induced dimension vector, respectively. According to section 2, we obtain an isomorphism \( \text{Gr}_e(M) \cong \text{Gr}(e_{i+1} - e_i, e_{i+2} - e_i) \times \text{Gr}_e(\hat{M}) \) when we remove the vertex \( i+1 \). Thus for the Euler characteristic we get
\[
\chi(\text{Gr}_e(M)) = \chi(\text{Gr}(e_{i+1} - e_i, e_{i+2} - e_i)) \chi(\text{Gr}_e(\hat{M})) = \left( e_{i+2} - e_i \right) \chi(\text{Gr}_e(\hat{M})).
\]

This yields the following easy relation between the corresponding \( F \)-polynomials.

Lemma 4.4. Let \( M \) be a representation of \( \tilde{D_n} \) which can be reduced to a representation \( \hat{M} \) by reduction of type one. Then we have
\[
F_M(x) = \sum_{\hat{\varepsilon} \in \mathbb{N}Q_0} \chi(\text{Gr}_e(\hat{M})) x^{\hat{\varepsilon}_{i+2}} (1 + x_{i+1})^{{\hat{\varepsilon}_i - \hat{\varepsilon}_{i+2}}}. \]

In other words, considering the variable transformation \( x_q \mapsto x_q' \) where
\[
x_i' := x_i (1 + x_{i+1}), \quad x_{i+2}' := x_{i+2} x_{i+1} (1 + x_{i+1})^{-1}, \quad x_q' = x_q \text{ for all } q \notin \{i, i+2\},
\]
we have \( F_M(x) = F_{\hat{M}}(x') \).

Moreover, we obtain an analogous statement for the second instance of reduction of type one.

Finally, in order to pass to preinjective representations, we can pass to the opposite quiver and dual representations. On the level of \( F \)-polynomials this can be described by the following formula:
Thus we get

\[ \chi_a \quad \text{for some} \quad \lambda \]

and

\[ \lambda \quad \text{for every} \quad q \in \mathbb{Q}. \]

4.2. Counting admissible subsets. In order to determine \( F \)-polynomials for any orientation of \( \tilde{D}_n \), we first determine the \( F \)-polynomials for representations of \( \tilde{D}_n \) in subspace orientation. Applying BGP-reflections, we obtain the corresponding formula for every orientation. To do so and to fix notation, we proceed with recalling some well known procedure which can be used to obtain the generating functions explicitly.

Let \( f_0, f_1 \in k[x_i \mid i \in I] \) and let \( f_j \) for \( j \geq 2 \) be recursively defined by

\[
\begin{pmatrix}
  f_{2n} \\
  f_{2n+1}
\end{pmatrix} =
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}^n
\begin{pmatrix}
  f_0 \\
  f_1
\end{pmatrix}
\]

for some \( a, b, c, d \in k[x_i \mid i \in I] \) and \( n \geq 1 \). The eigenvalues of \( A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) are the zeroes of \( \chi_A = (X-a)(X-d) - bc \), i.e.

\[ \lambda_{\pm} = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} - ad + bc}. \]

Define \( z := \sqrt{\frac{(a+d)^2}{4} - ad + bc} \). We have \( \lambda_+ \lambda_- = ad - bc \) and \( \lambda_+ - \lambda_- = 2z \). Assuming that \( z \neq 0 \), for the eigenspaces, we get

\[ E_{\lambda_{\pm}} = \left\langle \begin{pmatrix} -b \\ a - \lambda_{\pm} \end{pmatrix} \right\rangle. \]

For

\[ T := \begin{pmatrix} -b & -b \\ a - \lambda_+ & a - \lambda_- \end{pmatrix}, \]

we have

\[ T^{-1} = \frac{1}{\det T} \begin{pmatrix} a - \lambda_- & b \\ \lambda_+ - a & -b \end{pmatrix}. \]

Then we have

\[
\begin{pmatrix}
  f_{2n} \\
  f_{2n+1}
\end{pmatrix} =
\frac{1}{-2bz} \begin{pmatrix}
  -b & -b \\
  a - \lambda_+ & a - \lambda_-
\end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}^n
\begin{pmatrix}
  a - \lambda_- & b \\
  \lambda_+ - a & -b
\end{pmatrix}
\begin{pmatrix}
  f_0 \\
  f_1
\end{pmatrix}
\]

\[
= \frac{1}{-2bz} \begin{pmatrix} b(\lambda_+^n(a - \lambda_+) - \lambda_-^n(a - \lambda_-)) & b^2(\lambda_+^n - \lambda_-^n) \\ (a - \lambda_+)(a - \lambda_-)(\lambda_+^n - \lambda_-^n) & b(\lambda_+^n(a - \lambda_+) - \lambda_-^n(a - \lambda_-)) \end{pmatrix}
\begin{pmatrix}
  f_0 \\
  f_1
\end{pmatrix}.
\]

Thus we get

\[ f_{2n} = \frac{1}{2z}((a(\lambda_+^n - \lambda_-^n) - (ad - bc)(\lambda_+^{n-1} - \lambda_-^{n-1}))f_0 - b(\lambda_+^n - \lambda_-^n)f_1) \]

and

\[ f_{2n+1} = \frac{-1}{2bz}((a - \lambda_+)(a - \lambda_-)(\lambda_+^n - \lambda_-^n)f_0 + b(\lambda_+^n(a - \lambda_+) - \lambda_-^n(a - \lambda_-))f_1) \]
The quiver $A_m = 0 \leftarrow 1 \leftarrow \ldots \leftarrow m$ appears as a subquiver of $\tilde{D}_n$ and the coefficient quiver of the real root representation of dimension $1_m = (1,1,\ldots,1)$ as a subset of the vertex set of the coefficient quivers under consideration. If $X_{1_m}$ is the corresponding indecomposable representation, it is straightforward to check that we have

$$F_m := F_{X_{1_m}} = \sum_{i=-1}^{m} \prod_{j=0}^{i} x_j.$$  

Moreover, let $F_{1,m} := \sum_{i=0}^{m} \prod_{j=1}^{i} x_j$. By an easy induction, the following can be proved:

