THE TOTAL VARIATION FLOW*

Fuensanta Andreu and José M. Mazón†

Abstract

We summarize in this lectures some of our results about the Minimizing Total Variation Flow, which have been mainly motivated by problems arising in Image Processing. First, we recall the role played by the Total Variation in Image Processing, in particular the variational formulation of the restoration problem. Next we outline some of the tools we need: functions of bounded variation (Section 2), paring between measures and bounded functions (Section 3) and gradient flows in Hilbert spaces (Section 4). Section 5 is devoted to the Neumann problem for the Total variation Flow. Finally, in Section 6 we study the Cauchy problem for the Total Variation Flow.

1 The Total Variation Flow in Image Processing

We suppose that our image (or data) u_d is a function defined on a bounded and piecewise smooth open set D of $I\!R^N$ - typically a rectangle in $I\!R^2$. Generally, the degradation of the image occurs during image acquisition and can be modeled by a linear and translation invariant blur and additive noise. The equation relating u, the real image, to u_d can be written as

$$u_d = Ku + n, \tag{1}$$

where K is a convolution operator with impulse response k, i.e., $Ku = k * u$, and n is an additive white noise of standard deviation σ. In practice, the noise can be considered as Gaussian.

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†Departamento de Análisis Matemático, Universitat de Valencia, 46100 Burjassot (Valencia), Spain, Fuensanta.Andreu@uv.es, mazon@uv.es

The problem of recovering u from u_d is ill-posed. First, the blurring operator need not be invertible. Second, if the inverse operator K^{-1} exists, applying it to both sides of (1) we obtain

$$K^{-1}u_d = u + K^{-1}n. \tag{2}$$

Writing $K^{-1}n$ in the Fourier domain, we have

$$K^{-1}n = \left(\frac{\hat{n}}{\hat{k}}\right)^{\vee}$$

where $\hat{f}$ denotes the Fourier transform of f and $f^{\vee}$ denotes the inverse Fourier transform. From this equation, we see that the noise might blow up at the frequencies for which $\hat{k}$ vanishes or it becomes small.

The typical strategy to solve this ill-conditioning is regularization. Then the solution of (1) is estimated by minimizing a functional

$$J_\gamma(u) = \| Ku - u_d \|_2^2 + \gamma \| Qu \|_2^2, \tag{3}$$

which yields the estimate

$$u_\gamma = (K^tK + \gamma Q^tQ)^{-1}K^tu_d, \tag{4}$$

Q being a regularization operator.

The first regularization method consisted in choosing between all possible solutions of (2) the one which minimized the Sobolev (semi) norm of u

$$\int_D |Du|^2\,dx,$$

which corresponds to the case $Qu = \nabla u$. Then the solution of (3) given by (4) in the Fourier domain is given by

$$\hat{u} = \frac{\overline{\hat{k}}}{|\hat{k}|^2 + 4\gamma\pi^2|\xi|^2}\hat{u}_d.$$

From the above formula we see that high frequencies of u_d (hence, the noise) are attenuated by the smoothness constraint. This was an important step, but the results were not satisfactory, mainly due to the inability of the previous functional to resolve discontinuities (edges) and oscillatory textured patterns. The smoothness constraint is too restrictive. Indeed, functions in $W^{1,2}(D)$ cannot have discontinuities along rectifiable curves. These observations motivated the introduction of Total Variation in image restoration models by L. Rudin, S. Osher and E. Fatemi in their seminal work [23]. The a priori hypothesis is that functions of bounded variation (the BV model) [2],[13],[24]) are a reasonable functional model for many problems in image processing, in particular, for restoration problems ([22],[23]). Typically, functions of bounded variation have discontinuities along rectifiable curves, being continuous in some sense (in the measure theoretic sense) away from discontinuities. The discontinuities could be identified with edges.

On the basis of the BV-model, Rudin-Osher-Fatemi [23] proposed to solve the following constrained minimization problem

$$\begin{aligned} &\text{Minimize} \int_D |Du|\,dx \\ &\text{with} \quad \int_D Ku = \int_D u_d, \quad \int_D |Ku - u_d|^2\,dx = \sigma^2 |D|. \end{aligned} \tag{5}$$

The first constraint corresponds to the assumption that the noise has zero mean, and the second that its standard deviation is σ. The constraints are a way to incorporate the image acquisition model given in terms of equation (1). Under some assumption

$$\left\| u_d - \int_\Omega u_d \right\| \geq \sigma^2,$$

the constraint

$$\int_D |Ku - u_d|^2\,dx = \sigma^2 |D| \tag{6}$$

is equivalent to the constraint

$$\int_D |Ku - u_d|^2\,dx \leq \sigma^2 |D|,$$

which amounts to say that σ is an upper bound of the standard deviation of n. Moreover, assuming that $K1 = 1$, the constraint $\int_D Ku = \int_D u_d$ is automatically satisfied [11].

In practice, the above problem is solved via the following unconstrained minimization problem

$$\text{Minimize} \int_\Omega |Du|\,dx + \frac{\lambda}{2} \int_\Omega |Ku - u_d|^2\,dx \tag{7}$$

for some Lagrange multiplier λ.

The most successful analysis of the connections between (5) and (7) was given by A. Chambolle and P.L. Lions in [11]. Indeed, they proved that both problems are equivalent for some positive value of the Lagrange multiplier λ.

Let us define the functional $\Phi : L^2(\Omega) \to (-\infty, +\infty]$ by

$$\Phi(u) = \begin{cases} \int_\Omega \|Du\| & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{8}$$

Proposition 1 *If u is a solution of (5), then there is some $\lambda \geq 0$ such that*

$$-\lambda K^t(Ku - u_d) \in \partial\Phi(u). \tag{9}$$

In particular, the Euler-Lagrange equation associated with the denoising problem, that is, for problem (5) with $K = I$, is the equation

$$-\lambda(u - u_d) \in \partial\Phi(u). \tag{10}$$

Formally,

$$\partial\Phi(u) = -\text{div}\left(\frac{Du}{|Du|}\right).$$

Now, the problem is to give a sense to (10) as a partial differential equation, describing the subdifferential of Φ in a distributional sense.

Motivated by the image restoration problem we initiated in [3] the study of the minimizing total variation flow $u_t = \text{div}(\frac{Du}{|Du|})$. Indeed, this PDE is the gradient descent associated to the energy

$$\int_\Omega |Du|.$$

Observe that we are not considering the constraints given by the image acquisition model in this simplified energy. Thus our conclusions will not directly inform us about the complete model (5). Instead, our purpose was to understand how the minimizing total variation flow minimizes the total variation of a function. There are many flows which minimize the total variation of a function. Let us mention in particular the mean curvature motion ([21])

$$\frac{\partial u}{\partial t} = |Du|\text{div}\left(\frac{Du}{|Du|}\right). \tag{11}$$

Indeed, this flow corresponds to the motion of curves in $I\!R^2$ or hypersurfaces $S(t)$ in $I\!R^N$ by mean curvature, i.e.,

$$X_t = H\vec{N} \tag{12}$$

where X denotes a parametrization of $S(t)$, H denotes its mean curvature and $\vec{N}$ the outer unit normal. The classical motion given by (12) corresponds to the gradient descent of the area functional $\int_S dS$. Both flows, the classical mean curvature motion (12), and its viscosity solution (11) formulation have been studied by many authors, we refer in particular to the work by L.C. Evans and J. Spruck [14]. They proved, in particular, that the total variation of the (viscosity) solution of (11) decreases during the evolution, as it should happen since the flow decreases the $(N-1)$ Haussdoff measure of the level set surfaces of the solution u and the total variation corresponds to the integral of the (N-1) Hausdorff measure of the boundaries of the level sets. Let us compare the behaviour of the minimizing total variation flow with respect to the mean curvature motion flow. The viscosity solution formulation on the classical mean curvature motion has to be interpreted as follows. If $S(t)$ is a surface moving by mean curvature with initial condition $S(0)$, and $u(0,x)$ is the signed distance to $S(0)$, i.e., if $u(0,x) = d(x, S(0))$ when x is outside $S(0)$, and $u(0,x) = -d(x, S(0))$ if x is inside $S(0)$, then $S(t) = \{x : u(t,x) = 0\}$ for any $t \geq 0$, where $u(t,x)$ is the viscosity solution of (11). This is the level set formulation of the classical motion by mean curvature, initially proposed by S. Osher and J. Sethian in [21] and whose mathematical analysis was given in [14] and was followed by many other works. In particular, as it was shown by G. Barles, H.M. Soner and P. Souganidis [7], if instead of embedding $S(0)$ as the zero level set of a continuous function we just set $u(0,x) = \chi_{C(0)}$ where $C(0)$ is the region inside $S(0)$, and we assume that $S(0)$ is a smooth surface, then $u(t,x) = \chi_{C(t)}$ where $C(t)$ is the region inside $S(t)$. Thus, the mean curvature motion flow decreases the total variation of $\chi_{C(0)}$ by decreasing the $(N-1)$-Haussdorff measure of the boundary $S(t)$ of $C(t)$ [15]. Now, since the total variation of any function $u_0(x) = h\chi_C$ is

$$TV(h\chi_C) = hPer(C)$$

we see that two basic ways of minimizing the total variation of such a function are: either we decrease the heigth of $u_0(x)$ or we decrease the perimeter of its boundary. Our purpose was to explain which strategy was followed by the minimizing total variation flow. As we shall see below, under some geometric conditions for the sets $C(0)$, the strategy of the minimizing total variation flow consists in decreasing the heigth of the function without distortion of its boundary, while a distortion of the boundary will occur when these conditions are not satisfied, in particular, this will happen at points with a strong curvature. Thus the strategy followed by the minimizing total variation flow, compared to the one followed by the mean curvature motion is quite different. This gives an idea of the behavior of (5), at least what are the infinitesimal effects of (5) on the initial datum $u(0,x)$. The methods and results obtained can also be used to produce particular explicit solutions of the denoising problem which correspons to the kernel K in (7) being the identity, i.e., $K = I$.

2 Functions of Bounded Variation

Due to the linear growth condition on the Lagrangians associated with the total variation, the natural energy space to study them is the space of functions of bounded variation. In this section we collect some basic results of the theory of functions of bounded variation. For more information we refer the reader to [2], [13], [17], [24].

2.1 Definitions

Throughout this section, Ω denotes an open subset of $I\!R^N$.

Definition 1 A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The vector space of functions of bounded variation in Ω is denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and there are Radon measures $\mu_1, ..., \mu_N$ with finite total mass in Ω such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi d\mu_i \quad \forall \varphi \in C_0^\infty(\Omega),\ i = 1, ..., N.$$

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If $u \in BV(\Omega)$, the total variation of the measure Du is

$$\|Du\| = \sup \left\{ \int_\Omega u \,\mathrm{div}(\phi)\, dx : \ \phi \in C_0^\infty(\Omega, I\!R^N),\ |\phi(x)| \le 1 \text{ for } x \in \Omega \right\}.$$

The space $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV} = \|u\|_1 + \|Du\|,$$

is a Banach space. If $u \in BV(\Omega)$, the total variation $\|Du\|$ may be regarded as a measure, whose value on an open set $U \subseteq \Omega$ is

$$\|Du\|(U) = \sup \left\{ \int_U u \,\mathrm{div}(\phi)\, dx : \ \phi \in C_0^\infty(U, I\!R^N),\ |\phi(x)| \le 1 \text{ for } x \in U \right\}.$$

We also use

$$\int_U \|Du\|$$

to denote $\|Du\|(U)$.

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts

$$Du = D^a u + D^s u.$$

Then $D^a u = \nabla u \, \mathcal{L}^N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. There is also the polar decomposition $D^s u = \overrightarrow{D^s u}|D^s u|$ where $|D^s u|$ is the total variation measure of $D^s u$.

The total variation is lower semi-continuous. More concretely, we have the following result.

Theorem 1 *Suppose that $u_i \in BV(\Omega)$, $i = 1, 2, ...,$ and $u_i \to u$ in $L^1_{loc}(\Omega)$. Then*

$$\|Du\|(\Omega) \leq \liminf_{i \to \infty} \|Du_i\|(\Omega).$$

We say that $u \in L^1_{loc}(\Omega)$ is *locally of bounded variation* if $\varphi u \in BV(\Omega)$ for any $\varphi \in C_0^\infty(\Omega)$. We denote by $BV_{loc}(\Omega)$ the space of functions which are locally of bounded variation.

Here and in what follows we shall denote by $\mathcal{H}^\alpha$ the Hausdorff measure of dimension α in $I\!R^N$. In particular, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $\mathcal{H}^N$, the N-dimensional Hausdorff measure, coincides with the (outer) Lebesgue measure in $I\!R^N$. Given any Borel set $B \subseteq I\!R^N$ with $\mathcal{H}^\alpha(B) < \infty$, we denote by $\mathcal{H}^\alpha \llcorner B$ the finite Borel measure $\chi_B \mathcal{H}^\alpha$, i.e. $\mathcal{H}^\alpha \llcorner B(C) = \mathcal{H}^\alpha(B \cap C)$ for any Borel set $C \subseteq I\!R^N$. We recall that

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k\left(B \cap B(x, r)\right)}{r^k} = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in I\!R^N \setminus B \tag{13}$$

holds whenever $B \subseteq I\!R^N$ is a Borel set with finite k-dimensional Hausdorff measure (see for instance §2.3 of [13]).

2.2 Approximation by Smooth Functions

Theorem 2 *Assume that $u \in BV(\Omega)$. There exists a sequence of functions $u_i \in C^\infty(\Omega) \cap BV(\Omega)$ such that*

(i) $u_i \to u$ in $L^1(\Omega)$;

(ii) $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$ as $i \to \infty$.

Moreover,

(iii) if $u \in BV(\Omega) \cap L^q(\Omega)$, $q < \infty$, we cand find the functions u_i such that $u_i \in L^q(\Omega)$ and $u_i \to u$ in $L^q(\Omega)$;

(iv) if $u \in BV(\Omega) \cap L^\infty(\Omega)$, we can find the u_i such that $\|u_i\|_\infty \leq \|u\|_\infty$ and $u_i \to u$ in $L^\infty(\Omega)$-weakly.*

Finally,

(v) if $\partial\Omega$ is Lipschitz continuous one can find the u_i such that

$$u_i|_{\partial\Omega} = u|_{\partial\Omega} \qquad \text{for all} \;\; i.$$

Theorem 3 *Assume that $u \in BV(\Omega)$. There exists a sequence of functions $u_i \in C^\infty(\Omega) \cap BV(\Omega)$ such that*

(i) $u_i \to u$ in $L^1(\Omega)$;

(ii) if $U \subset\subset \Omega$ is such that $\|Du\|(\partial U) = 0$, then

$$\lim_{i\to\infty} \|Du_i\|(U) = \|Du\|(U).$$

Moreover, if $u \in L^q(\Omega)$, $1 \le q < \infty$ or $u \in L^\infty(\Omega)$, one can find the u_i satisfying (iii) or (iv), respectively, of the above result.

Definition 2 Let $u_i, u \in BV(\Omega)$, $i = 1, 2, \ldots$. We say that u_i *strictly converges* to u in $BV(\Omega)$ if both conditions (i), (ii) of Theorem 2 hold.

Definition 3 Let $u_i, u \in BV(\Omega)$, $i = 1, 2, \ldots$. We say that u_i weakly* converges to u in $BV(\Omega)$ if $u_i \to u$ in $L^1_{loc}(\Omega)$ and Du_i weakly* converges to Du as measures in Ω.

Proposition 2 *If $u_i, u \in BV(\Omega)$. Then $u_i \to u$ weakly* in $BV(\Omega)$ if and only if $\{u_i\}$ is bounded in $BV(\Omega)$ and converges to u in $L^1_{loc}(\Omega)$. Moreover, if*

$$\|Du_i\|(\Omega) \to \|Du\|(\Omega) \quad \text{as } i \to \infty,$$

and we consider the measures

$$\mu_i(B) = \int_{B\cap\Omega} Du_i, \quad \mu(B) = \int_{B\cap\Omega} Du,$$

for all Borel set $B \subset I\!R^N$. Then $\mu_i \rightharpoonup \mu$ weakly* as (vector valued) Radon measures in $I\!R^N$.

Theorem 4 *If $(u_k) \subseteq BV(\Omega)$ strictly converges to u and $f : I\!R^N \to I\!R$ is continuous and 1-positively homogeneous, we have*

$$\lim_{k\to\infty} \int_\Omega \phi f\left(\frac{Du_k}{\|Du_k\|}\right) d\|Du_k\| = \int_\Omega \phi f\left(\frac{Du}{\|Du\|}\right) d\|Du\|$$

for any bounded continuous function $\phi : \Omega \to I\!R$. As a consequence

$$f\left(\frac{Du_k}{\|Du_k\|}\right)\|Du_k\| \quad \text{weakly}^* \text{ converge in } \;\Omega\; \text{to} \;\; f\left(\frac{Du}{\|Du\|}\right)\|Du\|.$$

In particular, $\|Du_k\| \to \|Du\|$ weakly* in Ω.

