Exercise n 0.
Show that $\left(1-\frac{1}{n}\right)^n$ is bounded

$$0 \leq \left(1-\frac{1}{n}\right)^n < 1$$

Exercise n.1
Show that the sequence

$$x_n = \left(1-\frac{1}{n}\right)^n,$$

is an increasing sequence. We have

$$x_2 > x_1$$

For $n > 1$ consider the ratio

$$\frac{x_{n+1}}{x_n} = \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n}\right)^n} = \left(\frac{n}{n+1}\right)^{n+1}\left(\frac{n}{n-1}\right)^n$$

$$= \left(\frac{n}{n+1}\right)^{n+1}\left(\frac{n}{n-1}\right)^{n+1}\left(\frac{n-1}{n}\right) = \left(\frac{n}{(n+1)}\frac{n}{(n-1)}\right)^{n+1}\left(\frac{n-1}{n}\right)$$

$$= \left(\frac{n^2}{(n^2-1)}\right)^{n+1}\left(\frac{n-1}{n}\right) = \left(\frac{n^2-1+1}{(n^2-1)}\right)^{n+1}\left(\frac{n-1}{n}\right) = \left(\frac{n^2-1}{n^2-1}+\frac{1}{n^2-1}\right)^{n+1}\left(\frac{n}{n-1}\right)$$

By Bernoulli inequality with

$$0 < h = \frac{1}{n^2-1}, \quad \forall n > 1$$

Hence, substituing in the previous inequality

$$\frac{x_{n+1}}{x_n} = \left(1+\frac{1}{n-1}\right)\left(\frac{n}{n-1}\right) > 1, \quad \forall n > 1$$

Also, the sequence (x_n) defined by

$$x_n = \left(1-\frac{1}{n}\right)^n, \quad n = 1, 2, \ldots.$$

is increasing by an application of inequality between geometrical and arithmetic mean. Indeed, applying the inequality

$$\sqrt[n+1]{a_1 \ldots a_{n+1}} \leq \frac{a_1 + \cdots + a_{n+1}}{n+1}$$

with

$$a_1 = \cdots = a_n = 1 - \frac{1}{n} \quad \text{and} \quad a_{n+1} = 1,$$

we obtain

$$\sqrt[n+1]{\left(1-\frac{1}{n}\right)^n} \leq \frac{n}{n+1} = 1 - \frac{1}{n+1}.$$

This is equivalent to $x_n \leq x_{n+1}$. Hence (x_n) is increasing.

Exercise n 2. Compute

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n.$$

By Bernoulli inequality

$$1-\frac{1}{n}<\left(1-\frac{1}{n^2}\right)^n<1$$

Hence, passing to the limit

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n=1$$

Exercise n 3.

From the previous exercizes compute

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n.$$

We have

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\frac{1}{e}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

then

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\frac{\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

References

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