GENERAL REPRESENTATIONS OF QUIVERS

AIDAN SCHOFIELD

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ABSTRACT

We investigate general representations of a quiver \( Q \). There are two main questions that we shall address. The first, which was raised by Kac, is to find the dimension vectors of the summands of a general representation of dimension vector \( \alpha \); the second is to find the dimension vectors of the subrepresentations of a general representation of dimension vector \( \alpha \). These two questions are closely related. One consequence of our investigation is to find an algorithm to compute the canonical decomposition of an arbitrary dimension vector in terms of the Euler form of the quiver. We may also compute the dimension vectors of subrepresentations of a general representation and the minimal dimension of \( \text{Ext}(R, S) \) as \( R \) and \( S \) run through representations of dimension vectors \( \alpha \) and \( \beta \) respectively.

Introduction

A major goal in the representation theory of quivers is to classify all the representations of any given quiver. The first step in this process was taken by Kac who showed that the dimension vectors of indecomposable representations are precisely the positive roots of the Kac–Moody Lie algebra associated to the underlying graph of the quiver [5]. In the course of this investigation, he noted that the representations of a dimension vector \( \alpha \) of the quiver \( Q \) are parametrised by a vector space \( R(Q, \alpha) \) on which an algebraic group \( \text{GL}(Q, \alpha) \) acts such that the orbits of the group on the vector space are in 1–1 correspondence with the isomorphism classes of representations. Certain properties of the representation associated to a given point of \( R(Q, \alpha) \) will be independent of the point chosen in an open subset of \( R(Q, \alpha) \) and thus we say that such properties are true of the general representation. In particular, if \( R_p \) is the representation associated to the point \( p \) and \( R_p = \bigoplus S_{i,p} \) where each \( S_{i,p} \) is an indecomposable representation, Kac showed that there is an open subset \( U \) of \( R(Q, \alpha) \) such that for all \( p \in U \), the set of dimension vectors \( \{ \dim S_{i,p} \} \) is independent of \( p \); he called the corresponding sum \( \alpha = \sum \beta_i \) the canonical decomposition of \( \alpha \). It is a major aim of this paper to determine this canonical decomposition.

In Kac’s paper [7], he showed that \( \alpha = \sum \beta_i \) is the canonical decomposition if and only if each \( \beta_i \) is a Schur root; that is, a general representation of a dimension vector \( \beta \) has endomorphism ring \( k \); and for general representations of dimension vectors \( \beta_i \) and \( \beta_j \), \( R \) and \( S \) respectively, \( \text{Ext}(R, S) = 0 \). The major obstacle that prevents this from giving an inductive calculation of the canonical decomposition is the condition that \( \text{Ext}(R, S) = 0 \). We shall solve this by linking this question with the problem of the dimension vectors of subrepresentations of a general representation. First of all, we say that \( \text{Ext}(\alpha, \beta) \) vanishes generally if and only if there exist representations \( R \) and \( S \) of dimension vectors \( \alpha \) and \( \beta \) respectively such that \( \text{Ext}(R, S) = 0 \). We prove that \( \text{Ext}(\alpha, \beta) \) vanishes generally if and only if a general representation of dimension vector \( \alpha + \beta \) has a subrepresentation of dimension vector \( \alpha \). Next, we prove that \( \text{Ext}(\alpha, \beta) \) does not vanish generally if

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and only if there exists a dimension vector $\beta'$ such that every representation of dimension vector $\beta$ has a subrepresentation of dimension vector $\beta'$ and $\langle \alpha, \beta - \beta' \rangle < 0$. These two results, proved in §§3 and 5, respectively, clearly allow us to calculate inductively the dimension vectors of subrepresentations of a general representation of any dimension vector and its canonical decomposition. In § 4, we consider a slightly different approach to the canonical decomposition of a dimension vector for a quiver having no oriented cycles. This brings out the role of the real Schur roots in the canonical decomposition; it becomes clear that the majority of the terms in the canonical decomposition are real Schur roots. We show how the canonical decomposition of a dimension vector for an arbitrary quiver may be reduced to the canonical decomposition of a dimension vector for a quiver without oriented cycles. We also show that it is enough to calculate when $\text{Ext}(\alpha, \beta)$ vanishes generally if $\langle \alpha, \beta \rangle = 0$ and we discuss why this is a simple matter when one of $\alpha$ or $\beta$ are real. In the final section, we discuss what we know about the set of Schur roots as a subset of the roots, and we discuss a possible simplification of the results for the canonical decomposition of a root.

1. Notation and terminology

We fix an algebraically closed field $k$.

A quiver $Q$ has vertex set $V = \{v, w, \ldots\}$ and arrow set $A = \{a, b, \ldots\}$. An arrow has initial vertex $ia$ and terminal vertex $ta$. A representation of $Q$ is a collection of finite-dimensional vector spaces $\{R(v): v \in V\}$ and a collection of linear maps $\{R(a): a \in A, R(a): R(ia) \rightarrow R(ta)\}$. The dimension vector of $R$ is the function on the vertices $\dim R(v) = \dim R(v)$. So the dimension vector of $R$ is an element of $\mathbb{Z}^V$. Given $\alpha \in \mathbb{Z}^V$, we define the support of $\alpha$, $\text{supp} \alpha$, to be the full subquiver of $Q$ on the vertices $\{v: \alpha(v) \neq 0\}$. We say that $\alpha$ is divisible if $\text{h.c.f.}\{\alpha(v)\} > 1$.

If $R$ and $S$ are representations for $Q$, we define a homomorphism $\phi$ from $R$ to $S$ to be a collection of linear maps $\{\phi(v): R(v) \rightarrow S(v)\}$ such that $R(a)\phi(ia) = \phi(ia)S(a)$. So, $\text{Hom}(R, S)$ is the linear space of all such homomorphisms.

For integers $m$ and $n$, we define $m \times n$ to be the space of $m$ by $n$ matrices. Given a quiver $Q$ and a dimension vector $\alpha$, we define $R(Q, \alpha) = \bigtimes_a \alpha(ia)k^{\alpha(ia)}$. A point of $R(Q, \alpha)$ defines a representation $R_p$ of $Q$ in the obvious way; we see that every possible representation of $Q$ of dimension vector $\alpha$ is isomorphic to some representation $R_p$. We define $\text{Gl}(\alpha) = \bigtimes_v \text{Gl}(\alpha(v))$. Thus $\text{Gl}(\alpha)$ acts on $R(Q, \alpha)$ and the orbits of $\text{Gl}(\alpha)$ on $R(Q, \alpha)$ are in one-to-one correspondence with the isomorphism classes of representations of $Q$ of dimension vector $\alpha$.

Let $\alpha, \beta \in \mathbb{Z}^V$. We define

$\langle \alpha, \beta \rangle = \sum_v \alpha(v)\beta(v) - \sum_a \alpha(ia)\beta(ta)$ and $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

We note that $(\ , \ )$ is the bilinear form on $\mathbb{Z}^V$ defined by the symmetric Cartan matrix associated to the underlying graph of $Q$ obtained by forgetting the directions of the arrows. Let $R$ and $S$ be representations of $Q$ of dimension vectors $\alpha$ and $\beta$ respectively. Then

$\dim \text{Hom}(R, S) - \dim \text{Ext}(R, S) = \langle \alpha, \beta \rangle$
and $\text{Ext}^i(R, S)$ vanishes for $i > 1$ (see [4]). Hence, we call $\langle \ , \ \rangle$ the Euler inner product on the space of dimension vectors.

In [5], Kac shows that the set of dimension vectors of indecomposable representations is the set of positive roots for the Cartan matrix associated to the underlying graph of $Q$. We say that a root is real if $\langle \alpha, \alpha \rangle = 1$, imaginary if $\langle \alpha, \alpha \rangle < 0$, and isotropic if $\langle \alpha, \alpha \rangle = 0$. A root is a null root if its support is a quiver of tame type and $\langle \alpha, \alpha \rangle = 0$. Every root is either real or imaginary. We define $\alpha$ to be a Schur root if and only if $\alpha$ is the dimension vector of a representation having endomorphism ring $k$. Thus, we define real Schur roots, imaginary Schur roots and isotropic Schur roots in the obvious way. We say that $R$ is a Schur representation if $\text{End}(R) = k$.

Let $\alpha$ and $\beta$ be dimension vectors for $Q$. We define $\text{Hom}(Q, \alpha, \beta)$ to be the closed subvariety of $\times_v \alpha(v) k^{\beta(v)} \times R(Q, \alpha) \times R(Q, \beta)$:

$$\text{Hom}(Q, \alpha, \beta) = \{ (\phi, p, q) : \phi \in \text{Hom}(R_p, R_q) \}.$$  

There is a natural morphism from $\text{Hom}(Q, \alpha, \beta)$ to $R(Q, \alpha) \times R(Q, \beta)$ and the fibre above $(p, q)$ is $\text{Hom}(R_p, R_q)$. Hence, the function $\lambda(p, q) = \dim \text{Hom}(R_p, R_q)$ is upper semicontinuous on $R(Q, \alpha) \times R(Q, \beta)$. We define $\text{hom}(\alpha, \beta)$ to be its minimal value. Since $\dim \text{Hom}(R_p, R_q) - \dim \text{Ext}(R_p, R_q) = \langle \alpha, \beta \rangle$, the dimension of $\text{Ext}(R_p, R_q)$ is also an upper semicontinuous function and we define $\text{ext}(\alpha, \beta)$ to be the minimal value of this function. We note that $\text{hom}(\alpha, \beta) = \text{ext}(\alpha, \beta) = \langle \alpha, \beta \rangle$. We say that $\text{Hom}(\alpha, \beta)$ vanishes generally if $\text{hom}(\alpha, \beta) = 0$ and similarly we say that $\text{Ext}(\alpha, \beta)$ vanishes generally if $\text{ext}(\alpha, \beta) = 0$. In a similar way, we see that for fixed $R$, the functions $\dim \text{Hom}(R_p, R), \dim \text{Hom}(R, R_p)$ and $\dim \text{End}(R_p)$ are upper semicontinuous functions on $R(Q, \alpha)$. Hence, we define $\text{hom}(\alpha, R), \text{hom}(R, \alpha)$ and $\text{end}(\alpha)$ to be the minimal values of the respective functions. We define similarly, $\text{ext}(\alpha, R)$ and $\text{ext}(R, \alpha)$.