2.3 Traces and Extensions

Assume that Ω is open and bounded with $\partial\Omega$ Lipschitz. We observe that since $\partial\Omega$ is Lipschitz, the outer unit normal ν exists $\mathcal{H}^{N-1}$ a.e. on $\partial\Omega$.

Theorem 5 *Assume that Ω is open and bounded, with $\partial\Omega$ Lipschitz. There exists a bounded linear mapping*

$$T : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{N-1})$$

such that

$$\int_\Omega u \operatorname{div}(\varphi)\, dx = -\int_\Omega \varphi \cdot dDu + \int_{\partial\Omega} \varphi \cdot \nu Tu \, d\mathcal{H}^{N-1}$$

for all $u \in BV(\Omega)$ and $\varphi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Moreover, for any $u \in BV(\Omega)$ and for $\mathcal{H}^{N-1}$ a.e. $x \in \partial\Omega$, we have

$$\lim_{r\to 0^+} r^{-N} \int_{B(x,r)\cap\Omega} |u - Tu(x)|\, dy = 0.$$

Theorem 6 *Let Ω be an open bounded set, with $\partial\Omega$ Lipschitz. Then the trace operator $u \to Tu$ is continuous between $BV(\Omega)$, endowed with the topology induced by the strict convergence, and $L^1(\partial\Omega, \mathcal{H}^{N-1} \llcorner \partial\Omega)$.*

Theorem 7 *Assume that Ω is open and bounded, with $\partial\Omega$ Lipschitz. Let $u_1 \in BV(\Omega)$, $u_2 \in BV(\mathbb{R}^N \setminus \overline{\Omega})$. We define*

$$v(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega \\ \\ u_2(x) & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

Then $v \in BV(\mathbb{R}^N)$ and

$$\|Dv\|(\mathbb{R}^N) = \|Du_1\|(\Omega) + \|Du_2\|(\mathbb{R}^N \setminus \overline{\Omega}) + \int_{\partial\Omega} |Tu_1 - Tu_2| d\mathcal{H}^{N-1}.$$

In particular, if

$$Eu = \begin{cases} u(x) & \text{if } x \in \Omega \\ \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

then $Eu \in BV(\mathbb{R}^N)$ provided $u \in BV(\Omega)$.

2.4 Sets of Finite Perimeter and the Coarea Formula

Definition 4 An $\mathcal{L}^N$ measurable subset E of $I\!R^N$ has *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$. The perimeter of E in Ω is $P(E, \Omega) = \|D\chi_E\|(\Omega)$.

We shall denote the measure $\|D\chi_E\|$ by $\|\partial E\|$ and $P(E, I\!R^N)$ by $\operatorname{Per}(E)$.

Theorem 8 *Let E be a set of finite perimeter in Ω and let $D\chi_E = \nu_E \|D\chi_E\|$ be the polar decomposition of $D\chi_E$. Then the generalized Gauss-Green formula holds*

$$\int_E \operatorname{div}(\varphi)\, dx = -\int_\Omega \langle \nu_E, \varphi \rangle d\|D\chi_E\|$$

for all $\varphi \in C_0^1(\Omega, I\!R^N)$.

Theorem 9 *(***Coarea formula for BV-functions***)*
Let $u \in BV(\Omega)$. Then

(i) $E_{u,t} := \{x \in \Omega : u(x) > t\}$ *has finite perimeter for* $\mathcal{L}^1$ a.e. $t \in I\!R$ *and*

(ii) $\|Du\|(\Omega) = \displaystyle\int_{-\infty}^{\infty} P(E_{u,t}, \Omega) dt.$

(iii) Conversely, if $u \in L^1(\Omega)$ and

$$\int_{-\infty}^{\infty} P(E_{u,t}, \Omega) dt < \infty,$$

then $u \in BV(\Omega)$.

2.5 Isoperimetric Inequality

Theorem 10 *(***Sobolev inequality***)*
There exists a constant $C > 0$ such that

$$\|u\|_{L^{N/N-1}(I\!R^N)} \leq C\|Du\|(I\!R^N)$$

for all $u \in BV(I\!R^N)$.

If $u \in L^1(\Omega)$, the mean value of u in Ω is

$$u_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(x)\, dx.$$

Theorem 11 *(**Poincaré's inequality**)*
Let Ω be open and bounded with $\partial\Omega$ Lipschitz. Suppose that Ω is connected. Then

$$\int_{\Omega} |u - u_{\Omega}| \, dx \leq C\|Du\|(\Omega) \quad \forall u \in BV(\Omega)$$

for some constant C depending only on Ω.

Theorem 12 *Let $N > 1$. For any set E of finite perimeter in $\mathbb{R}^N$ either E or $\mathbb{R}^N \setminus E$ has finite Lebesgue measure and*

$$\min\left\{\mathcal{L}^N(E), \mathcal{L}^N(\mathbb{R}^N \setminus E)\right\} \leq C[\mathrm{Per}(E)]^{\frac{N}{N-1}}$$

for some dimensional constant C.

Theorem 13 *(**Embedding Theorem**)*
Let Ω be open and bounded, with $\partial\Omega$ Lipschitz. Then the embedding $BV(\Omega) \to L^{N/N-1}(\Omega)$ is continuous and $BV(\Omega) \to L^p(\Omega)$ is compact for all $1 \leq p < \frac{N}{N-1}$.

The continuity of the embedding of Theorem 13 and Theorem 11 imply the following *Sobolev-Poincaré inequality*

$$\|u - u_{\Omega}\|_p \leq C\|Du\|(\Omega) \quad \forall u \in BV(\Omega),\, 1 \leq p \leq \tfrac{N}{N-1} \tag{14}$$

for some constant C depending only on Ω.

3 Pairings Between Measures and Bounded Functions

In this section we give some of the main points of the results about pairing between measures and bounded functions given by G. Anzellotti in [6] (see also [18]).

3.1 Trace of the Normal Component of Certain Vector Fields

It is well known that summability conditions on the divergence of a vector field z in Ω yield trace properties for the normal component of z on $\partial\Omega$. In this section we define a function $[z,\nu] \in L^\infty(\partial\Omega)$ which is associated to any vector field $z \in L^\infty(\Omega, I\!R^N)$ such that $\mathrm{div}(z)$ is a bounded measure in Ω.

Let Ω be an open set in $I\!R^N$, $N \geq 2$, and $1 \leq p \leq N$, $\frac{N}{N-1} \leq q \leq \infty$. We shall consider the following spaces:

$$BV(\Omega)_q := BV(\Omega) \cap L^q(\Omega)$$

$$BV(\Omega)_c := BV(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$$

$$X(\Omega)_p := \{z \in L^\infty(\Omega, I\!R^N) \ : \ \mathrm{div}(z) \in L^p(\Omega)\}$$

$$X(\Omega)_\mu := \{z \in L^\infty(\Omega, I\!R^N) \ : \ \mathrm{div}(z) \text{ is a bounded measure in } \Omega\}.$$

In the next theorem we define a paring $\langle z, u\rangle_{\partial\Omega}$, for $z \in X(\Omega)_\mu$ and $u \in BV(\Omega)_c$. We need the following result, which can be easily obtained by the same technique that Gagliardo uses in [16] in proving his extension theorem $L^1(\partial\Omega) \to W^{1,1}(\Omega)$.

Lemma 1 *Let Ω be a bounded open set in $I\!R^N$ with Lipschitz boundary. Then, for any given function $u \in L^1(\partial\Omega)$ and for any given $\epsilon > 0$ there exists a function $w \in W^{1,1}(\Omega) \cap C(\Omega)$ such that*

$$w|_{\partial\Omega} = u$$

$$\int_\Omega |\nabla w|\, dx \leq \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \epsilon$$

$$w(x) = 0 \quad \text{if} \ \ \mathrm{dist}(x, \partial\Omega) > \epsilon.$$

Moreover, for any fixed $1 \leq q < \infty$, one can find the function w such that

$$\|w\|_q \leq \epsilon.$$

Finally, if one has also $u \in L^\infty(\partial\Omega)$, one can find w such that

$$\|w\|_\infty \leq \|u\|_\infty.$$

Theorem 14 *Assume that $\Omega \subset I\!\!R^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$. Denote by $\nu(x)$ the outward unit normal to $\partial\Omega$. Then there exists a bilinear map $\langle z, u\rangle_{\partial\Omega} : X(\Omega)_\mu \times BV(\Omega)_c \to I\!\!R$ such that*

$$\langle z, u\rangle_{\partial\Omega} = \int_{\partial\Omega} u(x)z(x)\cdot\nu(x)\, d\mathcal{H}^{N-1} \qquad \text{if} \quad z \in C^1(\Omega, I\!\!R^N) \tag{15}$$

$$|\langle z, u\rangle_{\partial\Omega}| \le \|z\|_\infty \int_{\partial\Omega} |u(x)|\, d\mathcal{H}^{N-1} \qquad \text{for all} \quad z, u. \tag{16}$$

Proof. For $u \in BV(\Omega)_c \cap W^{1,1}(\Omega)$ and $z \in X(\Omega)_\mu$, we define

$$\langle z, u\rangle_{\partial\Omega} := \int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega z\cdot\nabla u\, dx.$$

We remark that if $u, v \in BV(\Omega)_c \cap W^{1,1}(\Omega)$ and $u = v$ on $\partial\Omega$ then one has

$$\langle z, u\rangle_{\partial\Omega} = \langle z, v\rangle_{\partial\Omega} \qquad \text{for all} \quad z \in X(\Omega)_\mu.$$

In fact, by standard techniques in Sobolev spaces theory, we can find a sequence of functions $g_i \in \mathcal{D}(\Omega)$ such that, for all $z \in X(\Omega)_\mu$, one has

$$\langle z, u - v\rangle_{\partial\Omega} = \int_\Omega (u - v)\operatorname{div}(z)\, dx + \int_\Omega z\cdot\nabla(u - v)\, dx$$

$$= \lim_{i\to\infty}\left(\int_\Omega g_i \operatorname{div}(z)\, dx + \int_\Omega z\cdot\nabla g_i\, dx\right) = 0.$$

Now, we define $\langle z, u\rangle_{\partial\Omega}$ for all $u \in BV(\Omega)_c$ by setting

$$\langle z, u\rangle_{\partial\Omega} = \langle z, w\rangle_{\partial\Omega},$$

where w is any function in $BV(\Omega)_c \cap W^{1.1}(\Omega)$ such that $u = w$ on $\partial\Omega$. This is a valid definition, in view of the preceding remark and because of the Lemma 1.

To prove (16), we take a sequence $u_n \in BV(\Omega)_c \cap C^\infty(\Omega)$ converging to u as in Theorem 2 and we get

$$|\langle z, u\rangle_{\partial\Omega}| = |\langle z, u_n\rangle_{\partial\Omega}| \le \left|\int_\Omega u_n \operatorname{div}(z)\, dx\right| + \|z\|_\infty \int_\Omega |\nabla u_n|\, dx$$

for all z and for all n. Hence, taking limit when $n \to \infty$ we have

$$|\langle z, u\rangle_{\partial\Omega}| \le \left|\int_\Omega u \operatorname{div}(z)\, dx\right| + \|z\|_\infty \int_\Omega \|Du\|.$$

Now, for a fixed $\epsilon > 0$ we consider a function w as in Lemma 1. Then

$$|\langle z, u\rangle_{\partial\Omega}| = |\langle z, w\rangle_{\partial\Omega}| \leq \|w\|_\infty \int_{\Omega\setminus\Omega_\epsilon} |\mathrm{div}(z)| + \|z\|_\infty \Big(\int_{\partial\Omega} |u|\, dx + \epsilon \Big),$$

where $\Omega_\epsilon = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \epsilon\}$. Since $\mathrm{div}(z)$ is a measure of bounded total variation in Ω,

$$\lim_{\epsilon\to 0^+} \int_{\Omega\setminus\Omega_\epsilon} |\mathrm{div}(z)|\, dx = 0.$$

Consequently, (16) holds. $\square$

Theorem 15 *Let Ω be as in Theorem 14. Then there exists a linear operator $\gamma : X(\Omega)_\mu \to L^\infty(\partial\Omega)$ such that*

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty \tag{17}$$

$$\langle z, u\rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma(z)(x)u(x)\, d\mathcal{H}^{N-1} \quad \text{for all} \ \ u \in BV(\Omega)_c \tag{18}$$

$$\gamma(z)(x) = z(x)\cdot\nu(x) \qquad \text{for all} \ \ x \in \partial\Omega \ \ \text{if} \ \ z \in C^1(\overline{\Omega}, I\!R^N). \tag{19}$$

The function $\gamma(z)$ is a weakly defined trace on $\partial\Omega$ of the normal component of z. We shall denote $\gamma(z)$ by $[z, \nu]$.

Proof. Take a fix $z \in X(\Omega)_\mu$. Consider the functional $F : L^\infty(\partial\Omega) \to I\!R$ defined by

$$F(u) := \langle z, w\rangle_{\partial\Omega},$$

where $w \in BV(\Omega)_c$ is such that $w|_{\partial\Omega} = u$. By estimate (16),

$$|F(u)| \leq \|z\|_\infty \, \|u\|_1.$$

Hence there exists a function $\gamma(z) \in L^\infty(\partial\Omega)$ such that

$$F(u) = \int_{\partial\Omega} \gamma(z)(x)u(x)\, d\mathcal{H}^{N-1}$$

and the result follows. $\square$

Obviously, $X(\Omega)_p \subset X(\Omega)_\mu$ for all $p \geq 1$ and the trace $[z, \nu]$ is defined for all $z \in X(\Omega)_p$.

3.2 The Measure (z, Du)

Approximating by smooth functions and applying Green's formula, the following result can be deduced easily.

Proposition 3 *Let Ω be as in Theorem 14 and $1 \leq p \leq \infty$. Then, for all $z \in X(\Omega)_p$ and $u \in W^{1,1}(\Omega) \cap L^{p'}(\Omega)$, one has*

$$\int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega z \cdot \nabla u\, dx = \int_{\partial\Omega} [z, \nu] u\, d\mathcal{H}^{N-1}. \tag{20}$$

In the sequel we shall consider pairs (z, u) such that one of the following conditions holds

$$\begin{cases} a)\ u \in BV(\Omega)_{p'},\ z \in X(\Omega)_p \quad \text{and} \quad 1 < p \leq N; \\ \\ b)\ u \in BV(\Omega)_\infty,\ z \in X(\Omega)_1; \\ \\ c)\ u \in BV(\Omega)_c,\ z \in X(\Omega)_\mu. \end{cases} \tag{21}$$

Definition 5 Let z, u be such that one of the conditions (21) holds. Then we define a functional $(z, Du) : \mathcal{D}(\Omega) \to I\!R$ as

$$\langle (z, Du), \varphi \rangle := -\int_\Omega u\varphi \operatorname{div}(z)\, dx - \int_\Omega uz \cdot \nabla\varphi\, dx.$$

Theorem 16 *For all open set $U \subset \Omega$ and for all function $\varphi \in \mathcal{D}(U)$, one has*

$$|\langle (z, Du), \varphi \rangle| \leq \|\varphi\|_\infty \|z\|_{L^\infty(U)} \int_U \|Du\|, \tag{22}$$

hence (z, Du) is a Radon measure in Ω.

Proof. Take a sequence $u_n \in C^\infty(\Omega)$ converging to u as in Theorem 3. Take $\varphi \in \mathcal{D}(U)$ and consider an open set V such that $\operatorname{supp}(\varphi) \subset V \subset\subset U$. Then

$$|\langle (z, Du_n), \varphi \rangle| \leq \|\varphi\|_\infty \|z\|_{L^\infty(U)} \int_V \|Du_n\| \quad \text{for all} \quad n \in I\!N.$$

From here, taking limit as $n \to \infty$, the result follows. □

We shall denote by $|(z, Du)|$ the measure total variation of (z, Du) and by $\int_B |(z, Du)|$, $\int_B (z, Du)$ the values of these measures on every Borel set $B \subset \Omega$.

As a consequence of the above theorem, the following result holds.

