Nome, Cognome e Matricola

Esercizi Settimanali di Geometria 1
Ingegneria Chimica
Settimana 8
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Da consegnare Martedi 19 Novembre 2019

Esercizio 1. *Si consideri la seguente matrice reale:*

$$A = \begin{pmatrix} -2 & 4 \\ 3 & -2 \end{pmatrix}$$

1. *Dimostrare che A è invertibile.*
2. *Scrivere A come prodotto di matrici elementari.*

<u>Sol.</u>: 1. $\det A = 4-12 = -8 \neq 0 \Rightarrow A$ è invertibile

2. $A = \begin{pmatrix} -2 & 4 \\ 3 & -2 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\leadsto} \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix} = P_{12} A \underset{R_1 \mapsto R_1 + R_2}{\leadsto} \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = F_{12}(1) P_{12} A$

$\underset{R_2 \mapsto R_2 + 2R_1}{\leadsto} \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix} = F_{21}(2) F_{12}(1) P_{12} A$

$\underset{R_2 \mapsto \frac{1}{8} R_2}{\leadsto} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = D_2\left(\frac{1}{8}\right) F_{21}(2) F_{12}(1) P_{12} A$

$\underset{R_1 \mapsto R_1 - 2R_2}{\leadsto} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = F_{12}(-2) D_2\left(\frac{1}{8}\right) F_{21}(2) F_{12}(1) P_{12} A$

$\Rightarrow \mathbb{1}_2 = F_{12}(-2) D_2\left(\frac{1}{8}\right) F_{21}(2) F_{12}(1) P_{12} A$

$\Rightarrow A = \left(F_{12}(-2) D_2\left(\frac{1}{8}\right) F_{21}(2) F_{12}(1) P_{12}\right)^{-1} =$

$= P_{12} F_{12}(-1) F_{21}(-2) D_2(8) F_{12}(2)$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Esercizio 2. *Calcolare il determinante della seguente matrice:*

$$A = \begin{pmatrix} 1 & 4 & 4 & 3 & 3 \\ 3 & 1 & 3 & 2 & 1 \\ 3 & 1 & 5 & 1 & 2 \\ 4 & 2 & 4 & 4 & 1 \\ 4 & 3 & 5 & 4 & 2 \end{pmatrix}.$$

Sol.:

$$\det A = \det \begin{pmatrix} 1 & 4 & 3 & 3 & 3 \\ 3 & 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 4 & 2 & 0 & 4 & 1 \\ 4 & 3 & 1 & 4 & 2 \end{pmatrix} = \det \begin{pmatrix} -11 & -5 & 0 & -9 & -3 \\ 3 & 1 & 0 & 2 & 1 \\ -5 & -5 & 0 & -7 & -2 \\ 4 & 2 & 0 & 4 & 1 \\ 4 & 3 & 1 & 4 & 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} -11 & -5 & -9 & -3 \\ 3 & 1 & 2 & 1 \\ -5 & -5 & -7 & -2 \\ 4 & 2 & 4 & 1 \end{pmatrix} = \det \begin{pmatrix} -6 & -5 & -9 & -3 \\ 2 & 1 & 2 & 1 \\ 0 & -5 & -7 & -2 \\ 2 & 2 & 4 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & -2 & -3 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & -5 & -7 & -2 \\ 0 & 1 & 2 & 0 \end{pmatrix} = -2 \det \begin{pmatrix} -2 & -3 & 0 \\ -5 & -7 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= -4 \det \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} = -4 \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = 4 \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= 4$$

Esercizio 3. *Si consideri la seguente matrice complessa:*

$$A = \begin{pmatrix} 2-i & 1+i & 1-i \\ 2i & -i & 2+2i \\ -2+i & 1+i & 3i \end{pmatrix}.$$

1. *Sviluppare il determinante lungo la seconda colonna;*
2. *Sviluppare il determinante lungo la terza riga;*
3. *Sviluppare il determinante lungo la prima riga;*
4. *Sviluppare il determinante lungo la terza colonna.*