Let $P$ be some property of representations. We say that a general representation of dimension vector $\alpha$ has Property $P$ if and only if there exists some open subset $U$ of $R(Q, \alpha)$ such that for all $p \in U, R_p$ has Property $P$.

Let $\alpha$ be a dimension vector for $Q$. Kac [7] shows that there is an open subset $U$ of $R(Q, \alpha)$ such that for all $p \in U, R_p \cong \bigoplus S_{i,p}$ where each $S_{i,p}$ is indecomposable and the collection of dimension vectors $\{ \dim(S_{i,p}) \}$ is independent of the point $p$. Thus, if $\beta_i$ is the dimension vector of $S_{i,p}$, we obtain a sum $\alpha = \sum \beta_i$ such that a general representation of dimension vector $\alpha$ is a direct sum of indecomposable representations of dimension vectors $\beta_i$. We call the sum $\alpha = \sum \beta_i$ the canonical decomposition of $\alpha$. If a general representation of dimension vector $\alpha$ is a direct sum of representations of dimension vectors $\beta$ and $\gamma$, we write $\alpha = \beta \bigoplus \gamma$. For example, if $\alpha = \sum \beta_i$ is the canonical decomposition of $\alpha$, we may write $\alpha = \bigoplus \beta_i$. If a general representation of dimension vector $\alpha$ has a subrepresentation of dimension vector $\beta$, we write $\beta \hookrightarrow \alpha$. Similarly, if a general representation of dimension vector $\alpha$ has a factor of dimension vector $\gamma$, we write $\alpha \twoheadrightarrow \gamma$.

We say that $\alpha$ is positive if $\alpha(v) \geq 0$ for all $v$. Let $v_i$ be some vertex; we define $\sigma_i$ to be the dimension vector $\sigma_i(v_i) = \delta_{ij}$. Let $\Pi = \{ \sigma_i \}$. Let $r_i$ be the reflection in the hyperplane perpendicular to $\sigma_i$ in the space $\mathbb{R}^v$ for the symmetric form $\langle , \rangle$. We define the Weyl group $W$ of $Q$ to be the group generated by $\{ r_i \}$. From Kac [9], one knows that the set of real roots is $W\Pi$. Let $C = \{ \beta : (\beta, \sigma_i) \leq 0, \text{for all } i \}$.
all \( i \), \( \text{supp} \beta \) is connected). Then every vector in \( C \) is positive. Moreover, every vector in \( C \) is a positive imaginary root. Thus \( C \) is a fundamental region for the action of \( W \) on \( WC \) and \( WC \) is the collection of positive imaginary roots. The reader should consult Kac [9]. Given a real root \( \alpha \), we write \( r_\alpha \) for reflection in the hyperplane perpendicular to \( \alpha \); \( r_\alpha \in W \).

2. Preliminary results

The results in this section hold for arbitrary quivers.

As Kac showed [7], the subset of \( R(Q, \alpha) \) of points \( p \) such that \( R_p \) decomposes as a direct sum of representations \( S_{i,p} \) of dimension vector \( \alpha_i \) is a constructible subset with respect to the Zariski topology; therefore, there exist a sum \( \alpha = \sum \beta_i \) and an open subset \( U \) of \( R(Q, \alpha) \) such that for all \( p \) in \( U \), \( R_p \) is a direct sum of indecomposable representations \( S_{i,p} \) of dimension vector \( \beta_i \). In this case, we write \( \alpha = \sum \beta_i \) and we call this the canonical decomposition of \( \alpha \).

We begin by stating versions of Kac's theorems on general representations.

**Theorem 2.1 (Kac [7]).** Let \( R \) be a general representation of dimension vector \( \alpha \). Then \( R = \bigoplus S_i \) where \( \dim S_i = \alpha_i \) if and only if \( \text{Ext}(\alpha_i, \alpha_j) \) vanishes generally for \( i \neq j \).

**Theorem 2.2 (Kac [7]).** Let \( R \) be a general representation of dimension vector \( \alpha \); let \( R = \bigoplus S_i \) be its decomposition as a direct sum of indecomposable representations. Then each \( S_i \) is a Schur representation and \( \text{Ext}(S_i, S_j) = 0 \) for \( i \neq j \). In particular, \( \alpha \) is a Schur root if and only if a general representation is indecomposable.

Therefore, the sum \( \alpha = \sum \beta_i \) is the canonical decomposition if and only if each \( \beta_i \) is a Schur root and \( \text{Ext}(\beta_i, \beta_j) \) vanishes generally. It follows that \( \langle \beta_i, \beta_j \rangle \geq 0 \).

From these theorems we see that we shall be mainly interested in finding ways to show that \( \text{Ext}(\beta_i, \beta_j) \) vanishes generally. We shall see in the next section that \( \text{Ext}(\alpha, \beta) \) vanishes generally if and only if a general representation of dimension vector \( \alpha + \beta \) has a subrepresentation of dimension vector \( \alpha \).

There is a further combinatorial restriction that we may easily obtain on the canonical decomposition which follows quickly from a lemma due to Happel and Ringel [10, Lemma 4.1].

**Lemma 2.3.** Let \( R \) and \( S \) be indecomposable representations of a quiver \( Q \) such that \( \text{Ext}(R, S) = 0 \). Then a homomorphism from \( S \) to \( R \) is either injective or surjective.

We deduce the following from this lemma.

**Theorem 2.4.** Let \( \{ S_i \} \) be a family of non-isomorphic indecomposable Schur representations such that \( \text{Ext}(S_i, S_j) = 0 \) for \( i \neq j \). We define a relation \( i \rightarrow j \) if and only if there exists a non-zero homomorphism from \( S_i \) to \( S_j \). Then, the transitive relation generated by \( \rightarrow \) is a partial order. In particular, either \( \text{Hom}(S_i, S_j) = 0 \) or \( \text{Hom}(S_j, S_i) = 0 \).

Therefore, if \( \alpha = \sum \beta_i \) is the canonical decomposition, either \( \beta_i = \beta_j \) is a real Schur root or else \( \langle \beta_i, \beta_i \rangle \langle \beta_j, \beta_j \rangle = 0 \).
Proof. Any non-zero map from $S_i$ to $S_j$ is either surjective or injective. If there is a surjection from $S_i$ to $S_j$, there can be no injection from $S_i$ to $S_k$ for some $k \neq j$, since the composite is a non-zero homomorphism from $S_i$ to $S_k$ that is neither surjective nor injective. Hence, the first paragraph follows.

If $R$ is a general representation of dimension vector $\alpha$, then two summands $S_i$ and $S_j$ are isomorphic only if they are real Schur representations (since $\text{Ext}(S_i, S_j) = \text{Ext}(S_i, S_j) = 0$ follows), and the isomorphism classes of indecomposable summands of $R$ satisfy the conditions of the first paragraph. Hence, the second paragraph follows.

Kac suggested that a converse to the combinatorial part of Theorem 2.2 might hold for quivers without oriented cycles. That is, he asked whether the canonical decomposition might be characterised in the following way. Let $\alpha = \sum \beta_i$ where each $\beta_i$ is a Schur root and $\langle \beta_i, \beta_j \rangle \geq 0$; is this the canonical decomposition? A counter-example to this was known for quivers with oriented cycles due to Riedtmann. We give an example for a tame quiver of type $\tilde{D}_6$.

Let $Q$ be the quiver

\[
\begin{array}{ccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
& \uparrow & & \uparrow & & \uparrow & \\
& 1 & & 1 & & 1 & \\
\end{array}
\]

Consider the dimension vector

\[
\begin{pmatrix}
1 & 3 & 2 & 3 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} =
\begin{pmatrix}
222 & 1 & + & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

It is a simple matter to check that the first decomposition is the canonical decomposition whilst the second satisfies the requirements of the conjecture as well.

In a subsequent paper [17] we shall see that Kac's conjecture is true for the $n$-subspace quiver.

3. Subrepresentations of general representations

Given Kac's result that a general representation of dimension vector $\alpha + \beta$ is a direct sum of representations of dimension vector $\alpha$ and $\beta$ if and only if $\text{Ext}(\alpha, \beta)$ and $\text{Ext}(\beta, \alpha)$ vanish generally, we might expect that a general representation of dimension vector $\alpha + \beta$ has a subrepresentation of dimension vector $\alpha$ if and only if $\text{Ext}(\alpha, \beta)$ vanishes generally. This is the main aim of this section. The reason that this is useful for a discussion of the canonical decomposition has already been discussed in the introduction. There follows a number of consequences. We show that $\text{Hom}(\alpha, \alpha)$ vanishes generally if $\alpha$ is an imaginary Schur root; we deduce the canonical decomposition of $n\alpha$ if $\alpha$ is an imaginary Schur root and then we use these results to find the canonical decomposition of $n\alpha$ given that we know the canonical decomposition of $\alpha$. These results are not necessary for the main line of argument of the paper; however, it is important that they should be observed.