Corollary 1 *The measures (z, Du), $|(z, Du)|$ are absolutely continuous with respect to the measure $\|Du\|$ and*

$$\left|\int_B (z, Du)\right| \leq \int_B |(z, Du)| \leq \|z\|_{L^\infty(U)} \int_B \|Du\|$$

for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$. Moreover, by the Radon-Nikodym Theorem, there exists a $\|Du\|$-measurable function

$$\theta(z, Du, \cdot) : \Omega \to I\!R$$

such that

$$\int_B (z, Du) = \int_B \theta(z, Du, x)\, \|Du\| \qquad \text{for all Borel sets} \quad B \subset \Omega$$

and

$$\|\theta(z, Du, \cdot)\|_{L^\infty(\Omega, \|Du\|)} \leq \|z\|_\infty.$$

Assume u, z satisfy one of the conditions (21). By writing

$$z \cdot D^s u := (z, Du) - (z \cdot \nabla u)\, d\mathcal{L}^N,$$

we have that $z \cdot D^s u$ is a bounded measure. Furthermore, with an approximation argument to the one used in the proof of Theorem 16, we have that $z \cdot D^s u$ is absolutely continuous with respect to $\|D^s u\|$ (and, thus, it is a singular measure respect to $\mathcal{L}^N$), and

$$|z \cdot D^s u| \leq \|z\|_\infty |D^s u|. \tag{23}$$

Lemma 2 *Assume u, z satisfy one of the conditions (21). Let $u_n \in C^\infty(\Omega) \cap BV(\Omega)$ converging to u as in Theorem 2. Then we have*

$$\int_\Omega z \cdot \nabla u_n \, dx \to \int_\Omega (z, Du).$$

Proof. For a given $\epsilon > 0$, we take an open set $U \subset\subset \Omega$ such that

$$\int_{\Omega \setminus U} \|Du\| < \epsilon.$$

Let $\varphi \in \mathcal{D}(\Omega)$ be such that $\varphi(x) = 1$ in U and $0 \leq \varphi \leq 1$ in Ω. Then

$$\left|\int_\Omega (z, Du_n) - \int_\Omega (z, Du)\right| \leq$$

$$|\langle (z, Du_n), \varphi\rangle - \langle (z, Du), \varphi\rangle| + \int_\Omega |(z, Du_n)|(1 - \varphi) + \int_\Omega |(z, Du)|(1 - \varphi).$$

Since

$$\lim_{n\to\infty}\langle (z,Du_n),\varphi\rangle = \langle (z,Du),\varphi\rangle,$$

$$\limsup_{n\to\infty}\int_\Omega |(z,Du_n)|(1-\varphi) \le \|z\|_\infty \limsup_{n\to\infty}\int_{\Omega\setminus U}\|Du_n\| < \epsilon\,\|z\|_\infty,$$

$$\int_\Omega |(z,Du)|(1-\varphi) \le \epsilon\,\|z\|_\infty$$

and ϵ is arbitrary, the lemma follows. □

We give now the expected *Green's formula* relating the function $[z,\nu]$ and the measure (z,Du).

Theorem 17 *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let z, u be such that one of the conditions (21) holds, then we have*

$$\int_\Omega u\,\mathrm{div}(z)\,dx + \int_\Omega (z,Du) = \int_{\partial\Omega}[z,\nu]u\,d\mathcal{H}^{N-1}. \tag{24}$$

Proof. We assume that (21) (a) holds, in the general case an extension of Proposition 3 is nedeeded. Take a sequence of functions $u_n \in C^\infty(\Omega)\cap BV(\Omega)$ converging to u as in Theorem 2. Then, by Lemma 2 and Proposition 3, we have

$$\int_\Omega u\,\mathrm{div}(z)\,dx + \int_\Omega (z,Du) = \lim_{n\to\infty}\left(\int_\Omega u_n\,\mathrm{div}(z)\,dx + \int_\Omega z\cdot\nabla u_n\,dx\right)$$

$$= \lim_{n\to\infty}\int_{\partial\Omega}[z,\nu]u_n\,d\mathcal{H}^{N-1} = \int_{\partial\Omega}[z,\nu]u\,d\mathcal{H}^{N-1}.$$

□

Remark 1 Observe that with a similar proof to the one the above theorem, in the case $\Omega = \mathbb{R}^N$, the following integration by parts formula, for z and w satisfying one of the conditions (21), holds:

$$\int_{\mathbb{R}^N} w\ \mathrm{div}(z)\ dx + \int_{\mathbb{R}^N}(z,Dw) = 0. \tag{25}$$

In particular, if Ω is bounded and has finite perimeter in $\mathbb{R}^N$, from (25) it follows

$$\int_\Omega \mathrm{div}(z)\,dx = \int_{\mathbb{R}^N}(z,-D\chi_\Omega) = \int_{\partial^*\Omega}\theta(z,-D\chi_\Omega,x)\ d\mathcal{H}^{N-1}. \tag{26}$$

Notice also that as a consequence of Corollary 1, if $z_1, z_2 \in X(\mathbb{R}^N)_p$ and $z_1 = z_2$ almost everywhere on Ω, then $\theta(z_1,-D\chi_\Omega,x) = \theta(z_2,-D\chi_\Omega,x)$ for $\mathcal{H}^{N-1}$-almost every $x \in \partial^*\Omega$.

If Ω is a bounded open set with Lipschitz boundary, then (26) has a meaning also if z is defined only on Ω and not on the whole of $\mathbb{R}^N$, precisely when $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div}(z) \in L^N(\Omega)$. In this case we mean that $\theta(z, -D\chi_\Omega, \cdot)$ coincides with $[z, \nu]$.

Remark 2 Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, and let $z_{\text{inn}} \in L^\infty(\Omega; \mathbb{R}^2)$ with $\operatorname{div}(z_{\text{inn}}) \in L^2_{\text{loc}}(\Omega)$, and $z_{\text{out}} \in L^\infty(\mathbb{R}^2 \setminus \overline{\Omega}; \mathbb{R}^2)$ with $\operatorname{div}(z_{\text{out}}) \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$. Assume that

$$\theta(z_{\text{inn}}, -D\chi_\Omega, x) = -\theta(z_{\text{out}}, -D\chi_{\mathbb{R}^2 \setminus \overline{\Omega}}, x) \qquad \text{for } \mathcal{H}^1 - \text{a.e } x \in \partial\Omega.$$

Then if we define $z := z_{\text{inn}}$ on Ω and $z := z_{\text{out}}$ on $\mathbb{R}^2 \setminus \overline{\Omega}$, we have $z \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\operatorname{div}(z) \in L^2_{\text{loc}}(\mathbb{R}^2)$.

3.3 Representation of the Radon-Nikodym Derivative $\theta(z, Du, \cdot)$

This section is devoted to the problem of whether or not one can write

$$\theta(z, Du, x) = z(x) \cdot \frac{Du}{\|Du\|}(x) \tag{27}$$

where $\frac{Du}{\|Du\|}$ is the density function of the measure Du with respect to the measure $\|Du\|$.

For the sake of simplicity, we shall assume throughout this section that $z \in X(\Omega)_N$ and $u \in BV(\Omega)$, but it is clear that analogous results can be obtained for pairs (z, u) satisfying any of the conditions (21). First we have the following continuity result.

Proposition 4 *Assume that*

$$z_n \rightharpoonup z \qquad \text{in} - \quad L^\infty(U) - \text{weak}^* \tag{28}$$

$$\operatorname{div}(z_n) \rightharpoonup \operatorname{div}(z) \qquad \text{in} \quad L^N(U) - \text{weak} \tag{29}$$

for all open set $U \subset\subset \Omega$. Then, for all $u \in BV(\Omega)$, we have

$$(z_n, Du) \to (z, Du) \qquad \text{as measures in} \quad \Omega \tag{30}$$

and

$$\theta(z_n, Du, \cdot) \rightharpoonup \theta(z, Du, \cdot) \quad \text{in} \quad L^\infty(U) - \text{weak}^* \text{ for all } \ U \subset\subset \Omega. \tag{31}$$

Proof. By (28), for all $U \subset\subset \Omega$

$$\sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(U)} = c(U) < +\infty.$$

Moreover,

$$\int_U |(z_n, Du)| \le \|z_n\|_{L^\infty(U)} \int_U \|Du\|.$$

Hence, it is sufficient to check the weak convergence (30) on $\mathcal{D}(\Omega)$ functions. Now, if $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle (z_n, Du), \varphi\rangle = -\int_\Omega u\varphi \operatorname{div}(z_n)\, dx - \int_\Omega u z_n \cdot \nabla\varphi\, dx \to \langle (z, Du), \varphi\rangle$$

and (30) is proved.

By Corollary 1, we have

$$\|\theta(z_n, Du, \cdot)\|_{L^\infty(U,\|Du\|)} \le \|z_n\|_{L^\infty(U)} \le c(U).$$

Hence the convergence (31) has to be checked only on $C_c(\Omega)$ functions, now this is a consequence of (30). □

Using mollifiers it is easy to get the following result.

Lemma 3 *For every function $z \in X(\Omega)_N$, there exists a sequence of functions $z_n \in C^\infty(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ such that*

$$\|z_n\|_\infty \le \|z\|_\infty \qquad \text{for all } n \in \mathbb{N},$$

$$z_n \rightharpoonup z \quad \text{in } L^\infty(\Omega, \mathbb{R}^N) - \text{weak}^* \text{ and in } L^P_{loc}(\Omega, \mathbb{R}^N) \text{ for } 1 \le p < \infty,$$

$$z_n(x) \to z(x) \quad \text{at every Lebesgue point } x \text{ of } z, \text{and uniformly in sets of uniformly continuity for } z,$$

$$\operatorname{div}(z_n) \to \operatorname{div}(z) \quad \text{in } L^N_{loc}(\Omega).$$

Now we give the representation result for $\theta(z, Du, \cdot)$.

Theorem 18 *Assume that $z \in X(\Omega)_N$ and $u \in BV(\Omega)$. Then, we have*

$$\theta(z, Du, x) = z(x) \cdot \frac{Du}{\|Du\|}(x), \qquad \|D^a u\| - \text{a.e. in } \Omega. \tag{32}$$

Moreover, if $z \in C(\Omega, \mathbb{R}^N)$, we have

$$\theta(z, Du, x) = z(x) \cdot \frac{Du}{\|Du\|}(x), \qquad \|Du\| - \text{a.e. in } \Omega, \tag{33}$$

and consequently,

$$z \cdot D^s u = (z \cdot \overrightarrow{D^s u})\, |D^s u|. \tag{34}$$

Proof. Suppose first that $z \in C(\Omega, I\!\!R^N)$. (33) is equivalent to

$$\langle (z, Du), \varphi \rangle = \int_\Omega \varphi z \, Du \qquad \forall \, \varphi \in \mathcal{D}(\Omega). \tag{35}$$

Now, (35) is true by definition if $z \in C^1(\Omega, I\!\!R^N)$. If $z \in C(\Omega, I\!\!R^N)$, we take a sequence z_n as in Lemma 3, and by Proposition 4, for any $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle (z, Du), \varphi \rangle = \lim_{n \to \infty} \langle (z_n, Du), \varphi \rangle = \lim_{n \to \infty} \int_\Omega \varphi z_n \, Du = \int_\Omega \varphi z \, Du,$$

where, in the last step, we have used the fact that z_n converges uniformly to z on $\mathrm{supp}(\varphi)$.

Let us see now (32). This equality is equivalent to

$$\int_B \theta(z, Du, x) |\nabla u(x)| \, dx = \int_B z(x) \cdot \nabla u(x) \, dx \tag{36}$$

for all Borel set $B \subset \Omega$. Let E^a and E^s be two disjoint Borel sets such that $E^a \cup E^s = \Omega$ and

$$\int_{E^s} \|D^a u\| = \int_{E^a} \|D^s u\| = 0.$$

Let $\epsilon > 0$ fixed. Then, there exists a compact set $K \subset E^s$ such that

$$\int_{E^s \setminus K} \|D^s u\| < \epsilon. \tag{37}$$

Given a compact set $B_0 \subset E^a$, we can find an open set U with regular boundary, such that

$$B_0 \subset U \subset \Omega \setminus K, \qquad \int_{U \setminus B_0} \|Du\| < \epsilon$$

and by (37) it follows that

$$\int_U \|D^s u\| < \epsilon.$$

Take now a sequence $u_n \in C^\infty(U) \cap BV(U)$ approximating u as in Theorem 2. Then, by Lemma 2, it follows that

$$\left| \int_U \theta(z, Du, x) \, Du - \int_U z(x) \cdot \nabla u(x) \, dx \right|$$

$$= \lim_{n \to \infty} \left| \int_U z(x) \cdot \nabla u_n(x) \, dx - \int_U z(x) \cdot \nabla u(x) \, dx \right|$$

$$\leq \|z\|_\infty \lim_{n\to\infty} \int_U |\nabla u_n(x) - \nabla u(x)|\, dx \leq \|z\|_\infty \int_U \|D^s u\| \leq \epsilon \|z\|_\infty.$$

On the other hand, we have

$$\left| \int_U z(x)\cdot \nabla u(x)\, dx - \int_{B_0} z(x)\cdot \nabla u(x)\, dx \right| \leq \|z\|_\infty \int_{U\setminus B_0} \|Du\| \leq \epsilon \|z\|_\infty$$

and by Corollary 1, we also have

$$\left| \int_U \theta(z,Du,x)\, \|Du\| - \int_{B_0} \theta(z,Du,x)\, \|Du\| \right| \leq \|z\|_\infty \int_{U\setminus B_0} \|Du\| \leq \epsilon \|z\|_\infty.$$

Therefore, we obtain that

$$\left| \int_{B_0} \theta(z,Du,x)\, \|Du\| - \int_{B_0} z(x)\cdot \nabla u(x)\, dx \right| \leq 3\epsilon \|z\|_\infty.$$

Hence (36) is proved for all compact sets $B_0 \subset E^a$. From where it follows, having in mind the regularity of the Radon measures, that (36) holds for all Borel subset of Ω. □

For later use we recall that by the coarea formula (Theorem 9), if $u \in BV(\Omega)$ and $E_{u,t} := \{x \in \Omega : u(x) > t\}$, we have that

$$\frac{Du}{\|Du\|}(x) = \frac{D\chi_{E_{u,t}}}{\|D\chi_{E_{u,t}}\|}(x), \qquad \|D\chi_{E_{u,t}}\| - \text{a.e. in } \Omega$$

for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$.

In the next result we link the measure (z, Du) with the measure $(z, D\chi_{E_{u,t}})$.

Theorem 19 *If $z \in X(\Omega)_N$ and $u \in BV(\Omega)$, then we have:*

(i) for all function $\varphi \in C_c(\Omega)$, the function $t \mapsto \langle (z, D\chi_{E_{u,t}}), \varphi\rangle$ is $\mathcal{L}^1$-measurable and

$$\langle (z,Du), \varphi\rangle = \int_{-\infty}^{+\infty} \langle (z, D\chi_{E_{u,t}}), \varphi\rangle\, dt,$$

(ii) for all Borel set $B \subset \Omega$, the function $t \mapsto \int_B (z, D\chi_{E_{u,t}})$ is $\mathcal{L}^1$-measurable and

$$\int_B (z,Du) = \int_{-\infty}^{+\infty} \left(\int_B (z, D\chi_{E_{u,t}}) \right) dt,$$

(iii) $\theta(z,Du,x) = \theta(z, D\chi_{E_{u,t}}, x)$ $\|D\chi_{E_{u,t}}\|$-a.e. in Ω for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$.

Proof. (i) Take a sequence $z_n \in C^\infty(\Omega, I\!R^N) \cap L^\infty(\Omega, I\!R^N)$ converging to z as in Lemma 3. By the coarea formula we have

$$\langle (z_n, Du), \varphi \rangle = \int_\Omega z_n(x) \cdot \frac{Du}{\|Du\|}(x) \varphi(x) \, \|Du\|$$

$$= \int_{-\infty}^{+\infty} \left(\int_\Omega z_n(x) \cdot \frac{D\chi_{E_{u,t}}}{\|D\chi_{E_{u,t}}\|}(x) \varphi(x) \, \|D\chi_{E_{u,t}}\| \right) dt \tag{38}$$

$$= \int_{-\infty}^{+\infty} \langle (z_n, D\chi_{E_{u,t}}), \varphi \rangle \, dt.$$

Since

$$|\langle (z_n, D\chi_{E_{u,t}}), \varphi \rangle| \leq \|z\|_\infty \|\varphi\|_\infty \int_\Omega \|D\chi_{E_{u,t}}\|, \qquad \forall \, n \in I\!N,$$

having in mind Proposition 4, by the Dominated Convergence Theorem, taking limit in (38) we get (i).

