THE DYNAMIC PROGRAMMING PRICIPLE AND THE BELLMAN EQUATION NOTE MMII 2010-2011

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Here we give some known results to introduce the students to the subject.

1. Assumption

We consider the system

(1)
$$\begin{cases} \dot{X}(t) = b(X(t), \alpha(t)), \\ X(s) = x, \end{cases}$$

$\alpha(t)$ is the control function, measurable in $[0, +\infty)$ that takes its values in a compact set A.
In order to ensure the existence and uniqueness of the solution of (1) in the class Lip $([0, T]; \mathbb{R}^n)$, we assume that the mapping

$$b : \mathbb{R}^n \times A \to \mathbb{R}^n$$

is continuous, and satisfies the following standard assumptions:

There exist two positive constants C_1, C_2 such that for every $x, x' \in \mathbb{R}^n$, $a \in A$, we have

(2)
$$\begin{cases} \|b(x, a)\| \le C_1, \\ \|b(x, a) - b(x', a)\| \le C_2 \|x - x'\|. \end{cases}$$

The solution $X_x(t)$, starting in x at $t = 0$, is given by

$$X_x(t) = x + \int_0^{+\infty} b(X_x(s), \alpha(s)) ds$$

We assume that the mapping

$$f : \mathbb{R}^n \times A \to \mathbb{R}$$

is continuous, and satisfies the following assumptions:

There exist two positive constants C_1, C_2 such that for every $x, x' \in \mathbb{R}^n$, $a \in A$, we have

(3)
$$\begin{cases} |f(x, a)| \le C_1, \\ |f(x, a) - f(x', a)| \le C_2 \|x - x'\|. \end{cases}$$

The function f is called *the running cost.* Take $\lambda > 0$. The *functional cost* is

$$J(x,\alpha) = \int_0^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds$$

The value function is

$$u(x) = \inf_\alpha J(x,\alpha).$$

2. The Dynamic programming principle

DPP

$$u(x) = \inf_\alpha \big[\int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X_x(t))e^{-\lambda t}\big],$$

for all real x and for all positive t.
We show that

$$u(x) \geq \inf_\alpha \big[\int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X_x(t))e^{-\lambda t}\big]$$

Proof. Take any admissible control, then

$$\int_0^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds = \int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + \int_t^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds$$

$$\sigma = s - t$$

$$\int_t^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds = \int_0^{+\infty} f(X_x(\sigma+t),\alpha(\sigma+t))e^{-\lambda(\sigma+t)}d\sigma$$

We define

$$\overline{X}_x(\sigma) = X_x(\sigma+t)$$

$$\begin{cases} \dot{\overline{X}}_x(\sigma) = \dot{X}_x(t+\sigma) \\ \overline{X}(0) = X_t \end{cases} \tag{4}$$

$$\overline{X}_x(\sigma) = \overline{X}_x(0) + \int_0^\sigma \dot{\overline{X}}(s)ds = X(t) + \int_0^\sigma \dot{X}_x(t+s)ds =$$

$$X(t) + \int_0^\sigma b(X_x(t+s),\alpha(t+s)ds = X(t) + \int_0^\sigma b(\overline{X}_x(s),\overline{\alpha}(s))ds$$

We set

$$\overline{\alpha}(s) = \alpha(t+s)$$

verifies

$$\begin{cases} \dot{\overline{X}}_x(\sigma) = b(\overline{X}_x(s),\overline{\alpha}(s)) \\ \overline{X}(0) = X(t) \end{cases} \tag{5}$$

$$\int_t^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds = e^{-\lambda t}\int_0^{+\infty} f(X_x(\sigma+t),\alpha(\sigma+t))e^{-\lambda\sigma}d\sigma =$$

$$e^{-\lambda t}\int_0^{+\infty} f(\overline{X}_{X(t)}(\sigma),\overline{\alpha}(\sigma))e^{-\lambda\sigma}d\sigma \geq e^{-\lambda t}u(X(t))$$

Hence

$$u(x,t) = \inf_{\alpha} J(x,\alpha) \geq \inf_{\alpha} \int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X(t))e^{-\lambda t}$$

□

Proof. We show that

$$u(x) \leq \inf_{\alpha} \big[\int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X_x(t))e^{-\lambda t}\big]$$