We begin this section by defining a variety that parametrises representations of dimension vector $\alpha + \beta$ together with a distinguished subrepresentation of dimension vector $\alpha$. 
We define
\[
\text{Gr}
\left(\frac{\alpha + \beta}{\alpha}\right) = \times_v \text{Gr}
\left(\frac{\alpha(v) + \beta(v)}{\alpha(v)}\right),
\]
where \(\text{Gr}
\left(\frac{a + b}{a}\right)\) is the Grassmannian variety of \(a\)-dimensional vector subspaces of \(k^{a+b}\). We define \(R(Q, \alpha \subset \alpha + \beta)\) to be the closed subset of
\[
R(Q, \alpha + \beta) \times \text{Gr}
\left(\frac{\alpha + \beta}{\alpha}\right)
\]
of points \((p, q)\) such that the collection of vector subspaces determined by \(q\) define a subrepresentation of the representation \(R_p\). For each point \(q\), this is a linear condition on \(R(Q, \alpha + \beta)\) and since \(\text{Gl}(\alpha + \beta)\) acts transitively on \(\text{Gr}
\left(\frac{\alpha + \beta}{\alpha}\right)\), \(R(Q, \alpha \subset \alpha + \beta)\) is a vector subbundle of \(R(Q, \alpha + \beta) \times \text{Gr}
\left(\frac{\alpha + \beta}{\alpha}\right)\). There is a projective morphism from \(R(Q, \alpha \subset \alpha + \beta)\) to \(R(Q, \alpha + \beta)\) whose image is precisely the subvariety of points \(p\) such that \(R_p\) contains a subrepresentation of dimension vector \(\alpha\). The fibre of this morphism above a point \(p\) parametrises the collection of all subrepresentations of \(R_p\) of dimension vector \(\alpha\). We call this fibre \(\text{Gr}(R_p, \alpha)\). It is a projective variety which in general neither smooth, irreducible nor (considered as a scheme) reduced.

Lemma 3.1. Let \(D\) be the collection of points \(p\) of \(R(Q, \alpha + \beta)\) such that \(R_p\) has a subrepresentation of dimension vector \(\alpha\). Then \(D\) is a closed subset of \(R(Q, \alpha + \beta)\). In particular, a general representation of dimension vector \(\alpha + \beta\) has a subrepresentation of dimension vector \(\alpha\) if and only if all representations of dimension vector \(\alpha + \beta\) have a subrepresentation of dimension vector \(\alpha\).

Proof. As remarked in the preceding paragraph, \(D\) is the image of a projective morphism and hence it is a closed subvariety. The rest follows.

We need to be able to calculate the tangent space to a point \(q\) of \(\text{Gr}(R_p, \alpha)\). The point \(q\) determines a subrepresentation \(S_q\) of dimension vector \(\alpha\) of \(R_p\), and the analogy of the Grassmannian variety would suggest that the tangent space should be \(\text{Hom}(S_q, R_p/S_q)\). We shall see that this is true scheme-theoretically though false at the level of varieties. In order to carry this through we need to discuss what we mean by representations of dimension vector \(\alpha\) of a quiver \(Q\) over a commutative ring \(\Lambda\).

For simplicity of discussion we shall assume that \(\text{Spec}(\Lambda)\) is connected. The adjustments in general are simple to make but wordy. A representation \(R\) of the quiver \(Q\) over \(\Lambda\) is a collection \(\{R(v)\}\) of projective modules over \(\Lambda\) and a collection of module homomorphisms \(\{R(a): R(ia) \rightarrow R(\tau a)\}\). Its dimension vector is the function \(v \rightarrow \text{rank} R(v)\). A subrepresentation of \(R\) is given by a collection of projective summands \(S(v)\) of \(R(v)\) such that \(R(a)\) restricts to a map from \(S(ia)\) to \(S(\tau a)\) for each arrow \(a\). The reason for taking summands here rather than simply submodules is that the Grassmannian variety parametrises summands rather than submodules when considered as a functor on the category of commutative rings. The variety \(\text{Gr}(R_p, \alpha)\) is the set of \(k\)-points of a scheme \(\mathcal{G}(R_p, \alpha)\) whose points in the commutative ring \(\Lambda\) are the subrepresentations of \(R_p \otimes \Lambda\) of dimension vector \(\alpha\).
If $R$ is a representation of $Q$ over $k$, then a $k[\epsilon]$-point of $\mathcal{G}_\alpha(R, \alpha)$ is a subrepresentation $S$ of dimension vector $2\alpha$ of $R \otimes k[\epsilon]$ considered as a representation of $Q$ over $k$ such that it is stable under $\epsilon$ and $\dim(\epsilon S) = \alpha$. It follows that $\epsilon S = S \cap \epsilon(R \otimes k[\epsilon])$. Thus $S/\epsilon S$ is a subrepresentation of dimension vector $\alpha$ of $R$.

**Lemma 3.2.** Let $q$ be a point of $\text{Gr}(R_p, \alpha)$. The tangent space to $q$ in $\mathcal{G}_\alpha(R_p, \alpha)$ is $\text{Hom}(S_q, R_p/S_q)$ where $S_q$ is the subrepresentation determined by $q$.

**Proof.** We need to determine the $k[\epsilon]$-points of $\mathcal{G}_\alpha(R_p, \alpha)$ that lie over $q$. Let $S$ be the $k[\epsilon]$-subrepresentation of $R_p \otimes k[\epsilon]$ determined by such a point. Then $\epsilon S = S_q \otimes \epsilon$ and $S/\epsilon S = S_q \subset R_p$ so that we may choose a vector-space complement of $\epsilon S$ in $S$; to each vector $v$ in $S_q$ we have a vector $v + \phi(v) \otimes \epsilon$ in $S$ where $\phi \otimes \epsilon$ is determined modulo an element of $S_q \otimes \epsilon$; thus we have a vector space morphism $\tilde{\phi}$ from $S_q$ to $R_p/S_q$ where $\tilde{\phi}(v) = \phi(v) \mod S_q$. Since $S$ is a representation, it follows that $\tilde{\phi}$ is a homomorphism of representations of $Q$.

Conversely, given a homomorphism $\psi$ from $S_q$ to $R_p/S_q$, we lift this to a vector space map $\tilde{\psi}$ from $S_q$ to $R_p$ and we consider the $k$-linear span of $$\{v + \tilde{\psi}(v) \otimes \epsilon: v \in S_q\} \quad \text{and} \quad S_q \otimes \epsilon.$$ This is a $k[\epsilon]$-subrepresentation of dimension vector $\alpha$ in $R_p \otimes k[\epsilon]$ lying over the point $q$. It is clearly independent of the choice of $\tilde{\psi}$.

Thus we see that the fibre of the map from $\mathcal{G}_\alpha(R_p, \alpha)(k[\epsilon])$ to $\text{Gr}(R_p, \alpha)$ above the point $q$ is precisely $\text{Hom}(S_q, R_p/S_q)$, and this fibre is the tangent space to the functor $\mathcal{G}_\alpha(R_p, \alpha)$ at the point $q$.

**Theorem 3.3.** The following are equivalent:

(i) every representation of dimension vector $\alpha + \beta$ has a subrepresentation of dimension vector $\alpha$;

(ii) a general representation of dimension vector $\alpha + \beta$ has a subrepresentation of dimension vector $\alpha$;

(iii) $\text{Ext}(\alpha, \beta)$ vanishes generally.

**Proof.** The equivalence of the first two conditions was shown in 3.1.

We begin by calculating the dimension of $R(Q, \alpha \subset \alpha + \beta)$. The dimension of $\text{Gr}(\alpha + \beta)_\alpha$ is $\sum_v \alpha(v)\beta(v)$. The dimension of the fibre above some point of $\text{Gr}(\alpha + \beta)_\alpha$ is $\sum_v (\alpha(ia)\alpha(ta) + \beta(ia)(\alpha + \beta)(ta))$. Therefore the dimension of $R(Q, \alpha \subset \alpha + \beta)$ is

$$\sum_v \alpha(v)\beta(v) + \sum_a (\alpha(ia)\alpha(ta) + \beta(ia)(\alpha + \beta)(ta)).$$

It follows that $\dim R(Q, \alpha \subset \alpha + \beta) - \dim R(Q, \alpha + \beta) = \langle \alpha, \beta \rangle$.

Let us suppose that every representation of dimension vector $\alpha + \beta$ has a subrepresentation of dimension vector $\alpha$. Then the map from $R(Q, \alpha \subset \alpha + \beta)$ to $R(Q, \alpha + \beta)$ is surjective. We have seen that

$$\dim(R(Q, \alpha \subset \alpha + \beta)) - \dim(R(Q, \alpha + \beta)) = \langle \alpha, \beta \rangle.$$

Therefore, the dimension of a general fibre of this map is $\langle \alpha, \beta \rangle$. Since
$R(Q, \alpha \subset \alpha + \beta)$ is a reduced variety, a general fibre of this map is also generically reduced and it is possible to find a point $q$ in this fibre such that the tangent space of $\mathcal{S}(R_p, \alpha)$ at this point is of dimension $\langle \alpha, \beta \rangle$. However, $\langle \alpha, \beta \rangle = \dim(\text{Hom}(S_q, R_p/S_q)) - \dim(\text{Ext}(S_q, R_p/S_q))$ and by Lemma 3.2 this tangent space may be identified with $\text{Hom}(S_q, R_p/S_q)$. It follows that $\text{Ext}(S_q, R_p/S_q) = 0$. Hence, $\text{Ext}(\alpha, \beta)$ vanishes generally.