We shall prove (ii) after (iii). Let us prove (iii). For $a, b \in I\!R$, $a < b$, let $v = T_{a,b}(u)$ be. Then,

$$D\chi_{E_{u,t}} = D\chi_{E_{v,t}} \quad \text{and} \quad \frac{D\chi_{E_{u,t}}}{\|D\chi_{E_{u,t}}\|} = \frac{D\chi_{E_{v,t}}}{\|D\chi_{E_{v,t}}\|}, \quad \text{if} \quad a \leq t < b$$

and

$$D\chi_{E_{v,t}} = 0 \qquad \text{if} \quad t \geq b \quad \text{or} \quad t < a,$$

from where it follows that

$$\frac{Du}{\|Du\|}(x) = \frac{D\chi_{E_{u,t}}}{\|D\chi_{E_{u,t}}\|}(x) = \frac{D\chi_{E_{v,t}}}{\|D\chi_{E_{v,t}}\|}(x) = \frac{Dv}{\|Dv\|}(x),$$

$\|D\chi_{E_{v,t}}\|$-a.e in Ω for $\mathcal{L}^1$-almost all $t \in I\!R$. Hence,

$$\frac{Du}{\|Du\|}(x) = \frac{Dv}{\|Dv\|}(x), \qquad \|Dv\| - \text{a.e in } \Omega.$$

From here, taking again the sequence z_n of the first part, we get

$$\theta(z_n, Du, x) = z_n(x) \cdot \frac{Du}{\|Du\|}(x) = \theta(z_n Dv, x), \quad \|Dv\| - \text{a.e. in } \Omega, \ \forall \, n \in I\!N.$$

Then taking limit as $n \to \infty$, by the uniqueness of the limit in the $L^\infty(\Omega, \|Dv\|)$-weak* topology, we get

$$\theta(z, Du, x) = \theta(z, Dv, x), \qquad \|Dv\| - \text{a.e. in } \Omega. \tag{39}$$

Now, using statement (i) for v, we have, for a fixed $\varphi \in \mathcal{D}(\Omega)$,

$$\langle (z, Dv), \varphi \rangle = \int_{-\infty}^{+\infty} \langle (z, D\chi_{E_{v,t}}), \varphi \rangle \, dt.$$

From here, using (39) and the coarea formula, we obtain that

$$\int_a^b \left(\int_\Omega \theta(z, Du, x)\varphi(x) \, \|D\chi_{E_{v,t}}\| \right) dt$$
$$= \int_a^b \left(\int_\Omega \theta(z, D\chi_{E_{v,t}}, x)\varphi(x) \, \|D\chi_{E_{v,t}}\| \right) dt$$

and this implies that

$$\int_\Omega \theta(z, Du, x)\varphi(x) \, \|D\chi_{E_{v,t}}\| = \int_\Omega \theta(z, D\chi_{E_{v,t}}, x)\varphi(x) \, \|D\chi_{E_{v,t}}\|$$

for $\mathcal{L}^1$-almost all $t \in I\!R$. Then by a density argument, we finish the proof of (iii).

Finally, (ii) is a consequence of (iii) since

$$\int_b (z, du) = \int_B \theta(z, Du, x) \, \|Du\| = \int_{-\infty}^{+\infty} \left(\int_B \theta(z, Du, x) \, \|D\chi_{E_{u,t}}\| \right) dt$$
$$= \int_{-\infty}^{+\infty} \left(\int_B \theta(z, D\chi_{E_{u,t}}, x) \, \|D\chi_{E_{u,t}}\| \right) dt = \int_{-\infty}^{+\infty} \left(\int_B (z, D\chi_{E_{u,t}}) \right) dt.$$

□

Corollary 2 *Assume that $z \in X(\Omega)_N$ and $u \in BV(\Omega)$. If $f : I\!R \to I\!R$ is a Lipschitz continuous increasing function, then*

$$\theta(z, D(f \circ u), x) = \theta(z, Du, x), \qquad \|Du\| - \text{a.e. in } \Omega \tag{40}$$

Proof. Observe first that

$$E_{u,t} = \{x \in \Omega \ : \ u(x) > t\} = \{x \in \Omega \ : \ (f \circ u)(x) > f(t)\} = E_{f \circ u, f(t)}.$$

Hence, for almost all $t \in I\!R$, we have

$$D\chi_{E_{u,t}} = D\chi_{E_{f \circ u, f(t)}}.$$

Therefore,

$$\theta(z, Du, x) = \theta(z, D\chi_{E_{u,t}}, x) = \theta(z, D\chi_{E_{f \circ u, f(t)}}, x) = \theta(z, D(f \circ u), x),$$

$\|D\chi_{E_{u,t}}\|$-a.e. in Ω for $\mathcal{L}^1$-almost all $t \in I\!R$, and consequently (40) follows. □

4 Gradient Flows

One of the more important examples of maximal monotone operator in Hilbert spaces comes from the optimization theory, they are the subdifferentials of convex functions which we introduce next.

Hereafter H will denote a real Hilbert space, with inner product $(/)$ and norm $\| \|$.

4.1 Convex functions in Hilbert spaces

A function $\varphi : H \to]-\infty, +\infty]$ is *convex* provided

$$\varphi(\alpha u + (1-\alpha)v) \leq \alpha\varphi(u) + (1-\alpha)\varphi(v)$$

for all $\alpha \in [0,1]$ and $u, v \in H$.

We denote

$$D(\varphi) = \{u \in H \ : \ \varphi(u) \neq +\infty\} \qquad \text{(effective domain)}.$$

We say that φ is *proper* if $D(\varphi) \neq \emptyset$.

We say φ is *lower semi-continuous* (l.s.c) if $u_n \to u$ in H implies $\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n)$.

Some of the properties of φ are reflected in its epigraph:

$$\text{epi}(\varphi) := \{(u,r) \in H \times I\!R \ : \ r \geq \varphi(u)\}.$$

For instance, it is easy to see that φ is convex if and only if $\text{epi}(\varphi)$ is a convex subset of H; and φ is lower semi-continuous if and only if $\text{epi}(\varphi)$ is closed.

The *subdifferential* $\partial\varphi$ of φ is the operator defined by

$$w \in \partial\varphi(z) \iff \varphi(u) \geq \varphi(z) + (w/u - z) \qquad \forall\ u \in H.$$

We say $u \in D(\partial\varphi)$, the domain of $\partial\varphi$, provided $\partial\varphi(u) \neq \emptyset$.

Observe that $0 \in \partial\varphi(z) \Longleftrightarrow \varphi(u) \geq \varphi(z) \ \ \forall\ u \in H \Longleftrightarrow$

$$\varphi(z) = \min_{u \in D(\varphi)} \varphi(u).$$

Therefore, we have that $0 \in \partial\varphi(z)$ is the *Euler equation* of the variational problem

$$\varphi(z) = \min_{u \in D(\varphi)} \varphi(u).$$

If $(z,w),(\hat{z},\hat{w}) \in \partial\varphi$, then $\varphi(z) \geq \varphi(\hat{z}) + (\hat{w}/z - \hat{z})$ and $\varphi(\hat{z}) \geq \varphi(z) + (w/\hat{z} - z)$. Adding this inequalities we get

$$(w - \hat{w}/z - \hat{z}) \geq 0.$$

Thus, $\partial\varphi$ is a monotone operator.

Next we will discuss the relation between subdifferentials, directional derivatives and the Gâteaux derivative. Let $\varphi : H \to]-\infty, +\infty]$. The *directional derivative* $D_v\varphi(u)$ of φ at the point $u \in D(\varphi)$ in the direction $v \in H$ is defined by

$$D_v\varphi(u) = \lim_{\lambda \downarrow 0} \frac{\varphi(u+\lambda v) - \varphi(u)}{\lambda}$$

whenever the limit exists. If there exists $w \in H$ such that $D_v\varphi(u) = (v/w)$ for all $v \in H$, then φ is called *Gâteaux differentiable* at u, and w is called the *Gâteaux derivative* of φ at u, which will be denoted by $\varphi'(u)$.

Proposition 5 *Let $\varphi : H \to]-\infty, +\infty]$ be convex and proper. If φ is Gâteaux differentiable at u, then*

$$\partial\varphi(u) = \{\varphi'(u)\}.$$

Proof. Given $w \in H$, since φ is convex, we have

$$(\varphi'(u)/w - u) = D_{w-u}\varphi(u) = \lim_{\lambda \downarrow 0} \frac{\varphi(u + \lambda(w-u)) - \varphi(u)}{\lambda}$$

$$= \lim_{\lambda \downarrow 0} \frac{\varphi(\lambda w + (1-\lambda)u) - \varphi(u)}{\lambda} \leq \varphi(w) - \varphi(u).$$

Hence, $\varphi'(u) \in \partial\varphi(u)$.

On the other hand, if $v \in \partial\varphi(u)$, given $w \in H$ and $\lambda > 0$, we have

$$\frac{\varphi(u+\lambda w) - \varphi(u)}{\lambda} \geq \frac{1}{\lambda}(v/u + \lambda w - u) = (v/w),$$

from where it follows that

$$D_w\varphi(u) \geq (v/w) \quad \forall\, w \in H.$$

Moreover, taking $w = -w$, we have

$$-D_{-w}\varphi(u) \leq (v/w) \leq D_w\varphi(u).$$

Therefore, since φ is Gâteaux differentiable at u, we get

$$(\varphi'(u)/w) = -(\varphi'(u)/-w) \leq (v/w) \leq (\varphi'(u)/w) \quad \forall\, w \in H,$$

and consequently, $v = \varphi'(u)$. □

Remark 3 In the case φ is continuous at u, also the reciprocal is true (see [12]). That is, in this case we have

$$\varphi \text{ is Gateaux differentiable at } u \Leftrightarrow \ \partial\varphi(u) = \{v\},$$

and in this case $v = \varphi'(u)$.

Example 1 It is easy to see that if $\varphi : I\!\!R^N \to I\!\!R$ is defined by $\varphi(x) := \|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x = (x_1, \ldots, x_n) \in I\!\!R^N$, then

$$\partial\varphi(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \ x \neq 0 \\ \\ \overline{B_1(0)} & \text{if } \ x = 0. \end{cases}$$

Example 2 Let $\Omega \subset I\!\!R^N$ an open bounded set with smooth boundary. Consider the function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ defined by

$$\varphi(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 & \text{if } \ u \in W_0^{1,2}(\Omega) \\ \\ +\infty & u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

Then, it is well known (see for instance [8]) that

$$D(\partial\varphi) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

and

$$v \in \partial\varphi(u) \Leftrightarrow v = -\Delta u.$$

Hence, the following are equivalent:

(i) u is a solution of the variational problem

$$\varphi(u) = \min_{w \in L^2(\Omega)} \varphi(w).$$

(ii) u is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \ \Omega \\ \\ u = 0 & \text{on } \ \partial\Omega \end{cases}$$

Theorem 20 *Let $\varphi : H \to]-\infty, +\infty]$ be convex, proper and lower semi-continuous. Then, for each $w \in H$ and $\lambda > 0$, the problem*

$$u + \lambda\partial\varphi(u) \ni w$$

has a unique solution $u \in D(\partial\varphi)$.

Proof. Given $w \in H$ and $\lambda > 0$, consider the functional $J : H \to]-\infty, +\infty]$ defined by

$$J(u) := \frac{1}{2}\|u\|^2 + \lambda\varphi(u) - (u/w). \tag{41}$$

We intend to show that J attains its minimum over H. Let us first claim that J is weakly lower semi-continuous, that is,

$$u_n \rightharpoonup u \text{ weakly in } H \ \Rightarrow \ J(u) \leq \liminf_{n\to\infty} J(u_n). \tag{42}$$

Obviously, it is enough to show (42) for φ. Let u_{n_k} such that

$$l = \liminf_{n\to\infty} \varphi(u_n) = \lim_{k\to\infty} \varphi(u_{n_k}).$$

For each $\epsilon > 0$ the set $K_\epsilon := \{w \in H \ : \ \varphi(w) \leq l + \epsilon\}$ is closed and convex, and consequently is weakly closed. Since all but finitely many points $\{u_{n_k}\}$ lie in K_ϵ, $u \in K_\epsilon$, and consequently

$$\varphi(u) \leq l + \epsilon = \liminf_{n\to\infty} \varphi(u_n) + \epsilon.$$

Since the above inequality is true for all $\epsilon > 0$, (42) follows.

Next we assert that

$$\varphi(u) \geq -C - C\|u\| \qquad \forall\, u \in H \tag{43}$$

for some constant $C > 0$. To verify this claim we suppose to the contrary that for each $n \in \mathbb{N}$ there exists $u_n \in H$ such that

$$\varphi(u_n) \leq -n - n\|u_n\| \qquad \forall\, n \in \mathbb{N}. \tag{44}$$

If the sequence $\{u_n\}$ is bounded in H, there exists a weakly convergent subsequence $u_{n_k} \rightharpoonup u$. But then, (42) and (44) imply the contradiction $\varphi(u) = -\infty$. Thus we may assume, passing if necessary to a subsequence, that $\|u_n\| \to \infty$. Select $u_0 \in H$ so that $\varphi(u_0) < \infty$. Set

$$v_n := \frac{u_n}{\|u_n\|} + \left(1 - \frac{1}{\|u_n\|}\right) u_0.$$

Then, by the convexity of φ, we have

$$\varphi(v_n) \leq \frac{1}{\|u_n\|}\varphi(u_n) + \left(1 - \frac{1}{\|u_n\|}\right)\varphi(u_0)$$

$$\leq \frac{1}{\|u_n\|}(-n - n\|u_n\|) + |\varphi(u_0)| \leq -n + |\varphi(u_0)|.$$

As $\{v_n\}$ is bounded, we can extract a weakly convergent subsequence $v_{n_k} \rightharpoonup v$, and again derive the contradiction $\varphi(v) = -\infty$. Therefore, we establish the claim (43).

Choose a minimizing sequence $\{u_n\}$ so that

$$\lim_{n\to\infty} J(u_n) = \inf_{v\in H} J(v) = m.$$

By (43), it is not difficult to see that $m \in I\!R$. Then, having in mind (43), there exists $M > 0$, such that

$$M \geq J(u_n) \geq \frac{1}{2}\|u_n\|^2 - (\lambda C + \|w\|)\|u_n\| - \lambda C$$

$$= \frac{1}{2}(\|u_n\| - (\lambda C + \|w\|))^2 - \lambda C - \frac{1}{2}(\lambda C + \|w\|)^2.$$

Thus, we have $\{u_n\}$ is bounded. We may then extract a weakly convergent subsequence $u_{n_k} \rightharpoonup u$. Then, by (42) J has a minimum at u. Therefore, $0 \in \partial J$. Now, it is easy to see that $\partial J(u) = u - w + \lambda\partial\varphi(u)$, and so

$$u + \lambda\partial\varphi(u) \ni w.$$

Finally, to see the uniqueness, suppose as well

$$\overline{u} + \lambda\partial\varphi(\overline{u}) \ni w.$$

Then, $u + \lambda v = w$, $\overline{u} + \lambda\overline{v} = w$ for $v \in \partial\varphi(u)$, $\overline{v} \in \partial\varphi(\overline{u})$. Hence, by the monotony of $\partial\varphi$, we have

$$0 \leq (u - \overline{u}/v - \overline{v}) = \left(u - \overline{u}/\frac{\overline{u}}{\lambda} - \frac{u}{\lambda}\right) = -\frac{1}{\lambda}\|u - \overline{u}\|^2.$$

Since $\lambda > 0$, $u = \overline{u}$. □

Definition 6 Let $\varphi : H \to]-\infty, +\infty]$ be convex, proper and lower semi-continuous. For each $\lambda > 0$ define the *resolvent* J_λ^φ of $\partial\varphi$ as the operator $J_\lambda^\varphi : H \to D(\partial\varphi)$ defined by $J_\lambda^\varphi(w) := u$, where u is the unique solution of

$$u + \lambda\partial\varphi(u) \ni w.$$

The *Yosida approximation* is the operator $A_\lambda^\varphi : H \to H$ defined by

$$A_\lambda^\varphi(w) := \frac{1}{\lambda}(w - J_\lambda^\varphi(w)).$$

In the next result we collect some of the properties of the resolvent operator and the Yosida approximation.

Theorem 21 *Let $\varphi : H \to]-\infty, +\infty]$ be convex, proper and lower semicontinuous. For $\lambda > 0$, let $J_\lambda = J_\lambda^\varphi$ and $A_\lambda = A_\lambda^\varphi$. The following statements hold*

(i) $\|J_\lambda(w) - J_\lambda(\overline{w})\| \leq \|w - \overline{w}\|$ *for all* $w, \overline{w} \in H$.

(ii) $\|A_\lambda(w) - A_\lambda(\overline{w})\| \leq \frac{2}{\lambda}\|w - \overline{w}\|$ *for all* $w, \overline{w} \in H$.

(iii) $0 \leq (w - \overline{w}/A_\lambda(w) - A_\lambda)$, *i.e.,* A_λ *is a monotone operator.*

(iv) $A_\lambda(w) \in \partial\varphi(J_\lambda(w))$ *for all* $w \in H$.

