MAKING BAD CONJECTURES AND INCOMPLETE PROOFS WITH GOOD DRAWINGS

WITHIN A DYNAMIC GEOMETRY ENVIRONMENT¹

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We present some experiments with pre-service and in-service math teachers which show how the use of a good drawing with a dynamic geometry system sometimes "hides" some evident properties and therefore induces to miss some conjectures and to produce "proofs" with bugs.

INTRODUCTION

In order to give a geometric proof, which can be mastered by a student, it is important to be helped by a good drawing. There are many good examples of faulty geometric proofs based on wrong drawings (Dubnov, 1963; Ball, Coxeter, 1987). Sometimes in a proof one has to examine all the possible cases and therefore several drawings are required for a single theorem.

Dynamic geometry software is very useful because it gives several good drawings all at once. When used in a problem solving activity, dynamic geometry software displays all its strength: the student, accustomed to read in a text a geometric proof and to repeat it, begins to work and think like a mathematic researcher (Schattschneider, King, 1997; Furinghetti, Olivero, Paola, 2001; Furinghetti, Paola, 2003). A dynamic geometry software is also fruitful when used to introduce students to proof (Mariotti, 2000; Mariotti, 2001). A central role in problem solving activities is played by dragging. Arzarello, Olivero, Paola & Robutti, 2002 classified the different kinds of dragging while working on open problems. Sinclair, 2003 claims that sometimes students do not use the hints that an accurate dynamic drawing gives because they are usually warned by the teachers not to trust in figures that are not necessarily accurate.

We made many experiments with students and teachers working on open problems in a dynamic geometry environment. We claim that in some case a drawing, because it is accurate, can prevent students (and teachers, as well) from making conjectures and giving correct proofs. We have evidences that there is this problem with secondary school students (Accascina, G. et alii, 2004). In this paper we show that also pre-service teachers and in service teachers have the same problem.

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FIRST TOPIC

A) Experiments with pre-service high school teachers

Participants and organization. We made the experiment in Academic Years 2003-04 and 2004-05 with students who already got a four years degree in mathematics or physics and were attending the first of a two years course of postgraduate school in secondary (high school) teaching (SSIS). We repeated the experiments four times. Each year the students where in fact subdivided in two groups: one constituted by pre-service teachers in Math and Physics, the other by pre-service teachers in Applied Mathematics. The students of each group were enrolled on a 12 hours lab course subdivided in 4 meetings. Each meeting lasted 3 hours divided in sessions of 1 hour and half each with a 30 minutes break. The students were asked to write after each meeting at least half page with the description of the most significant points of it and to e-mail it to us before the next meeting. Students worked in pairs. Each pair had access to a PC. We made use of a projector linked to a PC.

The first 3 meetings focused on pros and cons using a Computer Algebra System in secondary teaching. Derive was used. The last meeting was on Dynamic Geometry Systems. Almost none of the students had worked with them. Cabri II was used. The activities of the last meeting were described in labsheets which were chosen from a proposal of an activity of problem solving with Cabri (Accascina, Margiotta, 2002 and 2003). Each labsheet was handed to students after the completion of the proceeding one.

Getting acquainted with Cabri II.

The first five labsheets were prepared to introduce students to Cabri II. They begin with the description of main commands of Cabri. Then some problems are posed:

- Draw two points *A* and *B* and one of the equilateral triangles which have an edge on the segment *AB* (Euclid's Elements, book 1, prop. 1). Write a macro for it.

- Draw two points A and B, the segment AB, its midpoint and its perpendicular bisector (Euclid's Elements, book 1, prop. 10). Do not use the Cabri commands for midpoint and perpendicular bisector.

- Draw a point A, a straight line r, the straight line passing through A and perpendicular to r (Euclid's Elements, book 1, prop. 11 and 12). Do not to use the Cabri commands for perpendicular line.

- Draw three points *A*, *B* and *C*, the triangle *ABC* and its circumcircle (Euclid's Elements, book 4, prop. Prop. 5). Write a macro for it.