**Lemma 4.6.** We have

(i) $\prod_{i=0}^{m} \begin{pmatrix} 1 & 0 \\ 1 & x_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F_{m-1} & \prod_{i=0}^{m} x_i \end{pmatrix}$

(ii) $\prod_{i=0}^{m-1} \begin{pmatrix} 0 & 1 \\ -x_{m-i} & x_{m-i}+1 \end{pmatrix} = \begin{pmatrix} -F_{1,m-1}+1 & F_{1,m-1} \\ -F_{1,m}+1 & F_{1,m} \end{pmatrix}$

In order to determine generating functions of preprojectives of defect $-1$, we consider the following snake-shaped coefficient quiver $\mathcal{Q}(s,n)$ (where $t := 2n-2$) where we omit the vertices $q_i$ in the notation:

![Diagram](image-url)

We will see that we can basically restrict our calculations to this case. Also the case of the exceptional tubes can be reduced to this case. We refer to the corresponding preprojective representation by $M(s,n)$.

We call a subgraph ramification subgraph if it is of the following form:

![Diagram](image-url)
We have the following recursive relations:

\[ e \text{ be the generating function counting the number } \chi \text{ of admissible subsets of } \mathcal{D}(s,n)_0 \text{ of a fixed type } e. \]  

**Definition 4.7.** We call a subset \( G_0 \) of \( \mathcal{D}(s,n)_0 \) admissible if the following holds:

(i) \( G_0 \) is extremal successor closed, i.e. if the tail of an extremal arrow is contained in \( G_0 \), the head is also contained in \( G_0 \).

(ii) For all ramification subgraphs, we have: if \( l + 1, l + 2 \in G_0 \), we have \( l \in G_0 \).

Note that we automatically have \( l + 3 \in G_0 \) if \( G \) is extremal successor closed and if \( l + 1 \in G_0 \) or \( l + 2 \in G_0 \). Every subset induces a dimension vector \( e \in \mathbb{N}Q_0 \), called the type of \( G_0 \) in what follows. The next step is to determine the number of admissible subsets of \( \mathcal{D}(s,n)_0 \) of a fixed type \( e \). By Theorem A, we have

**Theorem 4.8.** Let \( e \in \mathbb{N}Q_0 \). Then \( \chi(\text{Gr}_e(M(s,n))) \) is the number of admissible subsets of \( \mathcal{D}(s,n)_0 \) of type \( e \).

Consider \( I := \{0, \ldots, 2s + 1\} \) and \( J := \{0, \ldots, n - 4\} \). If we delete the sources of \( \mathcal{D}(s,n) \) corresponding to the ramification subgraphs, we can think of the remaining graph as a matrix having entries which are vertices, i.e. with every index \( (i,j) \) we associate the vertex in the \( i \)th row and \( j \)th column of the remaining graph. Note that we start the indexing by \( (0,0) \).

For \( (i,j) \in I \times J \), let \( \mathcal{G}(i,j) \) be the full (connected) subgraph of \( \mathcal{D}(s,n) \) which has vertices \( \{(0,n-4),(0,n-5),\ldots,(i,j)\} \) and where we add the subgraph \( 1 \leftarrow 0 \) and also all sources of ramification subgraphs whose remaining vertices are all contained in \( \{(0,n-4),(0,n-5),\ldots,(i,j)\} \). Let

\[ \mathcal{F}^j_i = \sum_{e \in \mathbb{N}Q_0} \chi(i,j,e)x^e \]

be the generating function counting the number \( \chi(i,j,e) \) of admissible subsets of \( \mathcal{G}(i,j)_0 \) of type \( e \). We define \( \mathcal{F}^{n-4}_{-1} := 1 \) and \( \mathcal{F}^{n-4}_0 := 1 + x_{n-4} + x_{n-4}x_d \).

**Lemma 4.9.** We have the following recursive relations:

(i) For all \( m \geq 0 \), \( j = n-5, \ldots, 0 \), we have \( \mathcal{F}^j_{2m} = x_j\mathcal{F}^{j+1}_{2m} + \mathcal{F}^{n-4}_{2m-1} \).

(ii) For all \( m \geq 0 \), we have

\[ \mathcal{F}^0_{2m+1} = (1 + x_0x_a + x_0x_b + x_0x_ax_b)\mathcal{F}^0_{2m} - x_0x_ax_b\mathcal{F}^{n-4}_{2m-1}. \]

(iii) For all \( m \geq 0 \), we have

\[ \mathcal{F}^1_{2m+1} = (x_1 + 1)\mathcal{F}^0_{2m+1} - x_1\mathcal{F}^0_{2m}. \]

(iv) For all \( m \geq 0 \), \( j = 2, \ldots, n-4 \), we have

\[ \mathcal{F}^j_{2m+1} = (x_j + 1)\mathcal{F}^{j-1}_{2m+1} - x_j\mathcal{F}^{j-2}_{2m}. \]

(v) For all \( m \geq 1 \), we have

\[ \mathcal{F}^{n-4}_{2m} = (1 + x_{n-4}x_c + x_{n-4}x_d + x_{n-4}x_c x_d)\mathcal{F}^{n-4}_{2m-1} - x_{n-4}x_c x_d \mathcal{F}^{n-5}_{2m-1}. \]

**Proof.** An admissible subset of \( \mathcal{G}(2m,j)_0 \) is obtained from one of \( \mathcal{G}(2m,j-1)_0 \) by adding the vertex corresponding to the index \( (2m,j) \) or it is given by an admissible subset \( \mathcal{G}(2m-1,n-4)_0 \). Note that if \( (2m,i) \) is a vertex of an admissible subset, then \( (2m,i-1) \) is forced to be part of the admissible subset because it is extremal successor closed for \( i = n-4, \ldots, 1 \). Thus we obtain the first statement. The third and forth statements can be obtained similarly.
An admissible subset of \(D(2m + 1, 0)_0\) is obtained by adding an admissible subset of the ramification subgraph which is glued. This corresponds to the first summand in the second statement. But because of the second property we have to drop those subsets containing the vertices \(sm + n - 2, sm + n - 1, sm + n\) but not containing \(sm + n - 3\). This gives the second summand. The last statement can be obtained by a similar argument.