(v) If $w \in D(\partial\varphi)$, *then*

$$\sup_{\lambda > 0} \|A_\lambda(w)\| \leq |(\partial\varphi)^0(w)| := \min\{\|u\| \ : \ u \in \partial\varphi(w)\}.$$

(vi) For each $w \in \overline{D(\partial\varphi)}$,

$$\lim_{\lambda \downarrow 0} J_\lambda(w) = w.$$

Proof. (i) Let $u = J_\lambda(w)$, $\overline{u} = J_\lambda(\overline{w})$. Then $u + \lambda v = w$, $\overline{u} + \lambda\overline{v} = \overline{w}$ for some $v \in \partial\varphi(u)$, $\overline{v} \in \partial\varphi(\overline{u})$. Therefore

$$\|w - \overline{w}\|^2 = \|u - \overline{u} + \lambda(v - \overline{v})\|^2$$

$$= \|u - \overline{u}\|^2 + 2\lambda(u - \overline{u}/v - \overline{v}) + \lambda^2\|v - \overline{v}\|^2 \geq \|u - \overline{u}\|^2.$$

This prove assertion (i). Assertion (ii) follows from (i) and the definition of Yosida approximation.

(iii) We have

$$(w - \overline{w}/A_\lambda(w) - A_\lambda) = \frac{1}{\lambda}(w - \overline{w}/w - \overline{w} - (J_\lambda(w) - J_\lambda(\overline{w}))$$

$$= \frac{1}{\lambda}\left(\|w - \overline{w}\|^2 - (w - \overline{w}/ - (J_\lambda(w) - J_\lambda(\overline{w}))\right)$$

$$\geq \frac{1}{\lambda}\left(\|w - \overline{w}\|^2 - \|w - \overline{w}\|\|J_\lambda(w) - J_\lambda(\overline{w}\|\right) \geq 0,$$

according to (i).

(iv) Note that $u = J_\lambda(w)$ if and only if $u + \lambda v = w$ for some $v \in \partial\varphi(u) = \partial\varphi(J_\lambda(w))$. But

$$v = \frac{1}{\lambda}(w-u) = \frac{1}{\lambda}(w - J_\lambda(w)) = A_\lambda(w).$$

(v) Assume $w \in D(\partial\varphi)$, $u \in \partial\varphi(w)$. Let $z = J_\lambda(w)$, so that $z + \lambda v = w$, where $v \in \partial\varphi(z)$. The, by monotonicity, we have

$$0 \le (w - z/u - v) = \Big(w - J_\lambda(w)/u - \frac{1}{\lambda}(w - J_\lambda(w))\Big) = (\lambda A_\lambda(w)/u - A_\lambda(w)).$$

Consequently

$$\lambda\|A_\lambda(w)\|^2 \le (\lambda A_\lambda(w)/u) \le \lambda\|A_\lambda(w)\|\,\|u\|,$$

and so

$$\|A_\lambda(w)\| \le \|u\|.$$

Since this estimate is valid for all $\lambda > 0$ and $u \in \partial\varphi(w)$, assertion (v) follows.

(vi) If $w \in D(\partial\varphi)$, by (v), we have

$$\|J_\lambda(w) - w\| = \lambda\|A_\lambda(w)\| \le \lambda|(\partial\varphi)^0(w)|,$$

and hence

$$\lim_{\lambda\downarrow 0} J_\lambda(w) = w.$$

Let $w \in \overline{D(\partial\varphi)}$. Given $\epsilon > 0$ there exists $\overline{w} \in D(\partial\varphi)$ such that $\|w - \overline{w}\| \le \frac{\epsilon}{4}$. Now, since $\overline{w} \in D(\partial\varphi)$, there exists $\lambda_0 >$ such that $\|J_\lambda(\overline{w}) - \overline{w}\| \le \frac{\epsilon}{2}$. Then,

$$\|J_\lambda(w) - w\| \le \|J_\lambda(w) - J_\lambda(\overline{w})\| + \|J_\lambda(\overline{w}) - \overline{w}\| + \|w - \overline{w}\|$$

$$\le 2\|w - \overline{w}\| + \|J_\lambda(\overline{w}) - \overline{w}\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

4.2 Gradient Flows in Hilbert spaces

Many problems in Physic and Mechanics can be written as a gradient system, that is a system of ordinary differential equation of the form

$$\begin{cases} u'(t) = -\nabla V(u(t)) & 0 < t < T \\ u(0) = u_0 \in I\!R^N, \end{cases}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a potential. In this section we are going to consider the generalization infinite dimensional (in the context of Hilbert spaces) of the gradient system. We propose now to study differential equation of the form

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0 \qquad t \geq 0 \\ \\ u(0) = u_0 \in H, \end{cases} \tag{45}$$

where $\varphi : H \to]-\infty, +\infty]$ is a convex, proper and lower semi-continuous function. A problem of the form (78) is called a *gradient flow*. Many partial differential equation can be rewritten as a gradient flow in an appropriate Hilbert space of functions. For example, as we see in the Example 2, if $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary, and we consider the function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ defined by

$$\varphi(u) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 \quad \text{if} \;\; u \in W_0^{1,2}(\Omega) \\ \\ +\infty \qquad\quad u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

Then,

$$D(\partial\varphi) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

and

$$v \in \partial\varphi(u) \Leftrightarrow v = -\Delta u.$$

Therefore, the initial valued problem for the heat equation

$$\begin{cases} u_t = \Delta u \qquad \text{in} \;\; (0, \infty) \times \Omega \\ \\ u = 0 \qquad\quad \text{on} \;\; (0, \infty) \times \partial\Omega \\ \\ u(0, x) = u_0(x) \qquad\quad x \in \Omega \end{cases}$$

can be rewritten as a gradient flow in $L^2(\Omega)$.

We have the following existence and uniqueness result for solutions of the gradient flows.

Theorem 22 *Let $\varphi : H \to]-\infty, +\infty]$ be convex, proper and lower semi-continuous. For each $u_0 \in D(\partial\varphi)$ there exists a unique function $u \in C([0, \infty[, H)$, with $u' \in L^\infty(0, \infty; H)$ such that $u(0) = u_0$, $u(t) \in D(\partial\varphi)$ for each $t > 0$ and*

$$u'(t) + \partial\varphi(u(t)) \ni 0, \qquad \text{for a.e.} \;\; t \geq 0.$$

Proof. For $\lambda > 0$, let $J_\lambda = J_\lambda^\varphi$ the resolvent of $\partial\varphi$ and $A_\lambda = A_\lambda^\varphi$ its Yosida approximation. By Theorem 21, $A_\lambda : H \to H$ is Lipschitz continuous mapping, and thus, by the classical Picard-Lindelöf Theorem there exists a unique solution $u_\lambda \in C^1([0,\infty[; H)$ of the problem

$$\begin{cases} u_\lambda'(t) + A_\lambda(u_\lambda(t)) = 0 \qquad t \geq 0 \\ \\ u_\lambda(0) = u_0. \end{cases} \tag{46}$$

Our plan is to show that as $\lambda \to 0^+$ the functions u_λ converge to a solutions of our problem. We divide the proof in several steps.

Step 1. Given $v \in H$, let v_λ the solution of the problem

$$\begin{cases} v_\lambda'(t) + A_\lambda(v_\lambda(t)) = 0 \qquad t \geq 0 \\ \\ v_\lambda(0) = v. \end{cases} \tag{47}$$

Then, by the monotony of A_λ, we have

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - v_\lambda(t)\|^2$$

$$= (u_\lambda'(t) - v_\lambda'(t)/u_\lambda(t) - v_\lambda)(t) = (-A_\lambda(u_\lambda(t)) + A_\lambda(v_\lambda(t))/u_\lambda(t) - v_\lambda(t)) \leq 0.$$

Hence, integrating we get

$$\|u_\lambda(t) - v_\lambda(t)\| \leq \|u_0 - v\| \qquad \forall\, t \geq 0. \tag{48}$$

In particular, if $h > 0$ and $v = u_\lambda(h)$, then by uniqueness $v_\lambda(t) = u_\lambda(t+h)$. Consequently, (48) implies

$$\|u_\lambda(t+h) - u_\lambda(t)\| \leq \|u_\lambda(h) - u_0\|.$$

Dividing by h, letting $h \to 0$, and having in mind Theorem 21 (v), we obtain that

$$\|u_\lambda'(t)\| \leq \|u_\lambda'(0)\| = \|A_\lambda(u_0)\| \leq |(\partial\varphi)^0(u_0)|. \tag{49}$$

Step 2. We next take $\lambda, \mu > 0$ and compute

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 = (u_\lambda'(t) - u_\mu'(t)/u_\lambda(t) - u_\mu(t)) \\ &= (-A_\lambda(u_\lambda(t)) + A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)). \end{aligned} \tag{50}$$

Now

$$u_\lambda(t) - u_\mu(t) = (u_\lambda(t) - J_\lambda(u_\lambda(t))) + (J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))) + (J_\mu(u_\mu(t)) - u_\mu(t))$$

$$= \lambda A_\lambda(u_\lambda(t)) + J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t)) - \mu A_\mu(u_\mu(t)).$$

Consequently

$$\begin{aligned} &(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \\ &= (A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))) \\ &+(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/\lambda A_\lambda(u_\lambda(t))) - \mu A_\mu(u_\mu(t))). \end{aligned} \tag{51}$$

Since $A_\lambda(u_\lambda(t)) \in \partial\varphi(J_\lambda(u_\lambda(t)))$ and $A_\mu(u_\mu(t)) \in \partial\varphi(J_\mu(u_\mu(t)))$, the monotonicity of $\partial\varphi$ implies that the first term of the right hand side of (51) is nonnegative. Thus

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \geq$$

$$\lambda\|A_\lambda(u_\lambda(t))\|^2 + \mu\|A_\mu(u_\mu(t))\|^2 - (\lambda+\mu)\|A_\lambda(u_\lambda(t))\|\,\|A_\mu(u_\mu(t))\|.$$

Since

$$(\lambda+\mu)\|A_\lambda(u_\lambda(t))\|\,\|A_\mu(u_\mu(t))\| \leq$$

$$\lambda\left(\|A_\lambda(u_\lambda(t))\|^2 + \frac{1}{4}\|A_\mu(u_\mu(t))\|^2\right) + \mu\left(\|A_\mu(u_\mu(t))\|^2 + \frac{1}{4}\|A_\lambda(u_\lambda(t))\|^2\right)$$

we deduce

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \geq -\frac{\lambda}{4}\|A_\mu(u_\mu(t))\|^2 - \frac{\mu}{4}\|A_\lambda(u_\lambda(t))\|^2.$$

Then, by (49), we get

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \geq -\frac{\lambda+\mu}{4}|(\partial\varphi)^0(u_0)|.$$

Therefore, by (50) and (51), we obtain the inequality

$$\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 \leq \frac{\lambda+\mu}{2}|(\partial\varphi)^0(u_0)|$$

and hence

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leq \frac{\lambda+\mu}{2}t|(\partial\varphi)^0(u_0)| \qquad \forall\, t \geq 0. \tag{52}$$

In view of the estimate (52) there exists a function $u \in C([0,\infty[, H)$ such that

$$u_\lambda \to u \quad \text{uniformly in} \quad C([0,T], H)$$

as $\lambda \downarrow 0$, for each time $T > 0$. Furthermore estimate (49) implies

$$u'_\lambda \rightharpoonup u' \quad \text{weakly in} \quad L^2(0,T;H) \tag{53}$$

for each $T > 0$, and

$$\|u'(t)\| \leq |(\partial\varphi)^0(u_0)| \qquad \text{a.e. } t. \tag{54}$$

Step 3. We must show $u(t) \in D(\partial\varphi)$ for each $t \geq 0$ and

$$u'(t) + \partial\varphi(u(t)) \ni 0, \qquad \text{for a.e. } \ t \geq 0.$$

Now, by (49)

$$\|J_\lambda(u_\lambda(t)) - u_\lambda(t)\| = \lambda\|A_\lambda u_\lambda(t)\| = \lambda\|u'_\lambda(t)\| \leq \lambda|(\partial\varphi)^0(u_0)|.$$

Hence

$$J_\lambda(u_\lambda) \to u \quad \text{uniformly in} \quad C([0,T],H) \tag{55}$$

for each $T > 0$.

On the other hand, for each $t \geq 0$,

$$-u'_\lambda(t) = A_\lambda(u_\lambda(t)) \in \partial\varphi(J_\lambda(u_\lambda(t))).$$

Thus, given $w \in H$, we have

$$\varphi(w) \geq \varphi(J_\lambda(u_\lambda(t))) - (u'_\lambda(t)/w - J_\lambda(u_\lambda(t))).$$

Consequently if $0 \leq s \leq t$,

$$(t-s)\varphi(w) \geq \int_s^t \varphi(J_\lambda(u_\lambda(\tau)))\,d\tau - \int_s^t (u'_\lambda(\tau)/w - J_\lambda(u_\lambda(\tau)))\,d\tau.$$

In view of (55), the lower semi-continuity of φ, and Fatou's Lemma, we conclude upon sending $\lambda \downarrow 0$ that

$$(t-s)\varphi(w) \geq \int_s^t \varphi(u(\tau))\,d\tau - \int_s^t (u'(\tau)/w - u(\tau))\,d\tau.$$

Therefore

$$\varphi(w) \geq \varphi(u(t)) - (u'(t)/w - u(t))$$

if t is a Lebesgue point of u', $\varphi(u)$. Hence, for almost all $t \geq 0$

$$\varphi(w) \geq \varphi(u(t)) - (u'(t)/w - u(t))$$

for all $w \in H$. Thus $u(t) \in D(\partial\varphi)$, with

$$u'(t) + \partial\varphi(u(t)) \ni 0, \qquad \text{for a.e. } \; t \geq 0.$$

Finally, we prove that $u(t) \in D(\partial\varphi)$ for each $t \geq 0$. To see this, fix $t \geq 0$ and choose $t_n \to t$ such that $u(t_n) \in D(\partial\varphi)$, $-u'(t_n) \in \partial\varphi(u(t_n))$. In view of (54) we may assume , upon passing to a subsequence, that

$$u'(t_n) \rightharpoonup v \qquad \text{weakly in } H.$$

Fix $w \in H$. Then

$$\varphi(w) \geq \varphi(u(t_n)) - (u'(t_n)/w - u(t_n)).$$

Let $t_n \to t$ and recall that $u \in C([0, \infty[, H)$ and φ is lower semi-continuous. We obtain that

$$\varphi(w) \geq \varphi(u(t)) - (v/w - u(t)).$$

Hence $u(t) \in D(\partial\varphi)$ and $-v \in \partial\varphi(u(t))$.

Step 4. To prove uniqueness assume $\overline{u}$ is another solution and compute

$$\frac{1}{2}\frac{d}{dt}\|u(t) - \overline{u}(t)\|^2 = (u'(t) - \overline{u}'(t)/u(t) - \overline{u}(t)) \leq 0 \qquad \text{for a.e. } t \geq 0,$$

since $-u'(t) \in \partial\varphi(u(t))$ and $-\overline{u}'(t) \in \partial\varphi(\overline{u}(t))$. Then, integrating we obtain that

$$\|u(t) - \overline{u}(t)\|^2 \leq \|u(0) - \overline{u}(0)\|^2.$$

□

Under the assumptions of the above theorem if for each $u_0 \in D(\partial\varphi)$ we define

$$S(t)u_0 := u(t) \qquad \forall\, t \geq 0,$$

$u(t)$ being the unique solution of problem

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0, & \text{for a.e. } \; t \geq 0 \\ u(0) = u_0, \end{cases} \tag{56}$$

we have the family of operator $(S(t))_{t \geq 0}$ satisfying

(i) $S(0) = I$,

(ii) $S(t + s) = S(t)S(s)$ for all $s, t \geq 0$,

(iii) the mapping $t \mapsto S(t)u_0$ is continuous from $[0, \infty[$ into H.

A family of operators $(S(t))_{t\geq 0}$ satisfying the conditions (i)-(iii) is called a *nonlinear semigroup of operators*.

Observe that as a consequence of the above theorem we have

$$\|S(t)u_0 - S(t)\overline{u_0}\| \leq \|u_0 - \overline{u_0}\|, \quad \forall\, t \geq 0, \text{ and } u_0, \overline{u_0} \in D(\partial\varphi). \tag{57}$$

Using this inequality the semigroup of nonlinear operators $(S(t))_{t\geq 0}$ can be extended to $\overline{D(\partial\varphi)}$. In the case $D(\partial\varphi)$ is dense in H, which happens in many applications, we have $(S(t))_{t\geq 0}$ is a nonlinear semigroup in H.

Theorem 22 is a particular case of the following general situation. Let $A \subset H \times H$ an operator (possible multivaluate) in the real Hilbert space H. We say that A is *monotone* if

$$(u - \overline{u}/v - \overline{v}) \geq 0 \quad \forall\, (u, \overline{u}), (v, \overline{v}) \in A.$$

Recall we have showed that $\partial\varphi$ is a monotone operator. Now, if φ is convex, lower semi-continuous and proper, it can be proved that $\partial\varphi$ is maximal monotone (see, [10], [8]), i.e., every monotone extension of $\partial\varphi$ coincides with $\partial\varphi$. The following theorem is a classical result due to G. Minty [20].