Sol.:

1. $-(1+i)\det\begin{pmatrix} 2i & 2+2i \\ -2+i & 3i \end{pmatrix} - i\det\begin{pmatrix} 2-i & 1-i \\ -2+i & 3i \end{pmatrix} - (1+i)\det\begin{pmatrix} 2-i & 1-i \\ 2i & 2+2i \end{pmatrix} =$

$= -(1+i)2i - i(3i+4) - (1+i)4 = 1-10i$

2. $(-2+i)\det\begin{pmatrix} 1+i & 1-i \\ -i & 2+2i \end{pmatrix} - (1+i)\det\begin{pmatrix} 2-i & 1-i \\ 2i & 2+2i \end{pmatrix} + 3i\det\begin{pmatrix} 2-i & 1+i \\ 2i & -i \end{pmatrix}$

$= (-2+i)(1+5i) - (1+i)4 + 3i(1-4i) = 1-10i$

3. $(2-i)\det\begin{pmatrix} -i & 2+2i \\ 1+i & 3i \end{pmatrix} - (1+i)\det\begin{pmatrix} 2i & 2+2i \\ -2+i & 3i \end{pmatrix} + (1-i)\det\begin{pmatrix} 2i & -i \\ -2+i & 1+i \end{pmatrix}$

$= (2-i)(3-4i) - (1+i)2i + (1-i)(-3) = 1-10i$

4. $(1-i)\det\begin{pmatrix} 2i & -i \\ -2+i & 1+i \end{pmatrix} - (2+2i)\det\begin{pmatrix} 2-i & 1+i \\ -2+i & 1+i \end{pmatrix} + 3i\det\begin{pmatrix} 2-i & 1+i \\ 2i & -i \end{pmatrix}$

$= (1-i)(-3) - (2+2i)(6+2i) + 3i(1-4i) = 1-10i$

Esercizio 4. *Si consideri la seguente matrice dipendente dal parametro k:*

$$A(k) = \begin{pmatrix} 0 & k & k^2 \\ 1 & (k-1)^2 & k-1 \\ -k-1 & k-1 & 1-k \end{pmatrix}$$

Utilizzare il teorema degli orlati per trovare i valori di $k \in \mathbb{R}$ per i quali

1. $rg(A(k)) = 1$;
2. $rg(A(k)) = 2$;
3. $rg(A(k)) = 3$.

Esercizio 4. *Stabilire la posizione reciproca delle seguenti coppie di rette di* $\mathbb{R}^2$

$$\frac{1}{\sqrt{3}} \qquad \frac{4\sqrt{3}}{3} = \frac{4}{\sqrt{3}}$$

$$\sqrt{3} \qquad 3\sqrt{3} = \frac{9}{\sqrt{3}}$$

1. $r : 2x + 3y = 1,\ s : 3x + 2y = -1;$
2. $r : 3x + 2y = -\sqrt{3},\ s = \begin{pmatrix} -1/\sqrt{3} \\ \sqrt{3}/3 \end{pmatrix} + \langle \begin{pmatrix} -2/9 \\ 1/3 \end{pmatrix} \rangle;$
3. $r = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + \langle \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rangle,\ s = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \langle \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$
4. $r = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{3} \end{pmatrix} + \langle \begin{pmatrix} \sqrt{2}/2 \\ 2/\sqrt{2} \end{pmatrix} \rangle,\ s = \begin{pmatrix} 4\sqrt{3}/3 \\ 3\sqrt{3} \end{pmatrix} + \langle \begin{pmatrix} \sqrt{2}/2\sqrt{3} \\ 2/\sqrt{6} \end{pmatrix} \rangle.$

Sol.: 1. Studiamo il sistema $\begin{cases} 2x+3y=1 \\ 3x+2y=-1 \end{cases}$:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 2 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 2 & -1 \\ 2 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 5 & 5 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right). \Rightarrow r \cap s = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