Let \(H(x, y, z) = 1 + x + xy + xz + xyz\). Together with the observations in the beginning of this section, Lemmas 4.6 and 4.9 give rise to the following recursive description of the generating functions:

**Corollary 4.10.**

\[
\begin{pmatrix}
\mathcal{F}_{n-4}^{2m+1} \\
\mathcal{F}_{2m+2}^{n-4}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & -F_{1, n-5} + 1 & F_{1, n-5} \\
-x_{n-4}x_c x_d & H(x_{n-4}, x_c, x_d) & F_{1, n-4} & -F_{1, n-4} + 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-x_{0}x_a x_b & H(x_0, x_a, x_b)
\end{pmatrix}
\begin{pmatrix}
F_{n-6} & 0 \\
\Pi_{i=0}^{n-5} x_i
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_{2m-1}^{n-4} \\
\mathcal{F}_{2m}^{n-4}
\end{pmatrix}
\]

For \(n = 4\), we get

\[
\begin{pmatrix}
\mathcal{F}_{2m+1}^4 \\
\mathcal{F}_{2m+2}^4
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & -F_{1, 2m-1} + 1 & F_{1, 2m-1} \\
-x_0 x_c x_d & H(x_0, x_c, x_d) & F_{1, 2m} & -F_{1, 2m} + 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-x_0 x_a x_b & H(x_0, x_a, x_b)
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-x_0 x_a x_b H(x_0, x_c, x_d) & -x_0 x_c x_d + H(x_0, x_c, x_d) H(x_0, x_a, x_b)
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_{2m-1}^4 \\
\mathcal{F}_{2m}^4
\end{pmatrix}
\]

With this tool in hand we can determine the \(F\)-polynomials of representations of \(\tilde{D}_n\) explicitly using formulae (4.1), (4.2). This is done in the following subsections.

### 4.3. The homogeneous tube.

The \(F\)-polynomials of the representations of the homogeneous tubes play an important role in the following. The imaginary Schur root \(\delta\) of \(\tilde{D}_n\) is independent of the orientation of \(\tilde{D}_n\). Using the methods of section 2.1 and 3, it is straightforward to check that the \(F\)-polynomial of a representation of dimension \(\delta\) lying in one of the homogeneous tubes does not depend on the tube. Thus we can fix a homogeneous tube without loss of generality. We denote the unique representation of dimension \(\delta\) in this tube by \(M_\delta\) and define \(F_\delta := F_{M_\delta}\). Moreover, for every \(r \geq 1\) there exists an almost split sequence of the form

\[
0 \rightarrow M_{r\delta} \rightarrow M_{(r-1)\delta} \oplus M_{(r+1)\delta} \rightarrow M_{r\delta} \rightarrow 0
\]

where \(M_{0\delta} = M_0 := 0\) and \(F_{M_0} = 1\). By applying Theorem 1.9, we obtain

**Lemma 4.11.** The \(F\)-polynomial of representations lying in one of the homogeneous tubes depends only on the dimension vector and satisfies the recursion

\[
F_{(r+1)\delta} F_{(r-1)\delta} = F_{r\delta}^2 - x^{r\delta}
\]

for \(r \geq 1\).
By the methods of this section, we also obtain an explicit formula for \( F_\delta \). Recall Corollary 1.23, saying that

\[
\begin{align*}
F_{r\delta} &= F_\delta F_{(r-1)\delta} - x^\delta F_{(r-2)\delta}.
\end{align*}
\]

This yields

\[
\begin{pmatrix}
F_{r\delta} \\
F_{(r+1)\delta}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 \\
-x^\delta & F_\delta
\end{pmatrix}^{r+1}
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

Defining

\[
z = \frac{1}{2} \sqrt{F_\delta^2 - 4x^\delta}, \quad \lambda_\pm = \frac{F_\delta}{2} \pm z,
\]

we thus get the following explicit formula:

**Corollary 4.12.** We have

\[
F_{r\delta} = \frac{1}{2z} (\lambda_+^{r+1} - \lambda_-^{r+1}).
\]

### 4.4. The exceptional tubes of rank two

In this section, we apply the developed methods to representations lying in the exceptional tubes of rank two. To do so we first restrict to \( \tilde{D}_4 \) in subspace orientation. Afterwards, we extend the results to \( \tilde{D}_n \) in subspace orientation and, finally, to any orientation.

If \( \alpha \) is a real root let \( F_\alpha := F_{M_\alpha} \). Similar to the case of preprojective representations of defect \(-1\), we obtain all coefficient quivers of representations lying in this tube by gluing the coefficient quivers

\[
\begin{array}{ccc}
& c & \\
a & & b \\
& & d
\end{array}
\]

We denote the representation on the left hand side by \( T_1 \) and the representation on the right hand side by \( T_2 \). Then we get for the generating functions

\[
F_{T_1} = 1 + x_0 + x_0x_a + x_0x_c + x_0x_ax_c, \quad F_{T_2} = 1 + x_0 + x_0x_b + x_0x_d + x_0x_bx_d.
\]

Without loss of generality we can assume that we start our gluing process with the coefficient quiver of \( T_1 \). Using the notation and results of subsection 4.2 and, moreover, by Theorem A, we have \( f_{-1} = 0, f_0 = 1 \) and

\[
f_{2r+1} = F_{T_1} f_{2r} - x_0x_ax_c f_{2r-1}, \quad f_{2r+2} = F_{T_2} f_{2r+1} - x_0x_bx_d f_{2r}
\]

where \( f_{2r+1} \) is the generating function of the unique indecomposable of dimension \( t(r) := \dim T_1 + r \cdot \delta \) and \( f_{2r+2} \) is the generating function of the unique indecomposable \( M_{(r+1)\delta}^1 \) of dimension \( (r+1) \cdot \delta \) such that \( T_1 \subset M_{(r+1)\delta}^1 \). Note that the recursion is up to permutation of arrows the same as the one in subsection 4.2. The only difference is that we start our gluing process in the present situation with the empty coefficient quiver while in subsection 4.2, we start with the coefficient quiver \( \bullet \leftarrow \bullet \).
Proposition 4.13. For $r \geq 0$, we have
\[
F_t(r) = f_{2r+1} = F_t F_r \delta = F_t(0) F_r \delta,
\]
\[
F_{M_t^1(r+1)\delta} = f_{2r+2} = F_{(r+1)\delta} + x_0 x_a x_c F_r \delta
\]