Theorem 23 (Minty Theorem) *Let A a monotone operator in the real Hilbert space H. Then, A is maximal monotone if and only if $Ran(I+\lambda A) = H$ for all $\lambda > 0$.*

Given an operator $A \subset H \times H$, consider the abstract Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, & \text{for a.e. } t \in (0, T) \\ \\ u(0) = u_0. \end{cases} \tag{58}$$

We say that a function $u \in C([0, T]; H)$ is a *strong solution* of problem (58) if $u(0) = u_0$, u is derivable a.e. $t \in (0, T)$, $u(t) \in D(A)$ and satisfies (58) for almost all $t \in (0, T)$.

Theorem 22 states that for every $u_0 \in D(\partial\varphi)$, $u(t) = S(t)u_0$ is a strong solution of the abstract Cauchy problem associated with $\partial\varphi$. Now, this result is also true in the general case in which A is a maximal monotone operator (see [10], [8]). Moreover, in the case $A = \partial\varphi$, with $\varphi : H \to]-\infty, +\infty]$ a convex, proper and lower semi-continuous, we also have (see [10]) that for all $u_0 \in \overline{D(\partial\varphi)}$, $u(t) = S(t)u_0$ is a strong solution.

5 The Neumann Problem for the Total Variation Flow

This section is devoted to prove existence and uniqueness of solutions for the Minimizing Total Variation Flow with Neumann boundary conditions, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in } Q = (0,\infty) \times \Omega \\ \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } S = (0,\infty) \times \partial\Omega \\ \\ u(0,x) = u_0(x) & \text{in } x \in \Omega, \end{cases} \tag{59}$$

where Ω is a bounded set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$ and $u_0 \in L^1(\Omega)$. As we saw in the previous section, this partial differential equation appears when one uses the steepest descent method to minimize the Total Variation, a method introduced by L. Rudin, S. Osher and E. Fatemi ([23]) in the context of image denoising and reconstruction. Then solving (59) amounts to regularize or, in other words, to filter the initial datum u_0. This filtering process has less destructive effect on the edges than filtering with a Gaussian, i.e., than solving the heat equation with initial condition u_0. In this context the given *image* u_0 is a function defined on a bounded, smooth or piecewise smooth open subset Ω of $\mathbb{R}^N$, typically, Ω will be a rectangle in $\mathbb{R}^2$. As argued in [1], the choice of Neumann boundary conditions is a natural choice in image processing. It corresponds to the reflection of the picture across the boundary and has the advantage of not imposing any value on the boundary and not creating edges on it. When dealing with the deconvolution or reconstruction problem one minimizes the Total Variation Functional, i.e., the functional

$$\int_\Omega |Du| \tag{60}$$

under some constraints which model the process of image acquisition, including blur and noise.

5.1 Strong Solutions in $L^2(\Omega)$

Consider the energy functional $\Phi : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\Phi(u) = \begin{cases} \displaystyle\int_\Omega \|Du\| & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{61}$$

Since the functional Φ is convex, lower semi-continuous and proper, then $\partial\Phi$ is a maximal monotone operator with dense domain, generating a contraction semigroup in $L^2(\Omega)$ (see subsection 4.2 or [10]). Therefore, we have the following result.

Theorem 24 *Let $u_0 \in L^2(\Omega)$. Then there exists a unique strong solution in the semigroup sense u of (59) in $[0,T]$ for every $T > 0$, i.e., $u \in C([0,T]; L^2(I\!R^N)) \cap W^{1,2}_{\rm loc}(0,T; L^2(\Omega))$, $u(t) \in D(\partial\Phi)$ a.e. in $t \in [0,T]$ and*

$$-u'(t) \in \partial\Phi(u(t)) \quad \text{a.e. in } t \in [0,T]. \tag{62}$$

Moreover, if u and v are the strong solutions of (59) corresponding to the initial conditions $u_0, v_0 \in L^2(\Omega)$, then

$$\|u(t) - v(t)\|_2 \le \|u_0 - v_0\|_2 \quad \text{for any } t > 0. \tag{63}$$

Our task will be to give a sense to (62) as a partial differential equation, describing the subdifferential of Φ in a distributional sense. To be precise we should not say distributional sense since the test functions will be functions in $BV(\Omega)$. To do that we need to recall first some results inspired in the duality theory of the Convex Analysis.

Let H be a real Hilbert space, with inner product $(\ /\ )$. Let $\Psi : H \to [0,\infty]$ be any function. Let us define $\tilde\Psi : H \to [0,\infty]$ by

$$\tilde\Psi(x) = \sup\left\{ \frac{(x/y)}{\Psi(y)} \ :\ y \in H \right\} \tag{64}$$

with the convention that $\frac{0}{0} = 0$, $\frac{0}{\infty} = 0$. Note that $\tilde\Psi(x) \ge 0$, for any $x \in H$. Note also that the supremum is attained on the set of $y \in H$ such that $(x/y) \ \ge 0$. Note also that we have the following Cauchy-Schwartz inequality

$$(x/y) \le \tilde\Psi(x)\Psi(y) \quad \text{if } \Psi(y) > 0.$$

The following Lemma is a simple consequence of the above definition.

Lemma 4 *Let $\Psi_1, \Psi_2 : H \to [0,\infty]$. If $\Psi_1 \le \Psi_2$, then $\tilde\Psi_2 \le \tilde\Psi_1$.*

Proposition 6 *If Ψ is convex, lower semi-continuous and positive homogeneous of degree 1, then $\tilde{\tilde\Psi} = \Psi$.*

Proof. Since $\frac{(y/x)}{\Psi(x)} \leq \tilde{\Psi}(y)$ for any $x, y \in H$, we also have that $\frac{(y/x)}{\tilde{\Psi}(y)} \leq \Psi(x)$ for any $x, y \in H$. This implies that $\tilde{\tilde{\Psi}}(x) \leq \Psi(x)$ for any $x \in H$. Assume that there is some $x \in H$ and $\epsilon > 0$ such that $\tilde{\tilde{\Psi}}(x)+\epsilon < \Psi(x)$, hence, in particular, $\Psi(x) > 0$ and $\tilde{\tilde{\Psi}}(x) < \infty$. Using Hahn-Banach's Theorem there is $y \in H$ separating x from the closed convex set $C := \{z \in H : \Psi(z) \leq \tilde{\tilde{\Psi}}(x) + \epsilon\}$. Since $0 \in C$ we may even assume that $(y/x) = 1$ and $(y/z) \leq \alpha < 1$ for any $z \in C$. Note that, from the definition of $\tilde{\tilde{\Psi}}$, we have

$$\tilde{\tilde{\Psi}}(x) \geq \frac{1}{\tilde{\Psi}(y)}. \tag{65}$$

Let us prove that $\tilde{\Psi}(y) \leq \frac{1}{\tilde{\tilde{\Psi}}(x)+\epsilon}$. For that it will be sufficient to prove that

$$\frac{(y/z)}{\Psi(z)} \leq \frac{1}{\tilde{\tilde{\Psi}}(x) + \epsilon} \tag{66}$$

for any $z \in H$ such that $(y/z) \geq 0$. Let $z \in H$, $(y/z) \geq 0$. If $\Psi(z) = \infty$, then (66) holds. If $\Psi(z) = 0$, then also $\Psi(tz) = 0$ for any $t \geq 0$. Hence $tz \in C$ for all $t \geq 0$, and we have that $0 \leq (y/tz) \leq 1$ for all $t \geq 0$. Thus $(y/z) = 0$, and, therefore, (66) holds. Finally, assume that $0 < \Psi(z) < \infty$. Let $t > 0$ be such that $\Psi(tz) = \tilde{\tilde{\Psi}}(x) + \epsilon$. Using that $tz \in C$, we have

$$\frac{(y/z)}{\Psi(z)} = \frac{(y/tz)}{\Psi(tz)} \leq \frac{1}{\tilde{\tilde{\Psi}}(x) + \epsilon}.$$

Both (65) and (66) give a contradiction. We conclude that $\tilde{\tilde{\Psi}}(x) = \Psi(x)$ for any $x \in H$. □

Lemma 5 *Assume that Ψ is convex, lower semi-continuous and positive homogeneous of degree* 1. *If $u \in D(\partial\Psi)$ and $v \in \partial\Psi(u)$, then $(v/u) = \Psi(u)$.*

Proof. Indeed, if $v \in \partial\Psi(u)$, then

$$(v/w - u) \leq \Psi(w) - \Psi(u), \quad \text{for all } w \in H.$$

To obtain the result it suffices to take $w = 0$ and $w = 2u$ in the above inequality. □

Theorem 25 *Assume that Ψ is convex, lower semi-continuous and positive homogeneous of degree* 1. *Then $v \in \partial\Psi(u)$ if and only if $\tilde{\Psi}(v) \leq 1$ and $(v/u) = \Psi(u)$ (hence, $\tilde{\Psi}(v) = 1$ if $\Psi(u) > 0$).*

Proof. When $(v/u) = \Psi(u)$, condition $v \in \partial\Psi(u)$ may be written as $(v/x) \leq \Psi(x)$ for all $x \in H$, which is equivalent to $\tilde{\Psi}(v) \leq 1$. □

Let Ω be a bounded domain in $I\!R^N$ with Lipschitz boundary. Let us consider the space (see Section 3)

$$X(\Omega)_2 := \left\{ z \in L^\infty(\Omega, I\!R^N) \; : \; \mathrm{div}(z) \in L^2(\Omega) \right\}.$$

Let us define for $v \in L^2(\Omega)$

$$\Psi(v) = \inf\{\| z \|_\infty : \; z \in X(\Omega)_2, \, v = -\mathrm{div}(z) \,\mathrm{in}\, \mathcal{D}'(\Omega), \, [z,\nu] = 0\}, \quad (67)$$

where ν denotes the outward unit normal to $\partial\Omega$ and $[z,\nu]$ is the trace of the normal component of z (see Section 3). We define $\Psi(v) = +\infty$ if does not exists $z \in X(\Omega)_2$ satisfying $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, $[z,\nu] = 0$.

Observe that Ψ is convex, lower semi-continuous and positive homogeneous of degree 1. Moreover, it is easy to see that, if $\Psi(v) < \infty$, the infimum in (67) is attained, i.e., there is some $z \in X(\Omega)_2$ such that $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, $[z,\nu] = 0$ and $\Psi(v) = \|z\|_\infty$.

Proposition 7 *We have that* $\Psi = \tilde{\Phi}$.

Proof. Let $v \in L^2(\Omega)$. If $\Psi(v) = \infty$, then we have $\tilde{\Phi}(v) \leq \Psi(v)$. Thus, we may assume that $\Psi(v) < \infty$. Let $z \in X(\Omega)_2$ be such that $v = -\mathrm{div}(z)$ and $[z,\nu] = 0$. Then

$$\int_\Omega vu \, dx = \int_\Omega (z, Du) \leq \| z \|_\infty \, \Phi(u) \quad \text{for all } u \in BV(\Omega) \cap L^2(\Omega).$$

Taking supremums in u we obtain $\tilde{\Phi}(v) \leq \| z \|_\infty$. Now, taking infimums in z, we obtain $\tilde{\Phi}(v) \leq \Psi(v)$.

To prove the opposite inequality, let us denote

$$D = \left\{ \mathrm{div}(z) \; : \; z \in C_0^\infty(\Omega, I\!R^N) \right\}.$$

Then

$$\begin{aligned} \sup_{v \in L^2} \frac{\int_\Omega uv \, dx}{\Psi(v)} &\geq \sup_{v \in D} \frac{\int_\Omega uv \, dx}{\Psi(v)} \geq \sup_{v \in D, \Psi(v) < \infty} \frac{\int_\Omega uv \, dx}{\Psi(v)} \\ &\geq \sup_{z \in C_0^\infty(\Omega, I\!R^N)} \frac{-\int_\Omega u \mathrm{div}(z) \, dx}{\| z \|_\infty} = \Phi(u). \end{aligned}$$

Thus, $\Phi \leq \tilde{\Psi}$. This implies that $\tilde{\tilde{\Psi}} \leq \tilde{\Phi}$, and, using Proposition 6, we obtain that $\Psi \leq \tilde{\Phi}$. □

We have the following characterization of the subdifferential $\partial\Phi$.

Theorem 26 *The following assertions are equivalent:*

(a) $v \in \partial\Phi(u)$*;*

(b)

$$u \in L^2(\Omega) \cap BV(\Omega),\ v \in L^2(\Omega), \tag{68}$$

$$\exists z \in X(\Omega)_2,\ \|z\|_\infty \le 1,\ \text{such that } v = -\mathrm{div}(z) \quad \text{in } \mathcal{D}'(\Omega), \tag{69}$$

and

$$\int_\Omega (z, Du) = \int_\Omega \|Du\|, \tag{70}$$

$$[z, \nu] = 0 \quad \text{on} \quad \partial\Omega; \tag{71}$$

(c) (68) and (69) hold, and

$$\int_\Omega (w-u)v\,dx \le \int_\Omega z \cdot \nabla w\,dx - \int_\Omega \|Du\|, \qquad \forall w \in W^{1,1}(\Omega) \cap L^2(\Omega); \tag{72}$$

(d) (68) and (69) hold, and

$$\int_\Omega (w-u)v\,dx \le \int_\Omega (z, Dw) - \int_\Omega \|Du\| \qquad \forall w \in L^2(\Omega) \cap BV(\Omega); \tag{73}$$

(e) (68) and (69) hold, and (73) holds with the equality instead of the inequality.

Proof. By Theorem 25, we have that $v \in \partial\Phi(u)$ if and only if $\tilde{\Phi}(v) \le 1$ and $\int_\Omega vu\,dx = \Phi(u)$. Since $\tilde{\Phi} = \Psi$, from the definition of Ψ and the observation following it, it follows that there is some $z \in X(\Omega)_2$ such that $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, $[z, \nu] = 0$ and $\tilde{\Phi}(v) = \|z\|_\infty$. Hence, we have $v \in \partial\Phi(u)$ if and only if there is some $z \in X(\Omega)_2$, with $\|z\|_\infty \le 1$, such that $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, $[z, \nu] = 0$ and $\int_\Omega vu\,dx = \Phi(u)$. Then, applying Green's formula (24) the equivalence of (a) and (b) follows.

To obtain (e) from (b) it suffices to multiply both terms of the equation $v = -\mathrm{div}(z)$ by $w - u$, for $w \in L^2(\Omega) \cap BV(\Omega)$ and to integrate by parts using Green's formula (24). It is clear that (e) implies (d), and (d) implies (c). To prove that (b) follows from (d) we choose $w = u$ in (73) and we obtain that

$$\int_\Omega \|Du\| \le \int_\Omega (z, Du) \le \|z\|_\infty \int_\Omega \|Du\| \le \int_\Omega \|Du\|.$$

To obtain (71) we choose $w = u \pm \varphi$ in (73) with $\varphi \in C^\infty(\overline{\Omega})$ and we obtain

$$\pm \int_\Omega v\varphi \, dx \leq \pm \int_\Omega z \cdot D\varphi = - \pm \int_\Omega \mathrm{div}(z) \, \varphi \, dx + \pm \int_{\partial\Omega} [z, \nu] \, \varphi \, d\mathcal{H}^{N-1},$$

which implies (71). In order to prove that (c) implies (d), let $w \in BV(\Omega) \cap L^2(\Omega)$. Using Theorem 2 we know that there exists a sequence $w_n \in C^\infty(\Omega) \cap BV(\Omega) \cap L^2(\Omega)$ such that

$$w_n \to w \ \text{ in } \ L^2(\Omega) \ \text{ and } \ \int_\Omega |\nabla w_n| \, dx \to \int_\Omega \|Dw\|.$$

Then

$$\begin{aligned} \int_\Omega z \cdot \nabla w_n \, dx &= -\int_\Omega \mathrm{div}(z) \, w_n \, dx + \int_{\partial\Omega} [z, \nu] w_n \, d\mathcal{H}^{N-1} \\ &\to -\int_\Omega \mathrm{div}(z) \, w \, dx + \int_{\partial\Omega} [z, \nu] w \, d\mathcal{H}^{N-1} = \int_\Omega (z, Dw). \end{aligned}$$

Now, we use w_n as test function in (72) and let $n \to \infty$ to obtain (73). □

Definition 7 *We say that $u \in C([0,T]; L^2(\Omega))$ is a strong solution of (59) if*

$$u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega)) \cap L^1_w(]0,T[; BV(\Omega)),$$

$u(0) = u_0$, and there exists $z \in L^\infty\left(]0,T[\times\Omega; \mathbb{R}^N\right)$ such that $\|z\|_\infty \leq 1$,

$$[z(t), \nu] = 0 \quad \text{in } \ \partial\Omega, \ \text{ a.e. } t \in [0,T]$$

satisfying

$$u_t = \mathrm{div}(z) \qquad \text{in } \mathcal{D}'\left(]0,T[\times\Omega\right)$$

and

$$\begin{gathered} \int_\Omega (u(t) - w) u_t(t) \, dx = \int_\Omega (z(t), Dw) - \int_\Omega \|Du(t)\| \\ \forall w \in L^2(\Omega) \cap BV(\Omega), \ \text{ a.e. } t \in [0,T]. \end{gathered} \tag{74}$$

Obviously, using Theorem 26, a strong solution of (59) is a strong solution in the sense of semigroups. The converse implication follows along the same lines, except for the measurability of $z(t,x)$. To ensure the joint measurability of z one takes into account that, by Theorem 22, semigroup solutions can be approximated by implicit in time discretizations of (62), and one constructs a function $z(t,x) \in L^\infty((0,T) \times \Omega)$ satisfying the requirements contained in Definition 7. We do not give the details of this proof here. We have obtained the following result.