On the fifth labsheet we gave the usual solution of the last problem. The circumcircle is constructed by drawing the circle with centre in the intersection of the perpendicular bisectors of two edges of the triangle and passing through one of the vertices of the triangle. We stressed the fact that usually secondary school students are able, using Cabri, to draw the circumcircle because they studied it in the middle school. But they are not able to prove that the constructed circle passes

also through the other two vertices of the triangle. They usually do not even understand the teacher's request since by the Cabri drawing it is "obvious". We pointed out that the proof rests on the fact that the 3 perpendicular bisectors are not in general position (i.e. they intersect in a point).

After working one hour and ten minutes on the five labsheets all the students were able to use properly Cabri.

The core of first topic.

Labsheet 6.

Draw triangle ABC. Construct the edge а on ABthe externally equilateral triangle ABR to triangle ABC, draw the Circle c1 which circumscribes ABR. Repeat the same construction on the edges BC and AC. You obtain the equilateral triangles BCP e ACQ and the circles c2which circumscribe them. Which and cЗ are the properties of this figure? Write the conjectures in the order you make them.

All the students used the macros they constructed before and quickly drew the following figure:



The problem is well known problem (cfr. for example Coxeter, 1961; Sinclair, 2002), but no one of the students knew it in advance.

The figure has many properties: the circles c1, c2 and c3 (which are called Fermat circles) intersect in a point F (which is called Fermat point); the triangle O1O2O3 is equilateral (Napoleon Theorem); the straight lines CR, AP and BQ intersect in F; the segments CR, AP and BQ are congruent; the segments CR, AP and BQ are perpendicular to the edges O2O3, O1O3, O1O2 respectively of the triangle O1O2O3; the point F is interior to the triangle ABC if and only if all the angles of

the triangle are less of 120° and in this case the point *F* has the property to be the point for which the sum of its distances from the vertices *A*, *B* and *C* is minimal.

Students spent the 15 minutes before the break looking for conjectures. All of them got very involved in the problem. Many continued investigating during the break. Almost all of them conjectured Napoleon Theorem.

We proposed this problem because we were interested in finding if students would notice that the Fermat circles intersect in a point. Many did not notice it. Many conjectured some properties of Fermat point without noticing explicitly that this point exists. They wrote, for example:

	2003-04		2004-05	
	Math. & Fis.	Appl. Math	Math. & Fis.	Appl. Math
Total	26	20	32	22
No conjectures on Fermat point	5	8	10	2
Conjectures on properties of Fermat point but without any conjecture on its existence	6	4	3	5
The conjecture on the existence of Fermat point is not the first in their list	3	4	10	7
The first conjecture in their list is on the existence of Fermat point	12	4	9	8

The point of intersection of the three circles is the incentre of the triangle ABC (Wrong). The circles intersect in a point interior to the triangle (Wrong).

Many students did not see that the 3 Fermat circles are not in general position (i.e. they do intersect in a point). Looking at this problem the pre-service teachers acted similarly to the high school students working with the problem of the intersection of perpendicular bisectors of a triangle, although they were advised just few minutes earlier about this kind of behaviour. We discussed with the students on their answer in the first 15 minutes after the break. They were very impressed by their failure. In the description of the meeting Silvia wrote: *Starting from the famous sentence "Geometry is the art of good reasoning on a bad picture", I would continue saying that Cabri offered me the possibility of not reasoning over a well done picture. Why do I say this? I do realise that the sentence is strong but this is what happened to me today in the laboratory. Looking at Fermat circles, I hardly understood that the triangle with vertices on the centres of those circles is equilateral and I was not able to see anything else. Well, maybe with a bad figure I would not even understand this.*

B) Experiments with in-service teachers.

A similar labsheet was given during two seminars for in-service teachers. In the first seminar the audience was of middle school teachers. All of them taught mathematics and sciences. Few of them graduated in mathematics or physics. In the second seminar the audience was of secondary school math teachers, all of them graduated in mathematics or physics. In both seminars, after one hour of seminar on Cabri 2D, it was given a worksheet with the description of the problem and the figure. The teachers had to write the conjectures. They had no computer to use. The speaker, using a projector connected to a PC, drew the picture with Cabri and dragged the points A, B and C following the teachers' requests. They worked for 10 minutes to give the following answers.