Conversely, let us suppose that $\text{Ext}(\alpha, \beta)$ vanishes generally. It follows that for a general point $p$ in the image of the map from $R(Q, \alpha \subset \alpha + \beta)$ to $R(Q, \alpha + \beta)$ and a general point $q$ of the fibre above $p$, $\text{Ext}(S_q, R_p/S_q) = 0$ and hence $\dim(\text{Hom}(S_q, R_p/S_q)) = \langle \alpha, \beta \rangle$. Thus a general fibre of this map has dimension $\langle \alpha, \beta \rangle$. Therefore, the image of the map has dimension $\dim(R(Q, \alpha \subset \alpha + \beta)) - \langle \alpha, \beta \rangle$ which is the dimension of $R(Q, \alpha + \beta)$. Since the map is projective, it follows that the image is all of $R(Q, \alpha + \beta)$. That is, every representation of dimension vector $\alpha + \beta$ has a subrepresentation of dimension vector $\alpha$.

The remaining results of this section are not directly required for the main line of argument of this paper and the reader may wish to skip them on a first reading; they do however contain information of importance for the understanding of the canonical decomposition.

**Theorem 3.4.** A general representation of dimension vector $\alpha + \beta$ is a direct sum of representations of dimension vector $\alpha$ and $\beta$ if and only if every representation of dimension vector $\alpha + \beta$ has subrepresentations of dimension vector $\alpha$ and $\beta$.

**Proof.** Every representation of dimension vector $\alpha + \beta$ has subrepresentations of dimension vector $\alpha$ and $\beta$ if and only if $\text{Ext}(\alpha, \beta)$ and $\text{Ext}(\beta, \alpha)$ vanish generally, which holds if and only if a general representation of dimension vector $\alpha + \beta$ is a direct sum of representations of dimension vector $\alpha$ and $\beta$ by 2.1.

We may deduce from Theorem 3.3 a number of useful results about general representations and Schur representations.

**Theorem 3.5.** Let $R$ and $S$ be general representations of dimension vector $\alpha + \beta$. Let $\phi: R \to S$ be a homomorphism such that $\dim(\text{im} \phi) = \beta$. Then, a general representation of dimension vector $\alpha + \beta$ is a direct sum of representations of dimension vector $\alpha$ and $\beta$.

In particular, if $\gamma = \alpha + \beta$ is a Schur root, and $R$ and $S$ are Schur representations, then either $\text{Hom}(R, S) = 0$ or else $R$ and $S$ are isomorphic. Hence, $\text{Hom}(\gamma, \gamma)$ vanishes generally for imaginary roots. If, in addition, $\gamma$ is isotropic, then $\text{Ext}(\gamma, \gamma)$ vanishes generally.

**Proof.** We note that $\text{im} \phi$ is a subrepresentation of $S$ of dimension vector $\beta$ and hence $\text{Ext}(\beta, \alpha)$ vanishes generally whilst $\ker \phi$ is a subrepresentation of dimension vector $\alpha$ in $R$ and hence $\text{Ext}(\alpha, \beta)$ vanishes generally. Therefore, the conclusion of the first paragraph follows.

If $\gamma$ is a Schur root, and hence $R$ and $S$ are Schur representations, then either $\alpha$ or $\beta$ is 0 and hence $\phi$ is either 0 or an isomorphism. However, for imaginary
roots, there is no open orbit for the action of $\text{GL}(\gamma)$ on $R(Q, \gamma)$ and hence there exist two non-isomorphic general representations $R$ and $S$ and we deduce that $\text{Hom}(R, S) = 0$. It follows that $\text{Hom}(\gamma, \gamma)$ vanishes generally. If, in addition, $\gamma$ is isotropic, then $\text{Ext}(R, S) = 0$ and $\text{Ext}(\gamma, \gamma)$ vanishes generally.

For a non-isotropic imaginary root, $\gamma$, $\text{Ext}(\gamma, \gamma)$ cannot vanish generally since $\langle \gamma, \gamma \rangle < 0$.

We note at once that this allows us to describe the canonical decomposition for a multiple of an isotropic Schur root.

**Theorem 3.6.** Let $\alpha$ be an isotropic Schur root. Then, $n\alpha = \alpha \oplus \cdots \oplus \alpha$ is the canonical decomposition of $n\alpha$.

**Proof.** Each $\alpha$ is a Schur root, and $\text{Ext}(\alpha, \alpha)$ vanishes generally. Hence, Theorem 2.2 applies.

Next, we discuss the case of a multiple of a non-isotropic imaginary Schur root. The argument we use in the proof of the next theorem is taken from [12].

**Theorem 3.7.** Let $\alpha$ be an imaginary Schur root, $\langle \alpha, \alpha \rangle < 0$. Then $n\alpha$ is also an imaginary Schur root.

**Proof.** Since $\alpha$ is an imaginary Schur root, we may find non-isomorphic representations $\{R_i: i = 1 \text{ to } n\}$ such that $\dim \text{Hom}(R_i, R_j) = \delta_{ij}$ by Theorem 3.5. Further, since $\langle \alpha, \alpha \rangle < 0$, $\text{Ext}(R_i, R_j) \neq 0$. It follows using the right exactness of $\text{Ext}$ that we may construct a representation $S_n$ with a filtration $0 = S_0 \subset S_1 \subset \cdots \subset S_n$ such that $S_i/S_{i-1} \cong R_i$ so that the short exact sequence

$$0 \rightarrow S_{i-1} \rightarrow S_i \rightarrow R_i \rightarrow 0$$

is non-split. By induction, we may assume that $\text{End}(S_{n-1}) = k$. Certainly, $\text{Hom}(S_{n-1}, R_n) = 0$; therefore, any endomorphism $\phi$ of $S_n$ restricted to $S_{n-1}$ induces an endomorphism of $S_{n-1}$ which must be a scalar $\lambda I$; therefore, $\phi - \lambda I$ is an endomorphism of $S_n$ trivial on $S_{n-1}$. However, $\text{Hom}(R_n, S_{n-1}) = 0$, $\text{Hom}(R_n, R_n) = k$ and the sequence

$$0 \rightarrow S_{n-1} \rightarrow S_n \rightarrow R_n \rightarrow 0$$

is non-split; hence $\phi - \lambda I = 0$. Thus, $\text{End}(S_n) = k$. Hence, $n\alpha$ is a Schur root.

Next we consider the canonical decomposition of $n\alpha$ in general given that we know the canonical decomposition of $\alpha$.

**Theorem 3.8.** Let $\alpha = \bigoplus \beta_i$ be the canonical decomposition of $\alpha$. Then, the canonical decomposition of $n\alpha$ is given by $n\alpha = \bigoplus (n\beta_i)$ where $(n\beta_i) = n\beta_i$ for an imaginary non-isotropic root $\beta_i$ and $(n\beta_i) = \beta_i \oplus \cdots \oplus \beta_i$ (n copies) for $\beta_i$ real or isotropic. In particular, an isotropic Schur root is not divisible.

**Proof.** Since $\text{Ext}(\beta_i, \beta_j)$ vanishes generally, $\text{Ext}(n\beta_i, n\beta_j)$ vanishes generally. So, $n\alpha = \bigoplus n\beta_i$. We have already determined the canonical decomposition of $n\beta_i$ in the cases where $\beta_i$ is isotropic or non-isotropic imaginary in Theorems 3.6 and 3.7; if $\beta_i$ is a real Schur root, $\text{Ext}(\beta_i, \beta_i)$ vanishes generally and this completes the first part. The second part follows at once.
4. The canonical decomposition

The results of the previous section and this section allow us to calculate the canonical decomposition for an arbitrary dimension vector. However, the results in the next section will allow us to simplify these methods in general and so we shall concentrate here only on those cases where most of the terms are real Schur roots. It turns out that the real Schur roots usually play a large role in the canonical decomposition; thus if $\alpha$ is a root and $\alpha = \bigoplus \beta_i$ is the canonical decomposition then up to multiplicity only one $\beta_i$ is an imaginary root and if it occurs with multiplicity then it is an isotropic root.

We begin this section by showing that if $\text{Ext}(\alpha, \beta)$ vanishes generally then at least one of $\text{Ext}(\beta, \alpha)$ and $\text{Hom}(\beta, \alpha)$ vanishes generally.

**Theorem 4.1.** Let $\alpha$ and $\beta$ be Schur roots. Assume that $\text{Ext}(\alpha, \beta)$ vanishes generally. Then either $\text{Hom}(\beta, \alpha)$ or $\text{Ext}(\beta, \alpha)$ vanishes generally. If both $\alpha$ and $\beta$ are imaginary roots then $\text{Hom}(\beta, \alpha)$ vanishes generally.