Proof. Using the notation from subsection 4.2, we have
\[
a = -x_0 x_a x_c, \quad b = F_T_1, \quad c = -x_0 x_a x_c F_T_2, \quad d = -x_0 x_b x_d + F_T_1 F_T_2.
\]
Then it is easy to check that we have
\[
a + d = F_\delta, \quad z = \frac{1}{2} \sqrt{F_\delta^2 - 4x_\delta}, \quad ad - bc = \lambda_+ \lambda_- = x_\delta.
\]
Moreover, we get
\[
\lambda_+ = \frac{1}{2}(F_\delta + \sqrt{F_\delta^2 - 4x_\delta}), \quad \lambda_- = \frac{1}{2}(F_\delta - \sqrt{F_\delta^2 - 4x_\delta}).
\]
Since $f_{-1} = 0$ and $f_0 = 1$, equation (4.1) yields
\[
f_{2r+1} = \frac{1}{2z}(F_T_1(\lambda_+^{r+1} - \lambda_-^{r+1})).
\]
Thus it remains to show that
\[
F_\delta = \frac{1}{2z}(\lambda_+^{r+1} - \lambda_-^{r+1}).
\]
For $r = 0$, this is clearly true. Since $a + d = F_\delta$ this is also true for $r = 1$. By Lemma 4.11, it suffices to show that
\[
\left(\frac{\lambda_+^{r+1} - \lambda_-^{r+1}}{2z}\right) \left(\frac{\lambda_+^{r-1} - \lambda_-^{r-1}}{2z}\right) = \left(\frac{\lambda_+ - \lambda_-}{2z}\right)^2 - x^{(r-1)\delta}
\]
for $r \geq 1$, what follows from
\[
2z = \lambda_+ - \lambda_-, \quad \lambda_+ \lambda_- = x_\delta.
\]
Using Equation (4.2), we have
\[
f_{2r+2} = \frac{1}{2z} \left(\lambda_+^{r+1}(-x_0 x_a x_c - \lambda_-) - (\lambda_+^{r+1}(-x_0 x_a x_c - \lambda_+))\right)
\]
\[= \frac{1}{2z} (\lambda_+^{r+2} - \lambda_-^{r+2}) + \frac{1}{2z} x_0 x_a x_c (\lambda_+^{r+1} - \lambda_-^{r+1})
\]
\[= F_{(r+1)\delta} + x_0 x_a x_c F_r \delta,
\]
which completes the proof of the proposition. \qed

Let us consider the tubes of rank two for general $n$ with arbitrary orientation. For a fixed tube, we denote by $t_1(0)$ and $t_2(0)$ the quasi-simple roots. The real roots in this tube are given by $t_i(r) = t_i(0) + r \delta$. Finally, we denote the representation of dimension $r \delta$ with subrepresentation $M_{t_i(0)}$ by $M'_{r \delta}$.
Theorem 4.14. For the indecomposable representations \( M_{\ell_1(r)} \) and \( M_{\ell_2} \) lying in one of the exceptional tubes of rank two of \( \tilde{D}_n \), we have:

(i) \( F_{\ell_1(r)} = F_{\ell_1(0)}F_{\delta} \)

(ii) \( F_{M_{\ell_2}} = F_{\delta} + x_{\ell_2(0)}F_{(r-1)\delta} \)

Proof. Under consideration of Lemma 4.4 it is straightforward to generalize Proposition 4.13 to arbitrary \( \tilde{D}_n \) in subspace orientation.

Assume that \( M \) with \( \dim M = t_i(r) + r\delta \) lies in one of the exceptional tubes of rank two of \( \tilde{D}_n \) (with arbitrary orientation) and satisfies \( F_M = F_{\ell_1(0)}F_{\delta} \). Applying Theorem 3.5, we have

\[ F_{\sigma_i M} = F_{\sigma_i\ell_1(0)}F_{\delta}. \]

Thus the first statement follows by induction.

For a fixed sink \( q \), of \( \tilde{D}_n \) with arbitrary orientation, it is straightforward to check that

\[ \sum_{p \in Q_0} a(p, q)t_i(0)_p = \delta_q. \]

Indeed, if \( q \in \{a, b, c, d\} \), both sides are one. Otherwise both sides are two. Assume that \( F_{M_{\ell_2}} = F_{\delta} + x_{\ell_2(0)}F_{(r-1)\delta} \). Then, again by Theorem 3.5 and Lemma 4.2, we have

\[ F_{\sigma_i M_{\ell_2}} = F_{\delta} + x_{\sigma_i\ell_2(0)}(1 + x_{\ell_2(0)}^{-1})\sum_{p \in Q_0} a(p, q)\delta_{(r-1)\delta} = F_{\delta} + x_{\sigma_i\ell_2(0)}F_{(r-1)\delta}. \]

Thus the second statement also follows by induction. \( \square \)

4.5. The exceptional tube of rank \( n-2 \). Let us consider the tube of rank \( n-2 \). In this tube, there exist \( n-2 \) indecomposable representations of dimension \( r\delta \), which we denote by \( M_{\ell_i} \) for \( i = 1, \ldots, n-2 \). If it is clear which of these representations is considered, we drop the \( i \).

We again first restrict to subspace orientation and treat the general case later. Let \( d_m(r) \) be the root given by

\[ d_m(r)_q = \begin{cases} 
  r & \text{for } q = a, b \\
  r - 1 & \text{for } q = c, d \\
  2r - 1 & \text{for } q = 0, \ldots, m \\
  2r - 2 & \text{for } i = m + 1, \ldots, n - 4 
\end{cases} \]

In this tube, there exists an infinite chain of irreducible inclusion

\[ M_{d_0(1)} \hookrightarrow M_{d_1(1)} \hookrightarrow \cdots \hookrightarrow M_{d_{n-4}(1)} \hookrightarrow M_{d_{n-2}(1)} \hookrightarrow M_{d_0(2)} \hookrightarrow \cdots \]

where the imaginary root representations \( M_{\ell_i} \) are uniquely determined by this chain.

