1. Approximation of e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Consider

$$\sum_{k=0}^{n+m} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \frac{1}{(n+m)!} =$$

$$\frac{1}{(n+1)!}\left(1+\frac{1}{(n+2)}+\dots \frac{1}{(n+2)\cdots(n+m)} \leq \frac{1}{(n+1)!}\left(1+\frac{1}{(n+2)}+\dots \frac{1}{(n+2)^{m-1}}\right) < \frac{1}{nn!}$$

Then,as $m \to +\infty$, we find

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{nn!}$$

$$e \approx 2,71828$$

2. Irrationality of e

We have

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{nn!},$$

assume, by contradiction, that e is a rational number, $e = \frac{p}{q}$ with p, q relative prime, hence

$$0 < \frac{p}{q} - \sum_{k=1}^{q} \frac{1}{k!} < \frac{1}{qq!},$$

and

$$0 < q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right) < \frac{1}{q},$$

and we arrive to a contradiction, since the first is an integer and the second cannot be.

3. Trascendence of e

We recall from literature a proof of Hermite's theorem on the transcendence of the number e. The assert is the following

The number e is transcendental, that is it does not satisfy any algebraic equation of integer coefficients.

Proof. If f is a polynomial of degree n, then integrating by parts we obtain that

$$\int_0^a f(x)e^{-x}\,dx + \left[e^{-x}(f(x) + f'(x) + \cdots + f^{(n)}(x))\right]_0^a = 0.$$

Putting

$$F(x) = f(x) + f'(x) + \cdots + f^{(n)}(x)$$

for brevity, it follows that

$$e^a F(0) = F(a) + e^a \int_0^a f(x)e^{-x}\ dx$$

for all real a.

Assume by contradiction that

$$c_0 + c_1 e + \cdots + c_m e^m = 0$$

for some integers $c_0, \ldots, c_m$ such that $c_0 \neq 0$. Then we deduce from the above formula the following identity:

$$0 = c_0 F(0) + c_1 F(1) + \cdots + c_m F(m) + \sum_{i=0}^{m} c_i e^i \int_0^i f(x)e^{-x}\ dx.$$

We shall arrive at a contradiction by constructing a polynomial f such that

$$|c_0 F(0) + c_1 F(1) + \cdots + c_m F(m)| \geq 1 \tag{1}$$

but

$$\Big| \sum_{i=0}^{m} c_i e^i \int_0^i f(x)e^{-x}\ dx \Big| < 1. \tag{2}$$

Fix a large prime number p, satisfying $p > m$ and $p > |c_0|$, and consider the polynomial

$$f(x) = \frac{1}{(p-1)!} x^{p-1}(x-1)^p(x-2)^p \ldots (x-m)^p.$$

Then

$$F(1),\ F(2), \ldots, F(m) \quad \text{are integer multiples of} \quad p. \tag{3}$$

Indeed, f, f',..., $f^{(p-1)}$ all vanish at 1, 2,..., m. Furthermore, developing f and then differentiating term by term we obtain that $f^{(p)}$, $f^{(p+1)}$,... are polynomials whose coefficients are integer multiples of p. Hence (3) follows.

The above reasoning also shows that

$$f(0) = f'(0) = \cdots = f^{(p-2)}(0) = 0$$

and that

$$f^{(p)}(0),\ f^{(p+1)}(0), \ldots$$

are integer multiples of p. On the other hand,

$$f^{(p-1)}(0) = (-1)^{mp}(m!)^p$$

is an integer, but not a multiple of p because $p > m$. Since $0 < |c_0| < p$, hence

$$F(0) \quad \text{is integer, but not a multiples of} \quad p. \tag{4}$$

Now (1) follows from (3) and (4).

For the proof of (2) first we remark that

$$|f(x)| \leq \frac{m^{mp+p-1}}{(p-1)!} \quad \text{if} \quad 0 \leq x \leq m.$$

Hence

$$\Big| \sum_{i=0}^{m} c_i e^i \int_0^i f(x)e^{-x}\ dx \Big| \leq \Big(\sum_{i=0}^{m} c_i e^i \Big) \frac{m^{mp+p-1}}{(p-1)!}$$

$$= \Big(\sum_{i=0}^{m} c_i e^i m^m\Big) \frac{(m^{m+1})^{p-1}}{(p-1)!}.$$

Since the last expression tends to zero as p tends to infinity, choosing a sufficiently large p hence (2) follows.

4. Stirling's Formula

James Stirling (Scotland, 1692-1770)
Approximation formula

$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n,$$

that is

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1.$$

□

References

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[2] E. Giusti, *Analisi Matematica I*, Boringhieri Ed, 1988.