**Proof.** Let $R$ and $S$ be Schur representations of dimension vectors $\alpha$ and $\beta$ respectively such that $\text{Ext}(R, S) = 0$. Assume that $\text{Hom}(\beta, \alpha)$ does not vanish generally. Then, $\text{Hom}(S, R) \neq 0$. Then, by Lemma 2.3, any homomorphism from $S$ to $R$ is either injective or surjective and hence either $\beta < \alpha$ or $\alpha < \beta$. We assume that $\beta < \alpha$ first. If $\beta$ is a real Schur root, then $\text{Ext}(S, S) = 0$; we argue that a general representation of dimension vector $\alpha$ contains a subrepresentation of dimension vector $\beta$ and hence $\text{Ext}(\beta, \alpha - \beta)$ vanishes generally. However, $\text{Ext}(S, S) = 0$. Therefore, $\text{Ext}(\beta, \alpha)$ vanishes generally since $\text{Ext}(S, (R/S) \oplus S) = 0$. If $\beta$ is not a real root, we note that for every general representation $S_p$ of dimension vector $\beta$ and for every general representation $R_q$ of dimension vector $\alpha$ (taken from an open set where $\text{Ext}(S_p, R_q)$ vanishes) there is an injective homomorphism $\phi_{p,q} : S_p \rightarrow R_q$. However, $\text{Ext}(S_p, S_p) \neq 0$, and hence $\text{Ext}(R_q, S_p) \neq 0$. Thus, $\text{Ext}(\alpha, \beta)$ does not vanish generally. A contradiction. Hence, it follows that $\text{Hom}(\beta, \alpha)$ vanishes generally. It remains to deal with the case that $\phi$ is surjective. We subdivide as before into the case where $\alpha$ is real or imaginary and proceed as in the previous case.

One should note that we can read off from $\langle \beta, \alpha \rangle$ which of $\text{Hom}(\beta, \alpha)$ and $\text{Ext}(\beta, \alpha)$ vanishes generally. If $\langle \beta, \alpha \rangle \geq 0$, then $\text{Ext}(\beta, \alpha)$ vanishes generally and if $\langle \beta, \alpha \rangle \leq 0$ then $\text{Hom}(\beta, \alpha)$ vanishes generally under the assumptions of Theorem 3.7.

We recall a certain polynomial semi-invariant discussed in [15]. Let $\alpha$ and $\beta$ be dimension vectors such that $\langle \alpha, \beta \rangle = 0$. Then, we define a polynomial function $P_{\alpha, \beta}$ on $R(Q, \alpha) \times R(Q, \beta)$ which is non-zero if and only if $\text{Ext}(\alpha, \beta)$ vanishes generally and hence $\text{Hom}(\alpha, \beta)$ vanishes generally. One sees from the definition that it is possible to calculate the polynomial $P_{\alpha, \beta}$ directly, though this would be an unreasonably difficult task in general.

**Theorem 4.2.** Let $Q$ be a quiver. Let $\alpha = \sum \beta_i$ where each $\beta_i$ is a Schur root. Then this is the canonical decomposition if and only if $\langle \beta_i, \beta_j \rangle \geq 0$ and $\langle \beta_i, \beta_j \rangle \langle \beta_j, \beta_i \rangle = 0$ for $i \neq j$ and in addition $\text{Ext}(\beta_i, \beta_j)$ vanishes generally when $\langle \beta_i, \beta_j \rangle = 0$ or equivalently $P_{\beta_i, \beta_j}$ does not vanish when $\langle \beta_i, \beta_j \rangle = 0$. 
Proof. We already know that if $\alpha = \sum \beta_i$ is the canonical decomposition then the various conditions follow. Conversely, let us suppose that our conditions hold. We know for $i \neq j$ that $\text{Ext}(\beta_i, \beta_j)$ vanishes generally provided that $\langle \beta_i, \beta_j \rangle = 0$, so assume that $\langle \beta_i, \beta_j \rangle > 0$, then we know that $\langle \beta_i, \beta_j \rangle = 0$ and hence $\text{Ext}(\beta_j, \beta_i)$ vanishes generally. Then by Theorem 4.1, $\text{Ext}(\beta_i, \beta_j)$ vanishes generally. Thus we have shown that $\text{Ext}(\beta_i, \beta_j)$ vanishes generally for $i \neq j$ and hence this must be the canonical decomposition.

This in principle allows us to calculate the canonical decomposition since we may assume by induction that we know the canonical decomposition of all smaller dimension vectors and hence which of them are Schur roots. If there is no such sum of smaller dimension vectors then $\alpha$ must itself be a Schur root. The major sticky point in this theorem is the condition that $P_{\beta_i, \beta_j}$ should not vanish. This is in general rather hard to calculate directly; however, in the case where one of $\beta_i$ and $\beta_j$ is real, and $Q$ is a quiver without oriented cycles, there is no necessity to calculate this polynomial semi-invariant explicitly. Until further notice we assume that $Q$ is a quiver without oriented cycles.

We see that it is enough to show that $\text{Ext}(\beta_i, \beta_j)$ vanishes generally whenever $\langle \beta_i, \beta_j \rangle = 0$. If $\beta_i$ or $\beta_j$ is a real Schur root, then the work in [16] shows that it is rather simple to find whether or not this happens. Thus if $\beta_i$ is a real Schur root and $\langle \beta_i, \beta_j \rangle = 0$, then $\text{Ext}(\beta_i, \beta_j)$ vanishes generally if and only if $\beta_j$ is a positive linear combination of the dimension vectors of the simple objects in $G(\beta_i)^\perp$, the right perpendicular category to $G(\beta_i)$ which we showed to be equivalent to the category of representations of a quiver having one fewer vertex. We have gone into some detail on how such a calculation may be carried out in [16] and so we refer the reader to § 2 of [16].

Next, we show how to reduce the calculation of the canonical decomposition of a dimension vector for any quiver to the canonical decomposition for some dimension vector for a quiver without oriented cycles. Let $Q$ be some quiver with vertex set $V$ and arrow set $A$; we define a quiver $\tilde{Q}$ with vertex set $\tilde{V}$ and arrow set $\tilde{A}$. If $V = \{v_1, v_2, \ldots \}$, then $\tilde{V} = V_1 \cup V_2$ where $V_i = \{v_i, w_i, \ldots \}$. If $A = \{a, b, \ldots \}$, then $\tilde{A} = \{a_v : v \in V \} \cup \{\tilde{a}, \tilde{b}, \ldots \}$ where $i(a_v) = v_1$ and $t(a_v) = v_2, i(\tilde{a}) = i(a), \text{ and } t(\tilde{a}) = t(a).$ Given a dimension vector $\alpha$ for $Q$ we define a dimension vector $\tilde{\alpha}$ for $\tilde{Q}$ by $\tilde{\alpha}(v_i) = \alpha(v)$. The category of representations of $\tilde{Q}$ is naturally equivalent to the category of representations of $\tilde{Q}$ such that the linear maps corresponding to $a_v$ are invertible. Moreover, under this equivalence the representations of dimension vector $\alpha$ correspond to representations of dimension vector $\tilde{\alpha}$. A general representation of dimension vector $\tilde{\alpha}$ inverts the arrows $a_v$; hence if the canonical decomposition of $\alpha$ is $\alpha = \sum \beta_i$, then the canonical decomposition of $\tilde{\alpha}$ is $\tilde{\alpha} = \sum \tilde{\beta}_i$. It is clear that $\tilde{Q}$ is always a quiver without oriented cycles.

The work we have done so far shows that the canonical decomposition is fairly simple to calculate provided that most of the terms in the canonical decomposition are real Schur roots. We wish to show next that this happens when we are considering the canonical decomposition of a root.

**Lemma 4.3.** Let $\alpha$ be a root, and let $\alpha = \sum \beta_i$ where each $\beta_i$ is a root and is not divisible isotropic, and $\langle \beta_i, \beta_j \rangle \geq 0$ for $i \neq j$. Then if $\beta_i$ and $\beta_j$ are imaginary roots for $i \neq j$, $\beta_i = q\beta_j$ is an isotropic root for some rational $q$. 
Proof. First we assume that $\alpha = \sum \beta_i$ where each $\beta_i$ is an imaginary root. After operating by a suitable element of the Weyl group, we may assume that $\beta_i$ lies in $C$ so that $(\beta_i, \gamma) \leq 0$, for all $\gamma > 0$. It follows that $(\beta_i, \beta_i) = 0$ for all $i$. If $\operatorname{supp} \beta_i$ contains a point adjacent to $\operatorname{supp} \beta_1$, then $(\beta_1, \beta_i) < 0$. It follows that either $\operatorname{supp} \beta_i$ lies in $\operatorname{supp} \beta_1$ or else it is totally disjoint from $\operatorname{supp} \beta_1$. Since $\alpha = \sum \beta_i$ is a root and therefore has connected support, we may deduce that $\operatorname{supp} \beta_i$ lies in $\operatorname{supp} \beta_1$ for all $\beta_i$. By Exercise 5.9 of Kac [9], the subdiagram with vertices \{v: $(\alpha_v, \beta_i) = 0$\} is a union of diagrams of finite type or else $\operatorname{supp} \beta_1$ is a quiver of tame type and $\beta_1$ is a null root. In the first case, there can be no $\beta_i$ such that $\operatorname{supp} \beta_i$ lies in $\operatorname{supp} \beta_1$, and $(\beta_i, \beta_1) = 0$. In the second case, it follows that $\beta_i$ is a multiple of the null root. Thus, all $\beta_i$ are proportional and null which deals with this case.

If some $\beta_i$ is a real root, then $(\alpha, \beta_i) > 0$ and it follows that $\alpha - \beta_i$ is still a root by Proposition 5.1 of Kac [9]. The result follows by induction for $\alpha - \beta_i$ and hence follows for $\alpha$.