Lemma 4.15. Let \( n = 5 \). Then we have

(i) \( F_{d_0(r)} = F_{d_0(1)}F_{\delta} + x_0x_1x_ay_bF_{(r-1)\delta} \)

(ii) \( F_{d_1(r)} = F_{d_1(1)}F_{\delta} \)

(iii) \( F_{M_{\delta}} = F_{\delta} + x_0x_1x_b(1 + x_1)F_{(r-1)\delta} \)
Proof. The last two statements follow from Proposition 4.13 together with Lemma 4.4. Considering the coefficient quivers of the respective representations pointed out in [14, Appendix B] and applying the methods of Lemma 4.9, we obtain

\[ F_{d_0}(r) = F_{d_0}(1)F_{M_{r,\delta}} - x_0 x_a x_b F_{d_1}(r-1) = F_{d_0}(1)F_{r,\delta} + x_0 x_a x_b F_{r,\delta}. \]

Here we use that \( F_{d_0}(1)(1 + x_1) - x_1 = F_{d_1}(1) \), see also Lemma 4.9.

For \( l \leq m \), we denote by \( S_{l,m} \) the exceptional representation with dimension vector \( \dim S_{l,m} = \Sigma_{i=l}^m q_i \). Note that

\[ \dim S_{m+2,n-4} = d_{n-4}(1) - d_0(1) - \tau^{-1}d_m(1) = d_{n-4}(1) - d_{m+1}(1). \]

Moreover, we have

\[ \prod_{i=1}^{m+1} x_i = x^{\tau^{-1}d_m(1)}, \quad x_0 x_a x_b = x^{d_0(1)}. \]

Lemma 4.16. For arbitrary \( n \), we have

(i) \( F_{d_m}(r) = F_{d_m}(1)F_{r,\delta} + x_0 x_a x_b \prod_{i=1}^{m+1} x_i F_{S_{m+2,n-4}}(r-1) \) for \( m = 0, \ldots, n-5 \).

(ii) \( F_{d_{n-4}}(r) = F_{d_{n-4}}(1)F_{r,\delta} \)

(iii) \( F_{M_{r,\delta}} = F_{r,\delta} + x_0 x_a x_b F_{S_{1,n-4}}(r-1) \)

Proof. This is obtained when combining Lemma 4.15 and Lemma 4.4.

For \( \tilde{D}_n \) with arbitrary orientation in the tube of rank \( n-2 \), there exist \( n-2 \) chains of irreducible morphisms of the form

\[ M_{0,1} \hookrightarrow M_{0,2} \hookrightarrow \ldots \hookrightarrow M_{0,n-3} \hookrightarrow M_{1,0} := M_{\delta} \hookrightarrow M_{1,1} \hookrightarrow \ldots \]

where the \( m_l(r) := \dim M_{r,l} \) are real roots and the imaginary root representations \( M_{r,0} := M_{r,\delta} \) are uniquely determined by this chain. In particular, for every real root \( \alpha \) in the tube of rank \( n-2 \) there exists an exceptional root \( m_l(0) \) such that \( \alpha = r\delta + m_l(0) \).

Lemma 4.17. For \( 1 \leq l \leq n-4 \), we have \( \delta = m_{n-3}(0) + \tau m_{l+1}(0) - m_l(0) \).

Proof. Considering the tube and its roots in detail, we obtain

\[ \delta = m_{n-3}(0) + \tau m_{n-3}(0) - m_{n-4}(0). \]

Since we also have

\[ m_l(0) + \tau m_{l+2}(0) = m_{l+1}(0) + \tau m_{l+1}(0) \Leftrightarrow \tau m_{l+2}(0) - m_{l+1}(0) = \tau m_{l+1}(0) - m_l(0), \]

the claim follows by induction.

Under the convention that \( F_{\alpha} = 0 \) if \( \alpha \in \mathbb{N}Q_0 \) has one negative coefficient, we obtain the following result:
Theorem 4.18. Set \( m_0(r) = m_{n-2}(r-1) = r\delta \). Let \( M \) be an indecomposable representation of \( \tilde{D}_n \) lying in the tube of rank \( n - 2 \) such that \( \dim M = m_l(r) \) for some \( l = 0, \ldots, n - 3 \). Then we have and, moreover, we have

\[
F_{m_l(r)} = F_{m_l(0)}F_{r\delta} + x^{m_{l+1}(0)}F_{m_{n-1}(0) - m_{l+1}(0)}F_{(r-1)\delta}
\]

for \( l = 0, \ldots, n - 3 \).

Proof. We proceed by induction. For every representation \( M \) lying in the tube of rank \( n - 2 \), there exists a sequence of reflections \( \sigma_1, \ldots, \sigma_l \) at sinks and a representation \( M_{d_m(r)} \) with \( m \in \{-1, \ldots, n - 4\} \) such that \( M = \sigma_1 \cdots \sigma_l M_{d_m(r)} \). Since the claim is true for \( M_{d_m(r)} \), by Lemma 4.16, we can assume that \( M = \sigma_qM_{r,l} \) and that the claim is true for \( M_{r,l} \), i.e. we have

\[
F_{m_l(r)} = F_{m_l(0)}F_{r\delta} + x^{m_{l+1}(0)}F_{m_{n-1}(0) - m_{l+1}(0)}F_{(r-1)\delta}
\]

Then applying Theorem 3.5 and Lemma 4.2, we have

\[
F_M = F_{\sigma_qm_l(0)}F_{r\delta} + (1 + x^{-1})^a(1 + x^{-1})^b x^{\sigma_qm_{l+1}(0)}F_{\sigma_qm_{n-1}(0) - m_{l+1}(0)}F_{(r-1)\delta}
\]

with

\[
a = -\delta_q - m_l(0)q + m_{n-3}(0)q - m_{l+1}(0)q, \quad b = \sum_{p \in Q_0} a(p,q)m_{l+1}(0)p.
\]

Again by Lemma 4.2 and, moreover, by Lemma 4.17, we have

\[
a + b = -\delta_q - m_l(0)q + m_{n-3}(0)q + \tau m_{l+1}(0)q = 0,
\]

which completes the proof of the theorem. \( \square \)

Note that Theorems 4.14 and 4.18 can be summarized as done in Theorem C.