	Middle S. Teachers	High S. Teachers
Total	15	19
No conjectures on Fermat point	12	13
Conjectures on properties of Fermat point but without any conjecture on its existence	3	2
The conjecture on the existence of Fermat point is not the first one in their list	0	0
The first conjecture in their list is on the existence of Fermat point	0	4

SECOND TOPIC. Experiments with pre-service high school teachers

Likewise the first topic, we repeated four times the experiment. All the times the experiment was done in the last hour and 15 minutes of the meeting after the conclusion and discussion of the first topic on Fermat circles. In this case we asked the students, instead to write their answers, to discuss all together. We gave the following problem:





Figure 1

Figure 2

Figure 3

On all the experiments no one, as expected, was able to give an answer. Therefore we gave them a hint with figure 2; later, since no one was able yet to give an answer, we gave another hint with figure 3. After it everything was clear. The arcs are part of Fermat circles. The angle in $A^*(B^*, C^*)$ is equal to the angle in A'(B', C'). Therefore it measures 60°. The construction of the equilateral triangle $A^*B^*C^*$ is easily done. We choose a point C^* on a Fermat circle and from it we draw the half straight lines passing through A and B respectively. There are therefore infinite triangles $A^*B^*C^*$. Then we asked.

Did we solve completely our problem?

On all the experiments all the students said "yes", as expected. Then we moved the point C^* either near the point B or near the point A. The point B^* (or A^*) disappeared.

We omit the discussion on which part of the arc AC'B the point C^* must be chosen and go to the core of the experiment. We asked:

Did we really found the equilateral triangle circumscribed to triangle ABC?

At the beginning, in all the experiments, the students did not reply anything. In one case after a while a student said

I would say "yes" but the proceeding experiences and the way you posed the question make me think that the replay is "no". But what else is missing?

In one case a student found the solution. Luca, another student, in its comments to the meeting, described what happened: To all of us the construction looked so beautiful that it should be exact ... [later] a student observed that, although all the angles of the triangle we constructed were of 60° , it should still be proved that the last constructed edge passes effectively through the vertex of the original triangle; this is evident in the drawing but from the drawings one has not a mathematical proof. Yet another time one must not be cheated by the drawing; if the teacher's apprentices fail, what will happen to students?

In the other two cases nobody found the solution.

THIRD TOPIC

We did this experiment in the Academic Year 2004-05 with 8 pre-service teachers who did the experiments on the first two topics in the Academic Year 2003-04. They were enrolled on 12 hours course on 3D geometry. The details on this course are in Accascina, Rogora. In the last 6 hours the students worked in a computer lab. Each student had access to a PC and we made use of a projector linked to a computer. We asked the students to open a Cabri3D file we prepared in advance. We showed how to change the point of view. Then we handed the following labsheet.



The drawing represents a regular tetrahedron and its four heights (height = straight line passing through a vertex and perpendicular to the opposed face). Which observations can you make on the heights of a regular tetrahedron? Write the conjectures in the order you find them.

The same experiment was done in the seminar with high school math teachers just after the first experiment on Fermat circles. The teachers had to write the conjectures. They had no computer to use. The speaker drew the picture with Cabri3D and changed the point of view following the teachers' requests. It is remarkable that in both experiments not all the people noticed the evident fact that the heights are not in general position. They graduated in mathematics or physics and should know that 4 straight lines in space usually do not intersect in a point.

	Pre-service teachers	In-service teachers
Total	8	19
No conjectures on the intersection of the heights	1	5
Conjectures on properties of the point of intersection of heights but without any conjecture on its existence	1	2
The conjecture on the intersection of heights is not the first one in their list	2	4
The conjecture is on the intersection of heights is the first in their list	4	10

CONCLUSIONS

We have shown drawings with Cabri 2D and 3D where some evident properties were not pointed out by the some of the users. All of them, being either pre-service or in-service math teachers, knew the role of drawings in geometric proofs. The properties not pointed out were not "obvious"; in the sense that no one of the users would have guessed them without a good drawing. On the other side we believe that all these properties were not been pointed out because with the drawings they were "evident". By "evident" here we mean that the property continues to hold while dragging (i.e. it is an invariant) and for seeing it in the drawing one does not need to add new elements or to measure segments or angles. Several consequences follow from the fact that good drawings "hide" some properties: students do not see what the teacher wants them to see and produce proofs with bugs.

In order to overcome these consequences the teacher should propose more general problems and drawings where the "evident" properties hold just in particular cases. Anyway we think that a teacher should expose the students to at least one drawing with "evident" properties to let students became aware that they cannot assume anything as granted.

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