1 Harmonic Functions

1.1 Definition in an open set of $\mathbb{R}^2$

A function $f$definited in an open set A of $\mathbb{R}^2$ and twice continuously differentiable in A is harmonic in A if satisfies the following partial differential equation:

$$f_{xx}(x,y) + f_{yy}(x,y) = 0 \quad (x,y) \in A$$

The above equation is called Laplace's equation. A function is harmonic if it satisfies Laplace's equation.

The operator $\Delta = \nabla^2$ is called the Laplacian $\Delta f = \nabla^2 f$ the laplacian of f. Constant functions and linear functions are harmonic functions. Many other functions satisfy the equation.

As example, we observe that in all the space $\mathbb{R}^2$ the following functions are harmonic

$$f(x,y) = x^2 - y^2$$
$$f(x,y) = \ln(x^2 + y^2)$$
$$f(x,y) = e^x \sin y$$
$$f(x,y) = e^x \cos y$$

Recall

$$e^z = e^x \cos y + ie^x siny.$$

From complex analysis we have

Let $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$.

If $f(z) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations on a region A then both u and v are harmonic functions on A. This is a consequence of the Cauchy-Riemann equations. Since $u_x = v_y$ we have $u_{xx} = v_{yx}$. Likewise, $u_y = -v_x$ implies $u_{yy} = -v_{xy}$. Since we assume $v_x = v_{yx}$ we have $u_{xx} + u_{yy} = 0$. Therefore u is harmonic. Similarly for v.

As example we may consider

$$e^z = e^x \cos y + ie^x siny$$

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1.2 Poisson formula in the circle

We consider the Laplace's equation in the circle $x^2 + y^2 < R^2$, with a prescribed function at the boundary $x^2 + y^2 = R^2$.

$$f_{xx}(x,y) + f_{yy}(x,y) = 0 \qquad x^2 + y^2 < R^2$$
$$f(x,y) = g(x,y) \qquad x^2 + y^2 = R^2.$$

This is the Dirichlet problem for the Laplace equation in the circle

Since we are looking for the solution in the circle we consider polar coordinates

$F(r,\theta) = f(r\cos\theta, r\sin\theta)$

Solving in polar coordinates we get

$$F_{rr}(r\,\theta) + \frac{1}{r}F_r(r,\theta) + \frac{1}{r^2}(r\cos\theta, r\sin\theta) = 0,$$
$$0 \le r < R\ 0 \le \theta \le 2\pi$$

$$F(R,\theta) = G(\theta) = g(r\cos\theta, r\sin\theta)$$

$0 \le \theta \le 2\pi$

We assume that the solution may be obtained as a product of two functions, one depending on r and the other one on θ.

$$F(R,\theta) = H(r)K(\theta)$$

K is bounded and 2πperiodic, and H bounded.

By substitution since K is assumed bounded and 2πperiodic, we have

(i) $K''(\theta) = -m^2K(\theta)$ $K(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$

(ii) $r^2H''(r) + rH'(r) - m^2H(r) = 0$

This is the most common Cauchy-Euler equation appearing in a number of physics and engineering applications, such as when solving Laplace's equation in polar coordinates.

Assuming the solution of the form r^α and substituting into the equation

(ii) $\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - m^2r^\alpha = 0$

$$\alpha - m^2 = 0$$

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Since H is bounded we obtain the solutions

$F_m(r,\theta) = r^m(a_m\cos(m\theta) + b_m\sin(m\theta))$,

and

$$F(r,\theta) = a_0 + \sum_{k=1}^{+\infty} r^m(a_m\cos(m\theta) + b_m\sin(m\theta))$$

Now taking the Fourier expansion of G

$$G(\theta) = \frac{1}{2}\alpha_0 + \sum_{m=1}^{+\infty}(\alpha_m\cos(m\theta) + \beta_m\sin(m\theta))$$

α_m and β_m are the Fourier coefficients of the function G

$$\alpha_m = \frac{1}{\pi}\int_0^{2\pi} G(\phi)\cos(m\phi)d\phi$$

$$\beta_m = \frac{1}{\pi}\int_0^{2\pi} G(\phi)\sin(m\phi)d\phi$$

Observe that from $F(R,\theta) = G(\theta)$. Hence we have the following

$$a_0 = \frac{1}{2}\alpha_0 \quad a_m = R^{-m}\alpha_m \quad b_m = R^{-m}\beta_m$$

Substituting the Fourier coefficients into the F

$$F(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} G(\theta)[\frac{1}{2} + \sum_{m=1}^{+\infty}\left(\frac{r}{R}\right)^m \cos(m(\phi-\theta))]d\theta,$$

Next we observe

$$\sum_{m=1}^{+\infty}\left(\frac{r}{R}\right)^m e^{im(\phi-\theta)} =$$

$$\frac{1}{1-\frac{r}{R}e^{i(\phi-\theta)}} - 1 = \frac{1}{1-\frac{r}{R}e^{i(\phi-\theta)}} = \frac{R}{R - r\cos(\phi-\theta) - ir\sin(\phi-\theta)}$$

Then

$$\frac{R(R - r\cos(\phi-\theta) + ir\sin(\phi-\theta))}{(R - r\cos(\phi-\theta) - ir\sin(\phi-\theta))(R - r\cos(\phi-\theta) + ir\sin(\phi-\theta))} =$$

$$\frac{R^2 - rR\cos(\phi-\theta) - iRr\sin(\phi-\theta))}{(R^2 - 2Rr\cos(\phi-\theta)) + r^2}$$

Taking the real part of the above computation

$$F(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} G(\phi)\left(\frac{R^2 - rR\cos(\phi-\theta)}{R^2 - 2Rr\cos(\phi-\theta) + r^2} - \frac{1}{2}\right)d\phi =$$
$$= \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{G(\phi)}{R^2 - 2Rr\cos(\phi-\theta) + r^2}d\phi$$

This is the Poisson formula for the Dirichlet problem of the Laplacian in the circle.