4.6. The preprojectives of small defect. Finally, we consider the preprojective roots. Also in this case, we obtain explicit formulae. Thanks to Theorem 1.17, the generating functions corresponding to the roots of defect \(-2\) can easily be obtained from those of defect \(-1\). Moreover, the generating functions for the preinjectives can be calculated when passing to the opposite quiver.

We again denote the projective representations corresponding to the outer vertices by \( P_a, P_b, P_c \) and \( P_d \). We follow the strategy of the last two subsections and first restrict to subspace orientation. Up to permutation of the sources, the preprojective roots for \( n = 4 \) are given by

\[
d_1(r) = (2r + 1, r + 1, r, r, r) \quad \text{and} \quad d_2(r) = (2r, r - 1, r, r, r).
\]

Proposition 4.19. Let \( n = 4 \). Then we have

(i) \( F_{d_1(r)} = F_{P_a}F_{r\delta} - x^{\dim \tau^{-1}P_a}F_{r\delta - \dim \tau^{-1}P_a}F_{(r-1)\delta} \)

(ii) \( F_{d_2(r)} = F_{\tau^{-1}P_a}F_{(r-1)\delta} - x^\delta F_{(r-2)\delta} \)
Proof. Initially, we consider the coefficient quiver of the preprojective representation of dimension 
$(1, 1, 0, 0, 0)$, i.e. $\bullet \overset{a}{\rightarrow} \bullet$. By gluing the coefficient quivers

in turn, we obtain the coefficient quivers of the representations under consideration pointed out in
[14, Appendix B], and we can apply Theorem A. We denote the corresponding representations by
$T_1$ and $T_2$ respectively. Using the notation and results from section 4.2, we have

$$f_0 = 1, f_1 = 1 + x_0 + x_0 a.$$ 

Moreover, we have $f_2 r = F_{d_2}(r)$ and $f_{2r+1} = F_{d_1}(r)$ and

$$a = -x_0 x_c x_d, \quad b = F_{T_1}, \quad c = -x_0 x_c x_d F_{T_2}, \quad d = -x_0 x_a x_b + F_{T_1} F_{T_2}.$$ 

Using $\lambda_+ \lambda_- = ad - bc = x^\delta$, we also have

$$(a - \lambda_+)(a - \lambda_-) = (a^2 + (ad - bc) - a(a + d)) = -bc.$$ 

Since we have

$$F_{r \delta} = \frac{1}{2 \delta}(\lambda_+^{r+1} - \lambda_-^{r+1}),$$

see Equation (4.3), (permuting $q_a$ and $q_b$) we obtain

$$f_{2r} = -x_0 x_c x_d F_{(r-1) \delta} - x^\delta F_{(r-2) \delta} + F_{T_1} F_{(r-1) \delta} f_1 = F_{r-1} P_a F_{(r-1) \delta} - x^\delta F_{(r-2) \delta},$$ 

$$f_{2r+1} = c F_{(r-1) \delta} - (a F_{(r-1) \delta} - F_{r \delta}) f_1$$

$$= F_{r-1} P_a F_{(r-1) \delta} - x F_{(r-1) \delta}.$$ 

Note that $\delta - \dim \tau^{-1} P_a = \dim S_a$ is the simple root (which is in this case also the injective root 
respectively) corresponding to the vertex $a$. \hfill $\Box$

Remark 4.20. Applying the reflection functor to $M_{d_1}(r)$ (resp. $F_{d_1}(r)$), it is straightforward to check that

$$F_{d_2}(r) = (1 + x^{-1}) F_{r \delta} - x^{\dim P_a} F_{r \delta} - \dim F_{(r-1) \delta}.$$ 

Note that $\tau d_1(r) = d_2(r)$, where $d_2(r)$ is defined below.

The following is straightforward to check:

Lemma 4.21. Considering the variable transformation of Lemma 4.1, we have

$$(1 + x) F_{r \delta} - \dim P_a (x') = F_{r \delta} - \dim P_a (x)$$

where the root $\delta - \dim P_a$ on the left hand side is the root $q_0 + q_a + q_c + q_d$ of $D_4$ and on the right 
hand side it is the root $q_0 + 2 q_1 + q_a + q_c + q_d$ of $D_5$.

For general $n$, the preprojective roots of defect $-1$ are given by:
\begin{itemize}
\item $d^m_i(r)$, $m = 0, \ldots, n - 4$, with $d^m_i(r)_{q_i} = 2r + 1$ for $i = 0, \ldots, m$, $d^m_i(r)_{q_i} = 2r$ for $i = m + 1, \ldots, n - 4$, $d^m_i(r)_a = d^m_i(r)_c = d^m_i(r)_d = r$ and $d^m_i(r)_b = r + 1$. We denote by $\hat{d}^m_i(r)$ the resulting root obtained when permuting $q_a$ and $q_b$.
\item $d_2(r)$ with $d_2(r)_{q_i} = 2r$ for $i = 0, \ldots, n - 4$, $d_2(n)_a = d_2(c) = d_2(d) = r$ and $d_2(b) = r - 1$. We denote by $\hat{d}_2(r)$ the resulting root obtained when permuting $q_d$ and $q_a$.
\item $d_3^m(r)$, $m = 0, \ldots, n - 4$, with $d_3^m(r)_{q_i} = 2r$ for $i = 0, \ldots, m$, $d_3^m(r)_{q_i} = 2r - 1$ for $i = m + 1, \ldots, n - 4$, $d_3^m(r)_a = d_3^m(r)_b = d_3^m(r)_c = r$ and $d_3^m(r)_d = r - 1$. We denote by $\hat{d}_3^m(r)$ the resulting root obtained when permuting $q_c$ and $q_d$.
\item $d_4(r)$ with $d_4(r)_{q_i} = 2r + 1$ for $i = 0, \ldots, n - 4$, $d_4(r)_a = d_4(r)_b = d_4(r)_c = r$ and $d_4(r)_d = r + 1$. We denote by $\hat{d}_4(r)$ the resulting root obtained when permuting $q_c$ and $q_d$.
\end{itemize}

Note that we have $\tau^{-1}d_1^m(r) = \hat{d}_1^{m+1}(r)$ for $m = 0, \ldots, n - 5$, $\tau^{-1}d_2^{m+4}(r) = \hat{d}_2(r + 1)$ and $\tau^{-1}\hat{d}_2(r + 1) = d_1^0(r + 1)$. The analogous statement holds for $d_3^m(r)$ and $d_4(r + 1)$.