We deduce at once the following theorem on the canonical decomposition of a root.

**Theorem 4.4.** Let $\alpha$ be a root and let $\alpha = \sum \beta_i$ be the canonical decomposition of $\alpha$. Then, up to multiplicity, at most one $\beta_i$ is an imaginary Schur root. If an imaginary Schur root occurs with multiplicity in this canonical decomposition then it must be a non-divisible isotropic Schur root.

**Proof.** This follows at once from Lemma 4.3 and Theorem 3.8.

Thus we see that the canonical decomposition of a root is fairly simple to determine: we know that when checking whether $\operatorname{Ext}(\beta_i, \beta_i)$ vanishes generally, at least one of $\beta_i$ and $\beta_j$ must be real, in which case our methods apply or else $\beta_i = \beta_j$ is an isotropic Schur root and we already know that $\operatorname{Ext}(\beta_i, \beta_i) = 0$.

We discuss briefly the remaining problem. Suppose that $\beta_i$ and $\beta_j$ are imaginary Schur roots such that $(\beta_i, \beta_j)$ and $(\beta_j, \beta_i)$ are non-negative. It follows that $(\beta_i, \beta_j)$ is non-negative. Let $w$ be an element of the Weyl group such that $w(\beta_i)$ lies in $C$; it follows that $\operatorname{supp}(\beta_j)$ is either totally disjoint from $\operatorname{supp}(w(\beta_i))$ or else is a subset of $\operatorname{supp}(w(\beta_j))$; in the first case, we find quickly that $w(\beta_i)$ is a null root and $w(\beta_j)$ is also a multiple of this null root which we have already dealt with; in the second case, we see that the quiver must be large enough to contain two disjoint quivers of tame or wild type. We also see that $(\beta_i, \beta_j) = 0 = (\beta_j, \beta_i)$. At any rate, there can be no further difficulties for quivers that do not contain disjoint quivers of tame or wild type.

We shall provide an algorithm in the next section which allows us to determine whether $\operatorname{Ext}(\gamma, \delta)$ vanishes generally for arbitrary dimension vectors and in particular for $\beta_i$ and $\beta_j$.

5. **Determining $\operatorname{ext}(\alpha, \beta)$**

Let $\alpha$ and $\beta$ be dimension vectors for the quiver $Q$. We wish to determine the minimal value of $\operatorname{Ext}(R, S)$ where $R$ and $S$ are representations of dimension vectors $\alpha$ and $\beta$ respectively. This minimal value is called $\operatorname{ext}(\alpha, \beta)$. Since
\[ \dim \text{Ext}(R_p, R_q) \] is an upper semicontinuous function as \((p, q)\) varies over \(R(Q, \alpha) \times R(Q, \beta)\), it is the value for an open subset of \(R(Q, \alpha) \times R(Q, \beta)\). In order to do this, we consider the rank of a general homomorphism between general representations of dimension vectors \(\alpha\) and \(\beta\). The rank of a homomorphism between representations of a quiver \(Q\) is the dimension vector of the image.

We define \(\text{Hom}_k(k^\alpha, k^\beta) = \times_v \text{Hom}(k^\alpha(v), k^\beta(v))\). If \(R\) is a representation of dimension vector \(\beta\) then \(\text{Hom}_k(k^\alpha, R) = \text{Hom}_k(k^\alpha, k^\beta)\).

**Lemma 5.1.** Let \(\alpha\) and \(\beta\) be dimension vectors for the quiver \(Q\). Then there are an open subset \(U\) of \(R(Q, \alpha) \times R(Q, \beta)\) and a dimension vector \(\gamma\) such that for all \((p, q)\) in \(U\), \(\dim \text{Hom}(R_p, R_q)\) is minimal and \((\phi \in \text{Hom}(R_p, R_q): \text{rank } \phi = \gamma\) is open and non-empty in \(\text{Hom}(R_p, R_q)\).

**Proof.** We define the variety
\[
\text{Hom}(\alpha, \beta) = \text{Hom}_k(k^\alpha, k^\beta) \times R(Q, \alpha) \times R(Q, \beta),
\]
\[
\text{Hom}(\alpha, \beta) = \{(\phi, p, q): \phi: R_p \rightarrow R_q \text{ is a homomorphism of representations}\}.
\]
There is the natural projection \(\Phi: \text{Hom}(\alpha, \beta) \rightarrow R(Q, \alpha) \times R(Q, \beta)\). The fibre above the point \((p, q)\) is \(\text{Hom}(R_p, R_q)\). Hence there is an open subset \(V\) of \(R(Q, \alpha) \times R(Q, \beta)\) such that \(\dim \text{Hom}(R_p, R_q)\) is minimal.

Let \(\delta\) be a dimension vector; we consider the subset \(\text{Hom}_\delta(Q, \alpha, \beta)\), in \(\text{Hom}(Q, \alpha, \beta)\):
\[
\text{Hom}_\delta(Q, \alpha, \beta) = \{(\phi, p, q): \text{rank } \phi = \delta\}.
\]
This is a constructible subset of \(\text{Hom}(Q, \alpha, \beta)\) and so there exists a dimension vector \(\gamma\) such that \(\text{Hom}_\gamma(Q, \alpha, \beta) \cap \Phi^{-1}(V)\) is constructible and dense in \(\Phi^{-1}(V)\); hence it contains an open subset of \(\Phi^{-1}(V)\). Thus \(\Phi(\text{Hom}_\delta(Q, \alpha, \beta) \cap \Phi^{-1}(V))\) is constructible and dense in \(V\); hence it contains an open subset \(U\) of \(V\). Therefore \(U\) and \(\gamma\) satisfy the requirements of the lemma.

We say that \(\gamma\) is the general rank of a homomorphism from a general representation of dimension vector \(\alpha\) to a general representation of dimension vector \(\beta\).

Our next theorem allows us to calculate \(\text{ext}(\alpha, \beta)\) in terms of the general rank of a homomorphism from a general representation of dimension vector \(\alpha\) to a general representation of dimension vector \(\beta\).

**Theorem 5.2.** Let \(\alpha\) and \(\beta\) be dimension vectors for \(Q\). Let \(\gamma\) be the general rank of a homomorphism from a general representation of dimension vector \(\alpha\) to a general representation of dimension vector \(\beta\). Then
\[
\alpha - \gamma \leftrightarrow \alpha \leftrightarrow \gamma \leftrightarrow \beta \leftrightarrow \beta - \gamma
\]
and
\[
\text{ext}(\alpha, \beta) = -\langle \alpha - \gamma, \beta - \gamma \rangle = \text{ext}(\alpha - \gamma, \beta - \gamma).
\]

**Proof.** Since the general rank of a homomorphism from a general representation of dimension vector \(\alpha\) to a general representation of dimension vector \(\beta\) is \(\gamma\), we know that a general representation of dimension vector \(\alpha\) has a factor of
dimension vector $\gamma$ and a general representation of dimension vector $\beta$ has a subrepresentation of dimension vector $\gamma$. Thus, $\alpha \to \gamma \to \beta$ and it follows that $\alpha - \gamma \to \alpha$ and $\beta \to \beta - \gamma$.

We fix a general representation $R$ of dimension vector $\beta$ such that $\text{hom}(\alpha, R) = \text{hom}(\alpha, \beta)$ and so that there exists an open subset $U'$ of $R(Q, \alpha)$ where for all $p \in U'$, the general rank of a homomorphism from $R_p$ to $R$ has rank $\gamma$. This is possible by Lemma 5.1.

Now Gr($R, \gamma$) is the variety of all subrepresentations of $R$ of dimension vector $\gamma$. By Theorem 3.3, we may also choose $R$ so that the dimension of Gr($R, \gamma$) is $\langle \gamma, \beta - \gamma \rangle$. If $p$ is a point of Gr($R, \gamma$) then $S_p$ is the corresponding subrepresentation of $R$. We consider a subbundle $B$ over Gr($R, \gamma$) of $\text{Hom}_k(k^\alpha, R) \times \text{Gr}(R, \gamma)$,

$$ B = \{ (\phi, p) : \text{image } \phi \subset S_p \}. $$

The fibre over the point $p$ may be identified with $\text{Hom}_k(k^\alpha, S_p)$ and so it has dimension $\sum \alpha(v)\gamma(v)$. We consider the open and dense subvariety $\text{Hom}(k^\alpha, \gamma, R)$ of $B$, $\text{Hom}(k^\alpha, \gamma, R) = \{ (\phi, p) : \text{image } \phi = S_p \}$. We note that $\text{Hom}(k^\alpha, \gamma, R)$ parametrises those elements of $\text{Hom}_k(k^\alpha, R)$ whose image is a subrepresentation of $R$ of dimension vector $\gamma$. It follows from the foregoing discussion that the dimension of $\text{Hom}(k^\alpha, \gamma, R)$ is $\sum \alpha(v)\gamma(v) + \langle \gamma, \beta - \gamma \rangle$.