2. $3\left(-\frac{1}{\sqrt{3}} - \frac{2}{9}t\right) + 2\left(\frac{\sqrt{3}}{3} + \frac{1}{3}t\right) = -\sqrt{3}$

$$\Leftrightarrow \left(-\frac{2}{3} + \frac{2}{3}\right)t = -\sqrt{3} + \frac{3}{\sqrt{3}} - \frac{2}{3}\sqrt{3} = 0 \Rightarrow r \equiv s.$$

3. $\begin{cases} -4 + 2t_1 = 2 + t_2 \\ 1 + 3t_1 = 4 + 3t_2 \end{cases} \Leftrightarrow \begin{cases} 2t_1 - t_2 = 6 \\ 3t_1 - 3t_2 = 3 \end{cases}$

$$\left(\begin{array}{cc|c} 2 & -1 & 6 \\ 3 & -3 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -1 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 4 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \Rightarrow r \cap s = \left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \end{pmatrix} \right\}$$

4. $r = \sqrt{3}\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} + \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle,\ s = \sqrt{3}\begin{pmatrix} \frac{4}{3} \\ 3 \end{pmatrix} + \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle$

$$\sqrt{3}\begin{pmatrix} 4/3 \\ 3 \end{pmatrix} - \sqrt{3}\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} = \sqrt{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow r \equiv s.$$

Esercizio 5. *Di ognuno dei seguenti sottospazi affini (dell'opportuno spazio vettoriale) trovare le equazioni cartesiane (se sono in forma parametrica) e parametriche (se sono in forma cartesiana):*

1. $\pi : 2x + 3y - 2z = 2$ *(in* $\mathbb{R}^3$*);*
2. $r : \begin{cases} -x + 2y + 3z = 1 \\ 2x + y - z = 3 \end{cases}$ *(in* $\mathbb{R}^3$*);*
3. $r = (1,1,1)^t + \langle (1,2,1)^t \rangle$ *(in* $\mathbb{R}^3$*);*
4. $\pi = (1,2,1)^t + \langle (2,2,1)^t, (1,-2,1)^t \rangle$ *(in* $\mathbb{R}^3$*);*
5. $r = (-1,1)^t + \langle (1,1)^t \rangle$ *(in* $\mathbb{R}^2$*);*
6. $r : 2x + 3y = -1$ *(in* $\mathbb{R}^2$*).*

Sol.:

1. $\pi : x = -\frac{3}{2}y + z + 1 \Rightarrow \pi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

2. $\left(\begin{array}{ccc|c} -1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & 3 & 1 \\ 0 & 5 & 5 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \Rightarrow r = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

3. $\left(\begin{array}{c|c} 1 & x_1 \\ 2 & x_2 \\ 1 & x_3 \end{array}\right) \sim \left(\begin{array}{c|c} 1 & x_1 \\ 0 & x_2 - 2x_1 \\ 0 & x_3 - x_1 \end{array}\right) \Rightarrow r : \begin{cases} 2x_1 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$

4. $\left(\begin{array}{cc|c} 2 & 1 & x_1 \\ 2 & -2 & x_2 \\ 1 & 1 & x_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & x_3 \\ 2 & -2 & x_2 \\ 2 & 1 & x_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & x_3 \\ 0 & -4 & x_2 - 2x_3 \\ 0 & -1 & x_1 - 2x_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & x_3 \\ 0 & 1 & 2x_3 - x_1 \\ 0 & -4 & x_2 - 2x_3 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} 1 & 1 & * \\ 0 & 1 & * \\ 0 & 0 & x_2 - 2x_3 + 8x_3 - 4x_1 \end{array}\right) \Rightarrow \pi : -4x_1 + x_2 + 6x_3 = 7$

5. $r : x - y = -2$

6. $r = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle$