Maximum principle for L^p-viscosity solutions of fully nonlinear elliptic/parabolic PDEs

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1 Introduction

In this talk, we discuss recent results in [3] with A. Święch on the maximum principle for L^p-viscosity solutions of fully nonlinear uniformly elliptic/parabolic equations with possibly superlinear Du terms, and with unbounded coefficients and inhomogenious terms. More precisely, we are concerned with the following PDEs:

$$\mathcal{P}^{\pm}(D^2u) \pm \mu(x)|Du|^m = f(x) \quad \text{in } \Omega, \tag{1}$$

$$u_t + \mathcal{P}^{\pm}(D^2u) \pm \mu(x,t)|Du|^m = f(x,t) \quad \text{in } \Omega \times (0,T), \tag{2}$$

where $\mathcal{P}^{\pm}(X) = \pm \max\{\mp \text{trace}(AX) \mid \lambda I \leq A \leq \Lambda I\}$ $(X \in S^n)$ for fixed $0 < \lambda \leq \Lambda$, $\Omega \subset \mathbf{R}^n$ is a bounded domain, and $T > 0$ given. Here, $m \geq 1$, and μ and f belong to L^p spaces.

We will denote by Q and $\partial_p Q$, respectively, $\Omega \times (0,T)$ and $\partial Q \setminus (\Omega \times \{T\})$.

In what follows, we only consider L^p-viscosity subsolutions of (1) and (2) for $\mathcal{P}^-$ because the other counterpart is trivially extended. Thus, we may suppose that μ and f are nonnegative.

We now mean by the maximum principle that the L^p-viscosity subsolutions of (1) and (2), respectively, satisfy

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + O(\|f\|_{L^p(\Omega)}), \qquad \text{for (1)}$$

$$\sup_{Q} u \leq \sup_{\partial_p Q} + O(\|f\|_{L^p(\Omega)}), \qquad \text{for (2)}$$

where $r \in [0,\infty) \to O(r) \in [0,\infty)$ are continuous functions (like polynomials) with $O(0) = 0$. For instance, if $p = n$, $m = 1$, $\mu \equiv 0$ and $f \in L^n(\Omega)$, we know $O(r) = Cr$ for some constant $C > 0$ as the famous Alexandrov-Bakelman-Pucci maximum principle though in this case, we may obtain more precise estimate with the upper contact set.

However, for the superlinear case, $m > 1$, there exists a counter-example even when $\mu, f \in L^\infty(\Omega)$. See our previous paper [2] for such an example when $m = 2$. We note that it is possible to get similar counter-examples also for $m > 1$ in the elliptic case (1).

2 Main results

We will use universal constants $p_1 \in [n/2, n)$ and $p_2 \in [(n+2)/2, n+1)$, respectively, for (1) and (2). In fact, for $p \in (p_1, n]$ (resp., $p \in (p_2, n+1]$), we can find strong solutions of some extremal PDEs associated with (1) (resp., (2)).

Here we only present typical results.

Theorem 1. Assume that $m > 1$, $p, q > n$, and that $\|f\|_p^{m-1}\|\mu\|_q$ is small. Then, the maximum principle holds for (1) with $O(r) = C_1(r + r^m\|\mu\|_q)$.

The next result shows a difference between elliptic and parabolic PDEs.

Theorem 2. Assume that $m \geq 1$, $p > n+2$, $f \in L^p(Q)$ and $\mu \in L^\infty(Q)$. Then, the maximum principle holds for (2) with $O(r) = C_2(r + \|\mu\|_\infty r^m)$.

Theorem 3. Assume that $m = 1$, $q > n+2$, $p \in (p_2, n+2]$. Let $\mu \in L^q(Q)$ and $f \in L^p(Q)$. Then, the maximum principle holds for (2) with $O(r) = C_3 r$.

Theorem 4. Assume that $m > 1$, $p > n+2$ and $q > n+2$. Let $f \in L^p(Q)$ and $\mu \in L^q(Q)$. Assume also that $\|f\|_p^{m-1}\|\mu\|_q$ is small. Then, the maximum principle holds for (2) with $O(r) = C_4(r + \|\mu\|_q r^m)$.

References

[1] L. A. Caffarelli, M. G. Crandall, M. Kocan & A. Święch, *Comm. Pure Appl. Math.*, **49** (1996), 365-397.

[2] S. Koike & A. Święch, *Nonlinear Differential Equations Appl.*, **11** (4) (2004), 491-509

[3] S. Koike & A. Święch, in preparation.