Finally, we consider the vector subbundle $\text{Hom}(Q, \alpha, \gamma, R)$ over $\text{Hom}(k^\alpha, \gamma, R)$ of $R(Q, \alpha) \times \text{Hom}(k^\alpha, \gamma, R)$,

$$ \text{Hom}(Q, \alpha, \gamma, R) = \{ (q, (\phi, p)) : \phi : R_q \to R \text{ is a homomorphism of representations} \}. $$

By our description of $\text{Hom}(k^\alpha, \gamma, R)$, the image of $\phi$ is $S_p$. The dimension of a fibre above the point $(\phi, p)$ of $\text{Hom}(k^\alpha, \gamma, R)$ is given by

$$ \sum_a (\alpha - \gamma)(ia)(\alpha - \gamma)(ta) + \sum_a \gamma(ia)(\alpha - \gamma)(ta) = \sum_a \alpha(ia)(\alpha - \gamma)(ta). $$

Hence, the dimension of $\text{Hom}(Q, \alpha, \gamma, R)$ is

$$ \langle \gamma, \beta - \gamma \rangle + \sum \alpha(v)\gamma(v) + \sum \alpha(ia)(\alpha - \gamma)(ta). $$

We consider the natural morphism of varieties $\Phi : \text{Hom}(Q, \alpha, \gamma, R) \to R(Q, \alpha)$. If $q \in R(Q, \alpha)$, then

$$ \Phi^{-1}(q) = \{ (q, (\phi, p)) : \phi : R_q \to R \text{ is a homomorphism of representations with image } S_p \}. $$

Thus we may identify $\Phi^{-1}(q)$ with $\{ \phi \in \text{Hom}(R_q, R) : \text{rank } \phi = \gamma \}$. We consider $\Phi^{-1}(U')$. By assumption on $R$ and $U'$, $U' \subset \text{im } \Phi$ and hence $\Phi^{-1}(U')$ is open and dense in $\text{Hom}(Q, \alpha, \gamma, R)$. Moreover, for $q$ in $Q$, $\Phi^{-1}(q)$ is open and dense in $\text{Hom}(R_q, R)$. Hence, we may calculate the dimension of $\text{Hom}(Q, \alpha, \gamma, R)$ in a different way to be $\text{dim}(R(Q, \alpha)) + \text{hom}(\alpha, R)$. Thus we obtain the equality

$$ \langle \gamma, \beta - \gamma \rangle + \sum \alpha(v)\gamma(v) + \sum \alpha(ia)(\alpha - \gamma)(ta) = \sum \alpha(ia)\alpha(ta) + \text{hom}(\alpha, \beta). $$
Hence, \( \langle \gamma, \beta - \gamma \rangle + \langle \alpha, \gamma \rangle = \langle \alpha, \beta \rangle + \text{ext}(\alpha, \beta) \) and we deduce that
\[
-\langle \alpha - \gamma, \beta - \gamma \rangle = \text{ext}(\alpha, \beta).
\]

Let \( q \in U' \) so that \( \dim \text{Ext}(R_q, R) = \text{ext}(\alpha, R) = \text{ext}(\alpha, \beta) \). Let \( S_1 \rightarrow R_q \) where \( \dim S_1 = \alpha - \gamma \). Let \( R \rightarrow S_2 \) where \( \dim S_2 = \beta - \gamma \). Then by the right exactness of \( \text{Ext} \) for a quiver, the natural map from \( \text{Ext}(R_q, R) \) to \( \text{Ext}(S_1, S_2) \) is surjective. However, \( \dim \text{Ext}(S_1, S_2) = -\langle \alpha - \gamma, \beta - \gamma \rangle = \dim \text{Ext}(R_q, R) \). Hence, the natural map from \( \text{Ext}(R_q, R) \) to \( \text{Ext}(S_1, S_2) \) is an isomorphism, \( \dim \text{Ext}(S_1, S_2) = -\langle \alpha - \gamma, \beta - \gamma \rangle \), \( \text{Hom}(S_1, S_2) = 0 \) and so \( \text{ext}(\alpha - \gamma, \beta - \gamma) = -\langle \alpha - \gamma, \beta - \gamma \rangle \).

We record a corollary of the argument in the proof of Theorem 5.2.

**Corollary 5.3.** The notation is as in Theorem 5.2. Let \( R \) and \( S \) be representations of dimension vectors \( \alpha \) and \( \beta \) respectively such that \( \dim \text{Ext}(R, S) = \text{ext}(\alpha, \beta) \). Let \( S_1 \) be any subrepresentation of \( R \) of dimension vector \( \alpha - \gamma \), and let \( S_2 \) be a factor of \( S \) of dimension vector \( \beta - \gamma \). Then the natural map from \( \text{Ext}(R, S) \) to \( \text{Ext}(S_1, S_2) \) is an isomorphism and \( \text{Hom}(S_1, S_2) = 0 \).

**Proof.** This is what was shown in the last paragraph of the proof of Theorem 5.2.

We may regard Theorem 5.2 as providing an explanation for the non-vanishing of \( \text{ext}(\alpha, \beta) \). It is clear that \( \text{ext}(\alpha, \beta) \neq 0 \) if \( \langle \alpha, \beta \rangle < 0 \). More generally, if \( \alpha' \rightarrow \alpha \) and \( \beta \rightarrow \beta' \) and \( \langle \alpha', \beta' \rangle < 0 \) then \( \text{ext}(\alpha', \beta') \neq 0 \) and since every representation of dimension vector \( \alpha \) has a subrepresentation of dimension vector \( \alpha' \) whilst every representation of dimension vector \( \beta \) has a factor of dimension vector \( \beta' \), it follows that \( \text{ext}(\alpha, \beta) \neq 0 \) too. Thus we have the following theorem which allows us to determine the value of \( \text{ext}(\alpha, \beta) \) in terms of the dimension vectors of subrepresentations of \( \beta \).

**Theorem 5.4.** Let \( \alpha \) and \( \beta \) be dimension vectors for the quiver \( Q \). Then
\[
\text{ext}(\alpha, \beta) = \max \{ -\langle \alpha', \beta' \rangle \} = \max \{ -\langle \alpha, \beta'' \rangle \} = \max \{ -\langle \alpha'', \beta \rangle \}.
\]

**Proof.** Let \( R \) and \( S \) be representations of dimension vectors \( \alpha \) and \( \beta \) respectively such that \( \text{Ext}(R, S) = \text{ext}(\alpha, \beta) \). Let \( R' \) be a subrepresentation of \( R \) of dimension vector \( \alpha' \) and let \( S' \) be a factor of \( S \) of dimension vector \( \beta' \). Then
\[
\text{ext}(\alpha, \beta) = \dim \text{Ext}(R, S) \geq \dim \text{Ext}(R', S') \geq -\langle \dim R', \dim S' \rangle.
\]
Hence, \( \text{ext}(\alpha, \beta) \) is certainly at least as large as the remaining terms in the equality.

Conversely, Theorem 5.2 shows that equality holds for the first two terms in the theorem. It remains to show that there is a factor \( S' \) of \( S \) such that \( \text{Ext}(R, S) \equiv \text{Ext}(R, S') \) and \( \text{Hom}(R, S') = 0 \) in order to show that the first and third terms are equal. A dual argument will prove the equality of the first and fourth terms.

We use the notation and the proof of Theorem 5.2 and Corollary 5.3. For every subrepresentation \( S_1 \) of dimension vector \( \alpha - \gamma \) in \( R \) and for every factor \( S_2 \)
of dimension vector $\beta - \gamma$ of $S$, the composite map $\text{Ext}(R, S) \rightarrow \text{Ext}(R, S_2) \rightarrow \text{Ext}(S_1, S_2)$ is an isomorphism whilst the first map is a surjection. It follows that $\text{Ext}(R, S)$ is isomorphic to $\text{Ext}(R, S_2)$ and so $\text{Ext}(\alpha, \beta - \gamma) \cong \text{ext}(\alpha, \beta) = \dim \text{Ext}(R, S)$. On the other hand, $\alpha - \gamma \mapsto \alpha$ and so 
\[
\text{ext}(\alpha, \beta) = \text{ext}(\alpha - \gamma, \beta - \gamma) \leq \text{ext}(\alpha, \beta - \gamma).
\]
Hence, $\text{ext}(\alpha, \beta) = \text{ext}(\alpha, \beta - \gamma)$. By induction on $\beta - \gamma$, there exists $\beta''$ such that $\beta - \gamma \mapsto \beta''$ and $\text{ext}(\alpha, \beta - \gamma) = -\langle \alpha, \beta'' \rangle$. It follows that $\beta \mapsto \beta''$ and $\text{ext}(\alpha, \beta) = -\langle \alpha, \beta'' \rangle$. Thus the first and third terms are equal and by a dual argument the first and fourth terms are equal too.

With the results of this section and the last we may determine the dimension vectors of subrepresentations of a general representation of dimension vector $\alpha$, and given two dimension vectors $\alpha$ and $\beta$, we may determine the value of $\text{ext}(\alpha, \beta)$. Thus, given a dimension vector $\alpha$ we may assume inductively that we know the value of $\text{ext}(\beta, \gamma)$ for all $\beta$ and $\gamma$ less than $\alpha$. Then, $\beta \mapsto \alpha$ if and only if $\text{ext}(\beta, \alpha - \beta) = 0$. Given two dimension vectors $\alpha$ and $\beta$, we may assume inductively that we know the dimension vectors of all subrepresentations of general representations of dimension vectors $\alpha$ and $\beta$; hence, we know the value of $\text{ext}(\alpha, \beta)$ by the formula in Theorem 5.4.

Thus together with Kac's theorems we have a way to calculate the canonical decomposition of an arbitrary dimension vector.

We should record a consequence of the kind of argument in Theorem 5.2.

**Theorem 5.5.** Let $R$ be a representation of dimension vector $\beta$ such that a general representation of dimension vector $\alpha$ has a factor isomorphic to $R$. Then $\text{ext}(\alpha, R) = 0 = \text{ext}(\alpha, \beta)$.