Take any admissible control. Then

$$\int_0^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds = \int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + \int_t^{+\infty} f(X_x(s),\alpha(s))e^{-\lambda s}ds$$

There exists $\hat{\alpha}$ such that

$$u(X(t)) \geq J(X(t),\hat{\alpha}) - \epsilon.$$

Take any admissible control α. Set

$$\overline{\alpha}(s) = \begin{cases} \alpha(s), & 0 \leq s \leq t \\ \hat{\alpha}(s-t) & s \geq t, \end{cases}$$

and

$$\overline{X}(s) = \begin{cases} X(s), & 0 \leq s \leq t \\ \hat{X}(s-t) & s \geq t, \end{cases}$$

where

$$\begin{cases} \hat{X}(s) = b(\hat{X}(s),\hat{\alpha}(s)) \\ \hat{X}(0) = X(t). \end{cases}$$

$$\int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X_x(t))e^{-\lambda t} \geq$$

$$\int_0^t f(\overline{X}_x(s),\overline{\alpha}(s))e^{-\lambda s}ds + e^{-\lambda t}J(X(t),\hat{\alpha}) - \epsilon e^{-\lambda t} =$$

$$\int_0^t f(\overline{X}_x(s),\overline{\alpha}(s))e^{-\lambda s}ds + e^{-\lambda t}\int_0^{\infty} f(\hat{X}_{X(t)}(s),\hat{\alpha}(s))e^{-\lambda s}ds - \epsilon e^{-\lambda t} =$$

$$\int_0^t f(\overline{X}_x(s),\overline{\alpha}(s))e^{-\lambda s}ds + e^{-\lambda t}e^{\lambda t}\int_t^{+\infty} f(\overline{X}_x(s),\overline{\alpha}(s))e^{-\lambda s}ds - \epsilon e^{-\lambda t} =$$

$$\int_0^{+\infty} f(\overline{X}_x(s),\overline{\alpha}(s))e^{-\lambda s}ds - \epsilon e^{-\lambda t} \geq$$

$$u(x) - \epsilon e^{-\lambda t} \geq u(x) - \epsilon$$

□

3. The Dynamic Programming Equation

The Bellman equation is

$$\lambda u(x) + \max_a\{-Du(x)\cdot b(x,a) - f(x,a)\} = 0.$$

All the computation are done under the assumption $u \in C^1(\mathbb{R}^n)$. By the dynamic programming principle, taking $\alpha(s) = a \quad a \in A$

$$u(x) \le \int_0^t f(X_x(s),a)e^{-\lambda s}ds + u(X_x(t))e^{-\lambda t}$$

Hence

$$u(x) - u(X_x(t) \le \int_0^t f(X_x(s),a)e^{-\lambda s}ds + u(X_x(t))(e^{-\lambda t}-1)$$

and

$$\frac{u(x) - u(X_x(t)}{t} \le \frac{1}{t}\int_0^t f(X_x(s),a)e^{-\lambda s}ds + \frac{u(X_x(t))(e^{-\lambda t}-1)}{t}$$

As $t \to 0^+$ we get

$$\lambda u - Du(x)\cdot b(x,a) - f(x,a) \le 0,$$

since a is chosen in an arbitrary way

$$\lambda u(x) + \max_a\{-Du(x)\cdot b(x,a) - f(x,a)\} \le 0.$$

On the other hand if it is untrue that

$$\lambda u(x) + \max_a\{-Du(x)\cdot b(x,a) - f(x,a)\} \ge 0,$$

this means that there exists x_1 and a positive number θ such that

$$\lambda u(x_1) + \max_a\{-Du(x_1)\cdot b(x_1,a) - f(x_1,a)\} < -\theta < 0.$$

Hence for any a

$$\lambda u(x_1) - Du(x_1)\cdot b(x_1,a) - f(x_1,a) < -\theta < 0.$$

For t small enough we have also

$$\lambda u(X_{x_1}(t)) - Du(X_{x_1}(t))\cdot b(X_{x_1}(t),a) - f(X_{x_1}(t),a) < -\theta < 0.$$