Theorem 27 *Let $u_0 \in L^2(\Omega)$. Then there exists a unique strong solution u of (59) in $[0,T] \times \Omega$ for every $T > 0$. Moreover, if u and v are the strong solutions of (59) corresponding to the initial conditions $u_0, v_0 \in L^2(\Omega)$, then*

$$\|u(t) - v(t)\|_2 \leq \|u_0 - v_0\|_2 \quad \text{for any } t > 0. \tag{75}$$

Remark 4 It is possible to obtain existence and uniqueness of solutions for any initial datum in $L^1(\Omega)$. In this case we need to use truncation functions of type T_k: $T_k(r) = [k - (k - |r|)^+]\mathrm{sign}_0(r)$, $k \geq 0$, $r \in I\!R$, and the concept of solution is the following

Definition 8 A measurable function $u : (0,T) \times \Omega \to I\!R$ is a *weak solution* of (59) in $(0,T) \times \Omega$ if $u \in C([0,T], L^1(\Omega)) \cap W^{1,1}_{loc}(0,T; L^1(\Omega))$, $T_k(u) \in L^1_w(0,T; BV(\Omega))$ for all $k > 0$ and there exists $z \in L^\infty((0,T) \times \Omega)$ with $\|z\|_\infty \leq 1$, $u_t = \mathrm{div}(z)$ in $\mathcal{D}'((0,T) \times \Omega)$ such that

$$\int_\Omega (T_k(u(t)) - w) u_t(t)\, dx \leq \int_\Omega z(t) \cdot \nabla w \, dx - \int_\Omega \|DT_k(u(t))\| \tag{76}$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and a.e. on $[0,T]$.

In [3] (see also [5]) we prove the following existence and uniqueness result.

Theorem 28 *Let $u_0 \in L^1(\Omega)$. Then there exists a unique weak solution of (59) in $(0,T) \times \Omega$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t), \hat{u}(t)$ are weak solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1, \tag{77}$$

for all $t \geq 0$.

To prove Theorem 28 we shall use the techniques of completely accretive operators and the Crandall-Liggett's semigroup generation Theorem. For that, we shall associate a completely accretive operator $\mathcal{A}$ to the formal differential expression $-\mathrm{div}(\frac{Du}{|Du|})$ together with Neumann boundary conditions. Then, using Crandall-Liggett's semigroup generation Theorem we conclude that the abstract Cauchy problem in $L^1(\Omega)$

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u \ni 0, \\ \\ u(0) = u_0 \end{cases} \tag{78}$$

has a unique strong solution $u \in C([0,T], L^1(\Omega)) \cap W^{1,1}_{loc}(0,T; L^1(\Omega))$ $(\forall T > 0)$ with initial datum $u(0) = u_0$, and we shall prove that strong solutions of (78) coincide with weak solutions of (59).

5.2 Asymptotic Behaviour of Solutions

To see that our concept of solution is useful we are going to compute explicitly the evolution of the characteristic function of a ball.

Theorem 29 *Let $\Omega = B(0,R)$ be the ball in $I\!R^N$ centered at 0 with radius R, and $u_0(x) = k\chi_{B(0,r)}$, where $0 < r < R$ and $k > 0$. Then, the strong solution of (59) for the initial datum u_0 is given by*

$$u(t) = \begin{cases} \left(k - \frac{N}{r}t\right)\chi_{B(0,r)} + \frac{Nr^{N-1}}{R^N - r^N} t \chi_{B(0,R)\setminus B(0,r)} & \text{if } 0 \le t \le T \\ \\ (k - \frac{N}{r}T)\chi_{B(0,R)} = \frac{Nr^{N-1}}{R^N - r^N} T \chi_{B(0,R)} & \text{if } t \ge T, \end{cases} \tag{79}$$

where T is given by

$$T\left(\frac{N}{r} + N\frac{r^{N-1}}{R^N - r^N}\right) = k. \tag{80}$$

Proof. We look for a solution of (59) of the form $u(t) = \alpha(t)\chi_{B(0,r)} + \beta(t)\chi_{B(0,R)\setminus B(0,r)}$ on some time interval $(0,T)$ defined by the inequalities $\alpha(t) > \beta(t)$ for all $t \in (0,T)$, and $\alpha(0) = k$, $\beta(0) = 0$. Then, we shall look for some $z \in L^\infty((0,T) \times B(0,R))$ with $\|z\|_\infty \le 1$, such that

$$\begin{aligned} &\alpha'(t) = \mathrm{div}(z(t)) && \text{in } (0,T) \times B(0,r) \\ &z(t,x) = -\frac{x}{|x|} && \text{on } (0,T) \times \partial B(0,r), \end{aligned} \tag{81}$$

$$\begin{aligned} &\beta'(t) = \mathrm{div}(z(t)) && \text{in } (0,T) \times (B(0,R) \setminus B(0,r)) \\ &z(t,x) = -\frac{x}{|x|} && \text{on } (0,T) \times \partial B(0,r) \\ &z(t) \cdot n = 0 && \text{on } (0,T) \times \partial B(0,R) \end{aligned} \tag{82}$$

and

$$\int_{B(0,R)} z(t) \cdot Du(t) = \int_{B(0,R)} |Du(t)| \qquad \text{for all } t \in (0,T). \tag{83}$$

Integrating equation (81) in $B(0,r)$ we obtain

$$\alpha'(t)|B(0,r)| = \int_{B(0,r)} \mathrm{div}(z(t))dx = \int_{\partial B(0,r)} z(t) \cdot n = -\mathcal{H}^{N-1}(\partial B(0,r)).$$

Thus

$$\alpha'(t) = -\frac{N}{r}.$$

and, therefore,

$$\alpha(t) = k - \frac{N}{r}t.$$

In this case we take $z = -\frac{x}{r}$ and (81) holds. Similarly, we deduce that

$$\beta'(t) = \mu := N\frac{r^{N-1}}{R^N - r^N},$$

hence,

$$\beta(t) = N\frac{r^{N-1}}{R^N - r^N}t.$$

Our first observation is that T is given by

$$T\left(\frac{N}{r} + N\frac{r^{N-1}}{R^N - r^N}\right) = k. \tag{84}$$

To construct z in $(0,T) \times (B(0,R) \setminus B(0,r))$ we shall look for z of the form $z(t,x) = \rho(|x|)\frac{x}{|x|}$ such that $\operatorname{div}(z(t)) = \beta'(t)$, $\rho(r) = -1$, $\rho(R) = 0$. Since

$$\operatorname{div}(z(t)) = \nabla\rho(|x|) \cdot \frac{x}{|x|} + \rho(|x|)\operatorname{div}(\frac{x}{|x|}) = \rho'(|x|) + \rho(|x|)\frac{N-1}{|x|},$$

we must have

$$\rho'(s) + \rho(s)\frac{N-1}{s} = N\frac{r^{N-1}}{R^N - r^N} \qquad\qquad s \in (r,R). \tag{85}$$

The solution of (104) such that $\rho(R) = 0$ is

$$\rho(s) = \frac{\mu s}{N} - \frac{\mu R^N}{N s^{N-1}}$$

which also satisfies $\rho(r) = -1$. Thus, in $B(0,R) \setminus B(0,r)$,

$$z(t,x) = \frac{\mu x}{N} - \frac{\mu R^N x}{N|x|^N}.$$

It is easy to check that (83) holds. Thus

$$u(t) = (k - \frac{N}{r}t)\chi_{B(0,r)} + \frac{N r^{N-1}}{R^N - r^N}t\chi_{B(0,R)\setminus B(0,r)}.$$

in $(0,T) \times B(0,R)$ where T is given by (84). On the other hand, we take

$$u(t) = (k - \frac{N}{r}T)\chi_{B(0,R)} = \frac{N r^{N-1}}{R^N - r^N}T\chi_{B(0,R)},$$

and $z(t,x) = 0$ in $(T,\infty) \times B(0,R)$. It is easy to check that $u(t)$ is the solution of (59) in $(0,\infty) \times B(0,R)$ with initial datum $u_0(x)$. □

Remark 5 The above result show that there is no spatial smoothing effect, for $t > 0$, similar to the linear heat equation and many other quasi-linear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity: $u(t) \in BV(\Omega) \setminus W^{1,1}(\Omega)$.

Respect to the asymptotic behaviour of solutions obtained in Theorem 28, using Lyapunov functionals methods we have proved in [3] (see also [5]) the following result.

Theorem 30 *Let $u_0 \in L^2(\Omega)$ and $u(t)$ the unique weak solution of (59) such that $u(0) = u_0$. Then*

$$\|u(t) - (u_0)_\Omega\|_1 \to 0 \quad \text{as} \quad t \to \infty,$$

where

$$(u_0)_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u_0(x)\ dx.$$

Moreover, if $u_0 \in L^\infty(\Omega)$ there exists a constant C, independent of u_0, such that

$$\|u(t) - (u_0)_\Omega\|_p \leq \frac{C\|u_0\|_2^2}{t} \qquad \text{for all} \quad t > 0, \quad \text{and} \quad 1 \leq p \leq \frac{N}{N-1}.$$

Now, we are going to prove, by energy methods that in the two dimensional case, in fact, this asymptotic state is reached in finite time.

Theorem 31 *Suppose $N = 2$. Let $u_0 \in L^2(\Omega)$ and $u(t,x)$ the unique strong solution of problem (59). Then there exists a finite time T_0 such that*

$$u(t) = (u_0)_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u_0(x)\ dx \qquad \forall\ t \geq T_0.$$

Proof: Since u is a strong solution of problem (59), there exists $z \in L^\infty(Q)$ with $\|z\|_\infty \leq 1$, $u_t = \mathrm{div}(z)$ in $\mathcal{D}'(Q)$ such that

$$\int_\Omega (u(t) - w) u_t(t)\, dx = \int_\Omega (z(t), Dw) - \int_\Omega \|Du(t)\| \tag{86}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$. Hence, taking $w = (u_0)_\Omega$ as test function in (86), it yields

$$\int_\Omega (u(t) - (u_0)_\Omega) u_t(t)\, dx = -\int_\Omega \|Du(t)\|.$$

Now, by Sobolev-Poincaré inequality for BV functions (14) and having in mind that we have conservation of mass, we obtain

$$\|u(t) - (u_0)_\Omega\|_2 \leq C \int_\Omega \|Du(t)\|.$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - (u_0)_\Omega)^2 \, dx + \frac{1}{C}\|u(t) - (u_0)_\Omega\|_2 \leq 0. \tag{87}$$

Therefore, the function

$$y(t) := \int_\Omega (u(t) - (u_0)_\Omega)^2 \, dx$$

satisfies the inequality

$$y'(t) + My(t)^{1/2} \leq 0,$$

from where it follows that there exists $T_0 > 0$ such that $y(t) = 0$ for all $t \geq T_0$. □

By Theorem 31, given $u_0 \in L^2(\Omega)$, if $u(t,x)$ is the unique strong solution of problem (59), then

$$T^*(u_0) := \inf\{t > 0 \ : \ u(t) = (u_0)_\Omega\} < \infty.$$

In [4] (see also [5]) we study of the behaviour of $u(t)$ near $T^*(u_0)$ establishing the following result.

Theorem 32 *Suppose $N = 2$. Let $u_0 \in L^2(\Omega)$ and let $u(t,x)$ be the unique strong solution of problem (59). Let*

$$w(t,x) := \begin{cases} \dfrac{u(t,x) - (u_0)_\Omega}{T^*(u_0) - t} & \text{if } \ 0 \leq t < T^*(u_0), \\ \\ 0 & \text{if } \ t \geq T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$, and a solution $v^* \neq 0$ of the stationary problem

$$(S_N) \begin{cases} -\text{div}\left(\dfrac{Dv}{|Dv|}\right) = v & \text{in} \quad \Omega \\ \\ \dfrac{\partial v}{\partial \eta} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w(t_n) = v^* \quad \text{in} \quad L^p(\Omega)$$

for all $1 \leq p < \infty$.

6 The Cauchy Problem for the Total Variation Flow

6.1 Initial Conditions in $L^2(\mathbb{R}^N)$

The purpose of this Subsection is to prove existence and uniqueness of the Minimizing Total Variation Flow in $\mathbb{R}^N$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \qquad \text{in }]0,\infty[\times \mathbb{R}^N, \tag{88}$$

coupled with the initial condition

$$u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \tag{89}$$

when $u_0 \in L^2(\mathbb{R}^N)$.

Definition 9 A function $u \in C([0,T]; L^2(\mathbb{R}^N))$ is called a *strong solution* of (88) if

$$u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(0,T; BV(\mathbb{R}^N))$$

and there exists $z \in L^\infty\left(]0,T[\times \mathbb{R}^N; \mathbb{R}^N\right)$ with $\|z\|_\infty \le 1$ such that

$$u_t = \mathrm{div}(z) \qquad \text{in } \mathcal{D}'(]0,T[\times \mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} (u(t) - w) u_t(t)\, dx = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} \|Du(t)\| \tag{90}$$

for all $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$, a.e. $t \in [0,T]$.

The main result of this subsection is the following existence and uniqueness theorem.

Theorem 33 *Let $u_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique strong solution u of (88), (89) in $[0,T] \times \mathbb{R}^N$ for every $T > 0$. Moreover, if u and v are the strong solutions of (88) corresponding to the initial conditions $u_0, v_0 \in L^2(\mathbb{R}^N)$, then*

$$\|u(t) - v(t)\|_2 \le \|u_0 - v_0\|_2 \quad \text{for any } t > 0. \tag{91}$$

Proof. Let us introduce the following multivalued operator $\mathcal{B}$ in $L^2(\mathbb{R}^N)$: a pair of functions (u,v) belongs to the graph of $\mathcal{B}$ if and only if

$$u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N),\ v \in L^2(\mathbb{R}^N), \tag{92}$$

$$\text{there exists } z \in X(\mathbb{R}^N)_2 \text{ with } \|z\|_\infty \leq 1, \text{ such that } v = -\text{div}(z) \tag{93}$$

and

$$\int_{\mathbb{R}^N} (w-u)v\,dx \leq \int_{\mathbb{R}^N} z \cdot \nabla w\,dx - \int_{\mathbb{R}^N} \|Du\|, \quad \forall w \in L^2(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N).$$

Let also $\Psi : L^2(\mathbb{R}^N) \to \ ]-\infty, +\infty]$ be the functional defined by

$$\Psi(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \|Du\| & \text{if} \quad u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N) \\ \\ +\infty & \text{if} \quad u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N). \end{cases} \tag{94}$$

Since Ψ is convex and lower semi-continuous in $L^2(\mathbb{R}^N)$, its subdifferential $\partial\Psi$ is a maximal monotone operator in $L^2(\mathbb{R}^N)$.

We divide the proof of the theorem into two steps.

Step 1. The following assertions are equivalent:

(a) $(u,v) \in \mathcal{B}$;

(b) (92) and (93) hold,

and

$$\int_{\mathbb{R}^N} (w-u)v\,dx \leq \int_{\mathbb{R}^N} (z, Dw) - \int_{\mathbb{R}^N} \|Du\| \tag{95}$$

for all $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$;

(c) (92) and (93) hold, and (95) holds with the equality instead of the inequality;

(d) (92) and (93) hold, and

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} \|Du\|. \tag{96}$$

It is clear that (c) implies (b), and (b) implies (a), while (d) follows from (b) taking $w = u$ in (95) and using (22). In order to prove that (a) implies (b) it is enough to use Theorem 2 and Lemma 2 as in the proof of Theorem 26. To obtain (c) from (d) it suffices to multiply both terms of the equation $v = -\text{div}(z)$ by $w - u$, for $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$, and to integrate by parts using (25).

Step 2. We also have $\mathcal{B} = \partial\Psi$. The proof is similar to the one given in Section 5.1 for the Neumann problem and we omit the details.