We denote by $P_m$ the projective corresponding to the vertex $q_m$ and by $s_q$ the simple root corresponding to $q$. We have $d_1^m(0) = \dim P_m + s_a$.

**Proposition 4.22.** Let $\tilde{D}_n$ be in subspace orientation. Then we have

\begin{enumerate}
\item $F_{d_1^m}(r) = F_{d_1^m(0)}F_{r\delta} - x_r^{-1}F_{d_1^m(0)}F_{\delta - \tau^{-1}d_1^m(0)}F_{(r-1)\delta}$
\item $F_{d_2}(r) = F_{d_2(1)}F_{(r-1)\delta} - x_{(r-2)\delta}$
\item $F_{d_3^m}(r) = F_{d_3^m(1)}F_{(r-1)\delta} - x_r^{-1}F_{d_3^m(1)}F_{\delta - \tau^{-1}d_3^m(1)}F_{(r-2)\delta}$
\item $F_{d_3^{m-4}}(r) = F_{d_3^{m-4}(1)}F_{(r-1)\delta} - x_{(r-2)\delta}F_{(r-2)\delta}$
\item $F_{d_4}(r) = F_{d_4(0)}F_{r\delta} - x_{(r-1)\delta}F_{d_4(0)}F_{\delta - \tau^{-1}d_4(0)}F_{(r-1)\delta}$
\end{enumerate}

**Proof.** First, we consider the case $n = 5$ and then we apply Lemma 4.1 afterwards in order to obtain the general case. By Lemma 4.4, and Proposition 4.19 we obtain

$$F_{d_1^1(r)} = F_{d_1^1(0)}F_{r\delta} - x_{\tau^{-1}d_1^1(0)}F_{\delta - \tau^{-1}d_1^1(0)}F_{(r-1)\delta}.$$ 

We have $\tau\hat{d}_1^1(r) = d_1^0(r)$ where $\tau\hat{d}_1^1(r)$ is obtained from $d_1^1(r)$ when reflecting successively at the sinks $q_0, q_a, q_b, q_1, q_c$ and $q_d$. Thus, keeping in mind Lemma 4.2 and Theorem 3.5, a straightforward calculation yields

$$F_{d_1^0(r)} = F_{\tau d_1^1(r)} = F_{\tau d_1^1(0)}F_{r\delta} - x_{\tau^{-1}\hat{d}_1^1(0)}F_{\delta - \tau^{-1}\hat{d}_1^1(0)}F_{(r-1)\delta} = F_{d_1^0(0)}F_{r\delta} - x_{\tau^{-1}d_1^0(0)}F_{\delta - \tau^{-1}d_1^0(0)}F_{(r-1)\delta}.$$ 

Again by Proposition 4.19 together with Lemma 4.4, we get

$$F_{d_2}(r) = F_{d_2(1)}F_{(r-1)\delta} - x_{(r-2)\delta}F_{(r-2)\delta}.$$ 

Moreover, we get

$$F_{d_3^1(r)} = F_{d_3^1(1)}F_{(r-1)\delta} - x_{(r-2)\delta}F_{(r-2)\delta}.$$
Remark 4.23. Let \( q \) be a sink. Finally, also the formula for \( F(i) \) is obtained in this way. In order to obtain the formulae for general \( n \geq 4 \), we can apply Lemma 4.4 starting with the case \( n = 5 \).

Remark 4.23. Using Lemma 4.21 together with Equation (4.1), we get

\[
F_{d_3}(r) = (1 + x_b^{-1})F_{r\delta} - x^{\dim P_b}F_{\delta - \dim P_b}F_{r(1)}F_{\delta}.
\]

It is likely that there is a similar formula for \( d_3^{n-4}(r) \).

- For the remaining part of the subsection, we should keep the following in mind, see [13]: if \( \alpha \leq \delta \) is a preinjective root such that \( \sigma_q(\alpha) > \delta \), then \( q \) is a source and \( \delta - \alpha \) is the injective simple root corresponding to \( q \). Indeed, \( \delta - \alpha \) is a preinjective root if \( 0 < \alpha < \delta \) is preprojective. Since the positive non-simple roots are invariant under the Weyl group, \( \delta - \alpha \) is forced to be simple. In particular, if \( \alpha < \delta \) and \( \tau^{-1} \alpha < \delta \), we have that \( \sigma_q \tau^{-1} \alpha > \delta \) and only if \( \delta - \tau^{-1} \alpha \) is the simple root corresponding to the source \( q \). The analogous statement holds if \( \alpha \) is preinjective.
- If \( q \) is a sink, we have that \( P_q = S_q \). If \( t_M \) is preprojective with \( \tau^{-1}t_M < \delta \), we have \( \sigma_q \tau^{-1}t_M < \delta \) because otherwise \( \delta - \tau^{-1}t_M = s_q \) were injective. In turn if \( \alpha < \delta \) and \( \sigma_q \tau^{-1}t_M > \delta \), we already have \( \tau^{-1}t_M > \delta \) in which case \( \delta - t_M \) is injective.
- If \( \alpha \) is a preprojective root of defect \( 1 \) of \( \tilde{D}_n \) in subspace orientation, we have that \( \delta - \alpha \) is injective if and only if \( \alpha = d_2(1) \) or \( \alpha = d_3^{n-4}(1) \).
- If \( I_q \) is the injective representation corresponding to \( q \) and \( q' \neq q \) is a source, we have that \( \sigma_q I_q \) is also injective. Note that since \( \tilde{D}_n \) is tree-shaped, we have \( \dim(I_q)_{q'} = 1 \) if and only if there exists a (unique) path from \( q' \) to \( q \) and \( \dim(I_q)_{q'} = 0 \) otherwise.