**Proof.** We consider the open subvariety of $\text{Hom}_k(k^\alpha, R)$,
\[
\text{Sur}_k(k^\alpha, R) = \{ \phi: \phi \text{ is surjective} \}.
\]

The dimension of $\text{Sur}_k(k^\alpha, R)$ is $\sum \alpha(v)\beta(v)$. We consider a vector subbundle $\text{Sur}(Q, \alpha, R)$ over $\text{Sur}_k(k^\alpha, R)$ of $R(Q, \alpha) \times \text{Sur}_k(k^\alpha, R)$:
\[
\text{Sur}(Q, \alpha, R) = \{(p, \phi): \phi: R_p \rightarrow R \text{ is a surjection of representations}\}.
\]

The dimension of a fibre of $\text{Sur}(Q, \alpha, R)$ over a point $\phi$ of $\text{Sur}_k(k^\alpha, R)$ is $\sum \alpha(ia)(\alpha - \beta)(ia)$. Hence the dimension of $\text{Sur}(Q, \alpha, R)$ is
\[
\sum \alpha(v)\beta(v) + \sum \alpha(ia)(\alpha - \beta)(ia).
\]

We consider the natural morphism $\Phi: \text{Sur}(Q, \alpha, R) \rightarrow R(Q, \alpha)$. A general representation of dimension vector $\alpha$ has a factor isomorphic to $R$ if and only if this morphism has dense image. In this case, the dimension of a general fibre of $\Phi$ is $\langle \alpha, \beta \rangle$. On the other hand, the fibre $\Phi^{-1}(p)$ is $\text{Sur}(R_p, R)$, the variety of surjections from $R_p$ to $R$. By assumption, for a general point $p$ of $R(Q, \alpha)$, $\text{Sur}(R_p, R)$ is an open subset of $\text{Hom}(R_p, R)$; hence $\langle \alpha, \beta \rangle = \text{Hom}(R_p, R)$ and $\text{Ext}(R_p, R) = 0$. Thus $\text{ext}(\alpha, R) = 0$ and $\text{ext}(\alpha, \beta) = 0$. 

6. Schur roots

In this section, we shall summarise what we have found about the Schur roots as a subset of the set of roots. We begin with a characterisation of Schur roots in terms of the dimension vectors of subrepresentations.

**Theorem 6.1.** Let \( \alpha \) be a dimension vector for the quiver \( Q \). Then \( \alpha \) is a Schur root if and only if for all \( \beta \mapsto \alpha \), \( \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle > 0 \).

**Proof.** If \( \alpha = \beta + \gamma \), then \( \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle = \langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle \). It follows that if \( \alpha \) is not a Schur root and hence \( \alpha = \beta \oplus \gamma \) for some \( \beta \) and \( \gamma \), then one of \( \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle \) and \( \langle \gamma, \alpha \rangle - \langle \alpha, \gamma \rangle \) is not positive.

Conversely, if \( \alpha \) is a Schur root, let \( \beta \mapsto \alpha \); then \( \text{ext}(\beta, \alpha - \beta) = 0 \), so \( \langle \alpha - \beta, \beta \rangle \geq 0 \). Since \( \alpha \) is a Schur root, \( \text{hom}(\alpha - \beta, \beta) = 0 \) and \( \text{ext}(\alpha - \beta, \beta) \neq 0 \); hence, \( \langle \alpha - \beta, \beta \rangle < 0 \). Thus, \( \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle = \langle \beta, \alpha - \beta \rangle - \langle \alpha - \beta, \beta \rangle > 0 \).

We should like to explain the significance of this result. Let \( \gamma \) be some dimension vector. We define \( R(\gamma) \) to be the full subcategory on the representations \( \{ R : \text{Hom}(R, \gamma) \text{ and Ext}(R, \gamma) \text{ vanish generally} \} \). This full subcategory is closed under extensions, direct sums, kernels and images as the reader may easily check. Hence it is an abelian category in which all objects have finite length. If \( \tau \) is the linear transformation on the space of dimension vectors induced by the Auslander–Reiten translate, we note that \( \langle \alpha, \beta \rangle = \langle \tau^{-1}(\beta), \alpha \rangle = \langle \beta, \tau \alpha \rangle \). Hence, \( \langle \alpha, \alpha + \tau^{-1}(\alpha) \rangle = 0 \). Theorem 6.1 may be interpreted as saying that \( \alpha \) is a Schur root if and only if the general representation of dimension vector \( \alpha \) is a simple object in \( R(\alpha + \tau^{-1}(\alpha)) \). It is possible to use the methods of geometric invariant theory to study representations of a given dimension vector, \( \alpha \). To each dimension vector \( \beta \) such that \( \langle \alpha, \beta \rangle = 0 \), there is associated a linearisation of the action of \( \text{Gl}(\alpha) \) on \( \mathbb{P}(\alpha(P)k^{\alpha(P)}) \); the category \( R(\beta) \) is the category of \( \beta \)-semistable objects with respect to this linearisation and hence we may also say that \( \alpha \) is a Schur root if and only if a general representation of dimension vector \( \alpha \) is \( \alpha + \tau^{-1}(\alpha) \)-stable. We summarise this theory here rather than developing it in detail since we have not made use of this machinery. However, the reader may wish to compare these methods with those in [17].

Next, we note certain combinatorial results on Schur roots.

**Theorem 6.2.** Let \( \alpha \) be a root that is not a Schur root. Then either \( \alpha \) is an isotropic divisible root or else there is a real Schur root in its canonical decomposition; thus in the latter case, there is a real Schur root \( \beta < \alpha \) such that \( \langle \beta, \alpha \rangle \) and \( \langle \alpha, \beta \rangle \) are positive.

**Proof.** This follows at once from Theorem 4.4.

We should point out that if \( \alpha \) is a root and \( \beta < \alpha \) is a real Schur root such that \( \langle \alpha, \beta \rangle \) and \( \langle \beta, \alpha \rangle \) are positive, the possibilities for \( \beta \) are very restricted. Thus if \( \alpha \) is real, then \( r_\alpha(\beta) < 0 \). Given an element of the Weyl group, \( w \), there are only finitely many positive roots \( \gamma \) such that \( w(\gamma) < 0 \); thus if \( w = r_{i_1}r_{i_2} \cdots r_{i_n} \), where the length of \( w \) in the Weyl group is \( n \) then for each \( m \leq n \), set
\[ \beta_m = r_{i_m} r_{i_{m-1}} \cdots r_{i_{m+1}}(\sigma_{i_m}); \text{ then } r_{\alpha}(\gamma) < 0 \text{ if and only if } \gamma = \beta_m \text{ for some } m. \] Given \( \alpha \) a real root, it is a simple matter to write \( r_{\alpha} \) as a product of the \( r_i \). If \( v_i \) is a vertex such that \( r_i(\alpha) = \alpha' < \alpha \) then \( r_{\alpha} = r_i r_{\alpha'} r_i \) and if \( r_{\alpha'} = r_i r_i \cdots r_i \) is a shortest expression for \( r_{\alpha'} \) then \( r_{\alpha} = r_i r_i \cdots r_i r_i \) is a shortest expression for \( r_{\alpha}. \) Hence we may inductively calculate a shortest expression for \( r_{\alpha}. \) Thus the possible real Schur roots in the canonical decomposition for \( \alpha \) are easy to find. We have already discussed the set of real Schur roots in [16] to which we refer the reader for further information on this topic.

If \( \alpha \) is an imaginary root, we may limit the possible real Schur roots in the canonical decomposition of \( \alpha \) in a similar fashion. If \( \beta \) is a real Schur root in the canonical decomposition of \( \alpha \), then \( \langle \alpha, \beta \rangle \) and \( \langle \beta, \alpha \rangle \) are positive; hence \( \langle \beta, \alpha \rangle \) is positive. Let \( w \) be an element of the Weyl group such that \( w(\alpha) \in C; \) then \( (w(\alpha), w(\beta)) > 0 \). It follows that \( w(\beta) < 0 \). As in the previous paragraph we may find all the possible candidates for \( \beta \) in terms of \( w \) very simply. Again, it is a simple matter to find \( w \) in terms of \( \alpha \); let \( v_i \) be a vertex such that \( r_i(\alpha) = \alpha' < \alpha \). Let \( w' \) be an element of the Weyl group such that \( w'(\alpha') \in C; \) then \( w'r_i(\alpha) \in C. \) Moreover, if \( w' = r_i r_i \cdots r_i \) is a shortest expression for \( w' \), then \( w = r_i r_i \cdots r_i r_i \) is a shortest expression for \( w. \) Once again we limit the possibilities for the real Schur roots in the canonical decomposition of \( \alpha. \)

At this point we wish to raise a question about the canonical decomposition. A positive answer to this would simplify our present picture of the Schur roots. Let \( \alpha \) be a root and let \( \beta \) be a real Schur root such that \( \beta < \alpha \) and \( \langle \alpha, \beta \rangle \) and \( \langle \beta, \alpha \rangle \) are both positive; does it follow that \( \alpha \) is not a Schur root? At present, we know only that this is true for small quivers which is not sufficient evidence for the general question. In [17], this is shown to hold for the \( n \)-subspace quiver; however, there are many results for this quiver which are simply false for general quivers.

**References**


_School of Mathematics_
_University of Bristol_
_University Walk_
_Bristol BS8 1TW_