As a consequence, the semigroup generated by $\mathcal{B}$ coincides with the semigroup generated by $\partial\Psi$ and therefore $u(t,x) = e^{-t\mathcal{B}}u_0(x)$ is a strong solution of

$$u_t + \mathcal{B}u \ni 0,$$

i.e., $u \in W^{1,2}_{\rm loc}(]0,T[;L^2(I\!\!R^N))$ and $-u_t(t) \in \mathcal{B}u(t)$ for almost all $t \in \;]0,T[$. Then, according to the equivalence proved in Step 1, we have that

$$\int_{I\!\!R^N} (u(t)-w)u_t(t)\,dx = \int_{I\!\!R^N} (z(t),Dw) - \int_{I\!\!R^N} \|Du(t)\| \tag{97}$$

for all $w \in L^2(I\!\!R^N)\cap BV(I\!\!R^N)$ and for almost all $t \in \;]0,T[$. Now, choosing $w = u-\varphi$, $\varphi \in C_0^\infty(I\!\!R^N)$, we see that $u_t(t) = {\rm div}(z(t))$ in $\mathcal{D}'(I\!\!R^N)$ for almost every $t \in \;]0,T[$. We deduce that $u_t = {\rm div}(z)$ in $\mathcal{D}'(]0,T[\times I\!\!R^N)$. We have proved that u is a strong solution of (88) in the sense of Definition 9.

The contractivity estimate (91) of Theorem 33 follows as in Theorem 28. This concludes the proof of the theorem. □

Given a function $g \in L^2(I\!\!R^N)\cap L^N(I\!\!R^N)$, we define

$$\|g\|_* := \sup\left\{\left|\int_{I\!\!R^N} g(x)u(x)\;dx\right| : u \in L^2(I\!\!R^N)\cap BV(I\!\!R^N),\ \int_{I\!\!R^N}\|Du\|\le 1\right\}.$$

Part (b) of the next Lemma gives a characterization of $\mathcal{B}(0)$. This was proved by Y. Meyer in [19] in the context of the analysis of the Rudin-Osher-Fatemi model for image denoising.

Lemma 6 *Let $f \in L^2(I\!\!R^N)\cap L^N(I\!\!R^N)$ and $\lambda > 0$. The following assertions hold.*

(a) The function u is the solution of

$$\min_{w\in L^2(I\!\!R^N)\cap BV(I\!\!R^N)} D(w), \quad D(w) := \int_{I\!\!R^N}\|Dw\| + \frac{1}{2\lambda}\int_{I\!\!R^N}(w-f)^2\;dx \tag{98}$$

if and only if there exists $z \in X(I\!\!R^N)_2$ satisfying (96) with $\|z\|_\infty \le 1$ and $-\lambda\,{\rm div}(z) = f-u$.

(b) The function $u \equiv 0$ is the solution of (98) if and only if $\|f\|_* \le \lambda$.

(c) If $N=2$, $\mathcal{B}(0) = \{f \in L^2(I\!\!R^2) : \|f\|_* \le 1\}$.

Proof. (a). Thanks to the strict convexity of D, u is the solution of (98) if and only if $0 \in \partial D(u) = \partial\Psi(u) + \frac{1}{\lambda}(u-f) = \mathcal{B}(u) + \frac{1}{\lambda}(u-f)$, where Ψ is defined in (94) and the last equality follows from Step 2 in the proof of Theorem 33. This means, recalling the definition of $\mathcal{B}$ in the proof of Theorem 33, that there exists $z \in X(I\!R^N)_2$ satisfying (96) with $\|z\|_\infty \leq 1$ and $-\lambda \operatorname{div}(z) = f - u$.

(b). The function $u \equiv 0$ is the solution of (98) if and only if

$$\int_{I\!R^N} \|Dv\| + \frac{1}{2\lambda}\int_{I\!R^N} (v-f)^2\, dx \geq \frac{1}{2\lambda}\int_{I\!R^N} f^2\, dx \qquad \forall v \in L^2(I\!R^N) \cap BV(I\!R^N). \tag{99}$$

Replacing v by ϵv (where $\epsilon > 0$), expanding the L^2-norm, dividing by $\epsilon > 0$, and letting $\epsilon \to 0+$ we have

$$\left| \int_{I\!R^N} f(x)v(x)\, dx \right| \leq \lambda \int_{I\!R^N} \|Dv\| \qquad \forall v \in L^2(I\!R^N) \cap BV(I\!R^N). \tag{100}$$

Since (100) implies (99), we have that (99) and (100) are equivalent. The assertion follows by observing that (100) is equivalent to $\|f\|_* \leq \lambda$.

(c) Let $N = 2$. We have

$$\mathcal{B}(0) = \left\{ f \in L^2(I\!R^2) \ : \ \exists z \in X(I\!R^2)_2,\ \|z\|_\infty \leq 1, -\operatorname{div}(z) = f \right\}.$$

On the other hand, from (a) and (b) it follows that $\|f\|_* \leq 1$ if and only if there exists $z \in X(I\!R^2)_2$ with $\|z\|_\infty \leq 1$ and such that $f = -\operatorname{div}(z)$. Then the assertion follows. □

Let us give a heuristic explanation of what the vector field z represents. Condition (96) essentially means that z has unit norm and is orthogonal to the level sets of u. In some sense, z is invariant under local contrast changes. To be more precise, we observe that if $u = \sum_{i=1}^p c_i \chi_{B_i}$ where B_i are sets of finite perimeter such that $\mathcal{H}^{N-1}((B_i \cup \partial^* B_i) \cap (B_j \cup \partial^* B_j)) = 0$ for $i \neq j$, $c_i \in I\!R$, and

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = f \in L^2(I\!R^N), \tag{101}$$

then also $-\operatorname{div}\left(\frac{Dv}{|Dv|}\right) = f$ for any $v = \sum_{i=1}^p d_i \chi_{B_i}$ where $d_i \in I\!R$ and $\operatorname{sign}(d_i) = \operatorname{sign}(c_i)$. Indeed, there is a vector field $z \in L^\infty(I\!R^N; I\!R^N)$ such that $\|z\|_\infty \leq 1$, $-\operatorname{div}(z) = f$ and (96) holds. Then one can check that

$$\|D\chi_{B_i}\| = \operatorname{sign}(c_i)(z, D\chi_{B_i})$$

as measures in $I\!R^N$ and, as a consequence, $(z, Dv) = \|Dv\|$ as measures in $I\!R^N$.

Let us also observe that the solutions of (101) are not unique. Indeed, if $u \in L^2(I\!R^N) \cap BV(I\!R^N)$ is a solution of (101) and $g \in C^1(I\!R)$ with $g'(r) > 0$

for all $r \in I\!R$, then $w = g(u)$ is also a solution of (101). In other words, a global contrast change of u produces a new solution of (101). In an informal way, the previous remark can be rephrased by saying that also local contrast changes of a given solution of (101) produce new solutions of it. To express this non-uniqueness in a more general way we suppose that $(u_1, v), (u_2, v) \in \mathcal{B}$, i.e., there are vector fields $z_i \in X(I\!R^N)_2$ with $\|z_i\|_\infty \leq 1$, such that

$$-\mathrm{div}(z_i) = v, \qquad \int_{I\!R^N} (z_i, Du_i) = \int_{I\!R^N} \|Du_i\|, \qquad i = 1, 2.$$

Then

$$\begin{aligned} 0 &= -\int_{I\!R^N} (\mathrm{div}(z_1) - \mathrm{div}(z_2))(u_1 - u_2)\, dx = \int_{I\!R^N} (z_1 - z_2, Du_1 - Du_2) \\ &= \int_{I\!R^N} \|Du_1\| - (z_2, Du_1) + \int_{I\!R^N} \|Du_2\| - (z_1, Du_2). \end{aligned}$$

Hence

$$\int_{I\!R^N} \|Du_1\| = \int_{I\!R^N} (z_2, Du_1) \quad \text{and} \quad \int_{I\!R^N} \|Du_2\| = \int_{I\!R^N} (z_1, Du_2).$$

In other words, z_1 is in some sense a unit vector field of normals to the level sets of u_2 and a similar thing can be said of z_2 with respect to u_1. Any two solutions of (101) should be related in this way.

6.2 Explicit Solutions

We are going to compute explicitly the evolution of the characteristic function of a ball and an annulus.

Lemma 7 *Let $u_0 = k\chi_{B_r(0)}$. Then the unique solution $u(t,x)$ of problem (88) with initial datum u_0 is given by*

$$u(t,x) = \mathrm{sign}(k)\frac{N}{r}\left(\frac{|k|r}{N} - t\right)^+ \chi_{B_r(0)}(x).$$

Observe that we may write

$$u(t,x) = \mathrm{sign}(k)\left(|k| - \frac{\mathcal{H}^{N-1}(\partial B_r(0))}{\mathcal{L}^N(B_r(0))}t\right)^+ \chi_{B_r(0)}(x).$$

Proof. Suppose that $k > 0$, the solution for $k < 0$ being constructed in a similar way. We look for a solution of (88) of the form $u(t,x) = \alpha(t)\chi_{B_r(0)}(x)$ on some time interval $(0,T)$. Then, we shall look for some $z(t) \in X(I\!R^N)_2$ with $\|z\|_\infty \leq 1$, such that

$$u'(t) = \mathrm{div}(z(t)) \qquad \text{in } \mathcal{D}'(I\!R^N), \tag{102}$$

$$\int_{I\!R^N} (z(t), Du(t)) = \int_{I\!R^N} \|Du(t)\|. \tag{103}$$

If we take $z(t)(x) = -\frac{x}{r}$ for $x \in \partial B_r(0)$, integrating equation (102) in $B_r(0)$ we obtain

$$\alpha'(t)\,\mathcal{L}^N(B_r(0))$$

$$= \int_{B_r(0)} \mathrm{div}(z(t))\,dx = \int_{\partial B_r(0)} z(t)\cdot\nu\,d\mathcal{H}^{N-1} = -\mathcal{H}^{N-1}(\partial B_r(0)).$$

Thus

$$\alpha'(t) = -\frac{N}{r},$$

and, therefore,

$$\alpha(t) = k - \frac{N}{r}t.$$

In that case, T must be given by $T = \frac{kr}{N}$.

To construct z in $(0,T)\times(I\!R^N \setminus B_r(0))$ we shall look for z of the form $z = \rho(\|x\|)\frac{x}{\|x\|}$ such that $\mathrm{div}(z(t)) = 0$, $\rho(r) = -1$. Since

$$\mathrm{div}(z(t)) = \nabla\rho(\|x\|)\cdot\frac{x}{\|x\|} + \rho(\|x\|)\mathrm{div}\left(\frac{x}{\|x\|}\right) = \rho'(\|x\|) + \rho(\|x\|)\frac{N-1}{\|x\|},$$

we must have

$$\rho'(s) + \rho(s)\frac{N-1}{s} = 0 \qquad \text{for} \quad s > r. \tag{104}$$

The solution of (104) such that $\rho(r) = -1$ is

$$\rho(s) = -r^{N-1}s^{1-N}.$$

Thus, in $I\!R^N \setminus B_r(0)$,

$$z(t) = -r^{N-1}\frac{x}{\|x\|^N}.$$

Consequently, the candidate for $z(t)$ is the vector field

$$z(t,x) := \begin{cases} -\frac{x}{r} & \text{if } x \in B_r(0) \text{ and } 0 \le t \le T \\ -r^{N-1}\frac{x}{\|x\|^N} & \text{if } x \in I\!R^N \setminus \overline{B_r(0)}, \text{ and } 0 \le t \le T \\ 0 & \text{if } x \in I\!R^N \text{ and } t > T, \end{cases}$$

and the corresponding function $u(t,x)$ is

$$u(t,x) = \left(k - \frac{N}{r}t\right)\chi_{B_r(0)}(x)\chi_{[0,T]}(t),$$

where $T = \frac{kr}{N}$. Let us check that $u(t,x)$ satisfies (102), (103). If $\varphi \in \mathcal{D}(I\!R^N)$ and $0 \le t \le T$, we have

$$\int_{I\!R^N} \frac{\partial z_i(t)}{\partial x_i}\varphi\, dx = -\frac{1}{r}\int_{B_r(0)} \varphi\, dx + \int_{\partial B_r(0)} \frac{x_i}{r}\frac{x_i}{r}\varphi\, d\mathcal{H}^{N-1}$$

$$-\int_{I\!R^N\setminus B_r(0)} \frac{\partial}{\partial x_i}\left(\frac{r^{N-1}x_i}{\|x\|^N}\right)\varphi\, dx - \int_{\partial B_r(0)} \frac{r^{N-1}}{r^N}x_i\frac{x_i}{r}\varphi\, d\mathcal{H}^{N-1}.$$

Hence

$$\int_{I\!R^N} \mathrm{div}(z(t))\varphi\, dx = -\frac{N}{r}\int_{B_r(0)} \varphi\, dx,$$

and consequently, (102) holds. Finally, if $0 \le t \le T$, by Green's formula , we have

$$\int_{I\!R^N} (z(t), Du(t)) = -\int_{I\!R^N} \mathrm{div}(z(t))u(t)\, dx =$$

$$-\int_{B_r(0)} \left(k - \frac{N}{r}t\right)\, \mathrm{div}(z(t))\, dx = \int_{B_r(0)} \left(k - \frac{N}{r}t\right)\frac{N}{r}\, dx =$$

$$\left(k - \frac{N}{r}t\right)\frac{N}{r}\mathcal{L}^N(B_r(0)) = \left(k - \frac{N}{r}t\right)\mathcal{H}^{N-1}\left(\partial B_r(0)\right) = \int_{I\!R^N} \|Du(t)\|.$$

Therefore (103) holds, and consequently $u(t,x)$ is the solution of (88) with initial datum $u_0 = k\chi_{B_r(0)}$. □

Lemma 8 *Let $\Omega = B_R(0) \setminus \overline{B_r(0)}$, $0 < r < R$ and $u_0 = k\chi_\Omega$. Then the unique solution $u(t,x)$ of problem (88) with initial datum u_0 is*

$$u(t,x) = \mathrm{sign(k)}\left(|k| - \frac{Per(\Omega)}{\mathcal{L}^N(\Omega)}t\right)\chi_\Omega(x) + \frac{\mathrm{Per}(B_r(0))}{\mathcal{L}^N(B_r(0))}t\chi_{B_r(0)}(x) \qquad (105)$$

$t \in [0,T_1]$, $x \in I\!R^N$, where T_1 is such that

$$T_1 \cdot \left(\frac{Per(\Omega)}{\mathcal{L}^N(\Omega)} + \frac{Per(B_r(0))}{\mathcal{L}^N(B_r(0))}\right) = |k|$$

and $u(t,x)$ evolves as the solution given in Lemma 7 until its extinction.

Proof. Let $\xi : I\!R^N \to I\!R^N$ be the vector field defined as

$$\xi(x) := \begin{cases} \dfrac{x}{r} & \text{for } x \in B_r(0), \\[2ex] \left((Rr)^{N-1} \dfrac{R+r}{\|x\|^N} - (R^{N-1} + r^{N-1}) \right) \dfrac{x}{R^N - r^N}, & x \in B_R(0) \setminus \overline{B_r(0)}, \\[2ex] -\dfrac{R^{N-1}}{\|x\|^N} x & \text{for } x \in I\!R^N \setminus \overline{B_R(0)}. \end{cases}$$

Then $\|\xi\|_\infty \leq 1$, $\mathrm{div}(\xi) = \frac{N}{r} = \frac{\mathrm{Per}(B_r(0)}{\mathcal{L}^N(B_r(0))}$ on $B_r(0)$, $\mathrm{div}(\xi) = -\frac{\mathrm{Per}(\Omega)}{\mathcal{L}^N(\Omega)}$ on $B_R(0) \setminus \overline{B_r(0)}$, $\mathrm{div}(\xi) = 0$ on $I\!R^N \setminus \overline{B_R(0)}$, and $\xi \cdot \nu^{B_r(0)} = 1$ on $\partial B_r(0)$, $\xi \cdot \nu^{B_R(0)} = -1$ on $\partial B_R(0)$. Therefore, one can check that the solution u of (88) with initial condition $u_0 = \chi_\Omega$ in $[0, T_1]$ is given by (105). At $t = T_1$, the two evolving sets reach the same height and $u(T_1, x) = \alpha \chi_{B_R(0)}$ for some $\alpha > 0$. For $t > T_1$ the solution u is equal to the solution starting from $\alpha \chi_{B_R(0)}$ (at time T_1) as it is described in Lemma 7. □

Remark 6 The above results show that there is no spatial smoothing effect, for $t > 0$, similar to the case of the linear heat equation and many other quasilinear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity: $u(t,.) \in BV(I\!R^N) \setminus W^{1,1}(I\!R^N)$.

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