Proposition 4.24. Let \( M \) be a preprojective representation of \( \tilde{D}_n \) such that \( t_M := \dim M - r \delta \leq \delta \).

Let \( q \) be a sink.

(i) Assume that

\[
F_M = F_{t_M}F_{r\delta} - x^{\tau^{-1}t_M}F_{\delta - \tau^{-1}t_M}F_{r(1)}F_{\delta}.
\]

If \( \tau^{-1}t_M < \delta \) and \( s_q \neq t_M \leq \delta \), we have

\[
F_{\sigma_q M} = F_{\sigma_q t_M}F_{r\delta} - x^{\sigma_q \tau^{-1}t_M}F_{\delta - \sigma_q \tau^{-1}t_M}F_{r(1)}F_{\delta},
\]

where we also have \( \sigma_q \tau^{-1}t_M < \delta \). If \( t_M = s_q \) we have

\[
F_{\sigma_q M} = F_{\delta - \dim I_q}F_{r(1)}F_{\delta} - x^{\delta}F_{r(2)}F_{\delta}.
\]
(ii) Assume that

\[ F_M = F_t M F_{r\delta} - x^\delta F_{(r-1)\delta} \]

and that \( \delta - t_M \) is the injective root corresponding to \( q' \). Then we have \( \delta - t_M = -\dim P_q \). Moreover, if \( q \neq q' \) is a sink

\[ F_{\sigma q M} = F_{\sigma q M} F_{r\delta} - x^\delta F_{(r-1)\delta}. \]

and if \( q = q' \) is a sink, (we have \( \delta - t_M \neq s_q \) and) we have

\[ F_{\sigma q M} = F_{\sigma q M} F_{r\delta} - x^{\sigma q \tau^{-1} M} M_{\delta - \sigma q \tau^{-1} M} F_{(r-1)\delta} \]

where \( \delta - \sigma q \tau^{-1} M = s_q \).

**Proof.** For a sink \( q \), by the second part of Lemma 4.2, we have

\[ \sum_{p \in \mathcal{Q}_0} a(p, q) \tau^{-1}(t_M)p - (\tau^{-1} t_M)_q = (t_M)_q. \]

Thus we get

\[ (t_M + r\delta)_q = \sum_{p \in \mathcal{Q}_0} a(p, q) \tau^{-1}(t_M)p + (\delta - \tau^{-1} t_M)_q + ((r - 1)\delta)_q. \]

Thus, since \( t_M \) is not the simple root corresponding to \( q \), the claim follows by Theorem 3.5. If \( t_M = s_q \), we have \( \tau t_M = -\dim I_q \). In particular, \( \delta - (\delta + \tau t_M) \) is injective. Thus by the second part of the proposition, we have

\[ F_{\tau M} = F_{\delta - \dim I_q M} F_{(r-1)\delta} - x^\delta F_{(r-2)\delta}. \]

Now there exists an admissible sequence

\[ \sigma := \prod_{q' \in \{D_0\} \setminus q \neq q} \sigma_{q'} \]

of reflections at sources such that \( \sigma \tau M = \sigma q M \). Since the first part of the second statement clearly also holds in the opposite direction, the claim follows.

For the second statement, note that \( \sigma q (\delta - M) \) is preinjective and \( s_q \) is the injective of \( \sigma q Q \). Indeed, \( q \) is a sink of \( Q \). For the first part, it suffices to show that

\[ \sum_{p \in \mathcal{Q}_0} a(p, q) \delta_p = (t_M)_q + \delta_q. \]

This can be deduced from

\[ \sum_{p \in \mathcal{Q}_0} a(p, q) \delta_p - \delta_q = \sum_{p \in \mathcal{Q}_0} a(p, q) (\tau^{-1} t_M)_p - (\tau^{-1} t_M)_q. \]

what actually follows from \( \tau^{-1} t_M - \delta = \dim P_q \). Indeed, since \( q \) is a sink and \( Q \) tree-shaped, we have \( \dim(P_q q)_p = 1 \) if and only if there is exactly one neighbor \( p \) such that \( \dim(P_q q)_p = 1 \).

Since \( q \) is a sink, we have \( \dim P_q = s_q \). Thus the last statement follows because

\[ x^\delta (1 + x^{-1}_q) = x^\delta - s_q F_{s_q} = x^{\sigma q \tau^{-1} M} M_{\delta - \sigma q \tau^{-1} M}. \]
and \((\delta - \tau^{-1}t_M)_q = 1\) in this case.

Clearly the formulae hold in the other direction as well. This leads to the main result of this section:

**Theorem 4.25.** Let \(M\) be preprojective of defect \(-1\) such that \(t_M = \dim M - r\delta \leq \delta\). If \(\delta - t_M\) is injective we have

\[
F_M = F_{tM}F_{r\delta} - x^{\delta}F_{(r-1)\delta}.
\]

If \(\delta - t_M\) is not injective we have

\[
F_M = F_{tM}F_{r\delta} - x^{\tau^{-1}t_M}F_{\delta - \tau^{-1}t_M}F_{(r-1)\delta}.
\]

**Proof.** Fix \(\tilde{D}_n\) in subspace orientation. Then the statement is true by Proposition 4.22 keeping in mind Remark 4.23.

Now assume that \(\alpha_1, \ldots, \alpha_l\) are all preprojective roots of defect \(-1\) of \(\tilde{D}_n\) with a fixed orientation such that \(\alpha_i \leq \delta\). If \(q\) is a sink, we can reflect at \(q\) to obtain all preprojective roots \(\sigma_q\alpha\) with \(\sigma_q\alpha \leq \delta\) of \(\sigma_qQ\) except \(\beta_q := \delta - s_q\). In particular, we can apply Proposition 4.24 to obtain the generating functions corresponding to preprojectives except those of the form \(r\delta + \beta_q\). But since the case of the preprojective roots of the form \(r\delta + \beta_q\) is covered by the second part of Proposition 4.24, the claim follows.

**References**


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