1. Exercise

1. Assume $f(x) = Ax \cdot x + b \cdot x$ A is a positive-definite matrix, b any vector in $\mathbb{R}^N$. Show that f is coercive, that is

$$\lim_{|x| \to +\infty} f(x) = +\infty$$

2. Write the matrix associated to the quadratic form

$$3\lambda_1^2 + 2\lambda_1\lambda_2 + 2\lambda_3\lambda_2 - \lambda_2^2 - \lambda_3^2$$

3. *The Dynamic programming principle* **DPP**

$$u(x) = \inf_{\alpha} \big[\int_0^t f(X_x(s), \alpha(s)) e^{-\lambda s} ds + u(X_x(t)) e^{-\lambda t} \big],$$

for all real x and for all positive t.

The dynamic programming principle has been shown taking $u \in C^1(\mathbb{R}^n)$.
Show that u continuous in $\mathbb{R}^n$ satisfying DPP, verifies the property
(a). For any $\varphi \in C^1(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum in x, then

$$\lambda u(x) + \max_a \{-D\varphi(x) \cdot b(x, a) - f(x, a)\} \leq 0.$$

Hint: Use $u(x) - \varphi(x) \geq u(X_x(s)) - \varphi(X_x(s))$ for s small for s small) to get $u(x) - u(X_x(s)) \geq \varphi(x) - \varphi(X_x(s))$ for s small......)

(b). For any $\varphi \in C^1(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum in x, then

$$\lambda u(x) + \max_a \{-D\varphi(x) \cdot b(x, a) - f(x, a)\} \geq 0.$$