By the dynamic programming principle there exists a control α such that

$$u(x) \ge \int_0^t f(X_{x_1}(s),\alpha(s))e^{-\lambda s}ds + u(X_{x_1}(s))e^{-\lambda t} - \frac{\theta t}{2}$$

$$u(x) - u(X_{x_1}(t)) \ge \int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds + u(X_{x_1}(t))(e^{-\lambda t}-1) - \frac{\theta t}{2}$$

$$u(x) - u(X_{x_1}(t)) = -\int_0^t \frac{d}{ds}u(X_{x_1}(s)) = -\int_0^t Du(X_{x_1}(s))b(u(X_{x_1}(s)),\alpha(s))ds$$

Hence

$$\int_0^t -Du(X_{x_1}(s))b(X_{x_1}(s),\alpha(s))ds - \int_0^t f(X_x(s),\alpha(s))e^{-\lambda s}ds \ge u(X_{x_1}(t))(e^{-\lambda t}-1) - \frac{\theta t}{2}$$

and

$$\int_0^t -Du(X_{x_1}(s))b(u(X_{x_1}(s)),\alpha(s))-f(X_x(s),\alpha(s))ds-\int_0^t f(X_x(s),\alpha(s))(e^{-\lambda s}-1)ds \geq$$

$$u(X_{x_1}(t))(e^{-\lambda t}-1)-\frac{\theta t}{2}$$

Since

$$\int_0^t -Du(X_{x_1}(s))b(u(X_{x_1}(s)),\alpha(s))-f(X_x(s),\alpha(s))ds \leq \int_0^t -\lambda u(X_{x_1}(s))ds-\theta t$$

we have

$$\int_0^t -\lambda u(X_{x_1}(s))ds-\theta t \geq u(X_{x_1}(t))(e^{-\lambda t}-1)-\frac{\theta t}{2}+\int_0^t f(X_x(s),\alpha(s))(e^{-\lambda s}-1)ds$$

Hence

$$0 \geq \frac{\theta t}{2}+\omega(t)$$

with $\omega(t) \to 0$ as $t \to 0$, a contradiction.

3.1. **Applications.** Take

$$b(X,\alpha) = -X\cdot\alpha \quad \text{for} \quad X,\alpha \in \mathbb{R},$$
$$J(x,\alpha) = \int_0^\infty \big(|X_x^\alpha(s)|+|\alpha(s)|\big)e^{-2s}\ ds.$$

In this case the principle of dynamic programming means that

$$u(x) = \inf_\alpha\Big(\int_0^t \big(|X_x^\alpha(s)|+|\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t}\Big) \tag{6}$$

for every $t > 0$. Hence we deduce the

Proposition 3.1. The value function $u : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(a) u is Lipschitzian;

(b) in every point $x \neq 0$ where u is differentiable, we have

$$2u(x) - |x| + \max_{|a|\leq 1}\{axu'(x) - |a|\} = 0.$$

Proof.

(a) For $x, y \in \mathbb{R}$ and $\varepsilon > 0$ fixed arbitrarily, there exists an admissible control such that

$$u(x) > \int_0^\infty \big(|X_x^\alpha(s)|+|\alpha(s)|\big)e^{-2s}\ ds - \varepsilon.$$

Since

$$u(y) \leq \int_0^\infty \big(\big|X_y^\alpha(s)\big|+|\alpha(s)|\big)e^{-2s}\ ds,$$

we have

$$\begin{aligned} u(y)-u(x) &< \int_0^\infty \big(\big|X_y^\alpha(s)\big| - |X_x^\alpha(s)|\big)e^{-2s}\ ds + \varepsilon \\ &\le \int_0^\infty \big|X_y^\alpha(s) - X_x^\alpha(s)\big|\, e^{-2s}\ ds + \varepsilon \\ &= |y-x| \int_0^\infty e^{-\int_0^s \alpha(t)\ dt} e^{-2s}\ ds + \varepsilon \\ &\le |y-x| \int_0^\infty e^{-s}\ ds + \varepsilon \\ &= |y-x| + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ and using the symmetry between x and y, we conclude that

$$|u(y)-u(x)| \le |y-x| \quad \text{for all} \quad x, y \in \mathbb{R}.$$

(b) For every sufficiently regular admissible control α we have

$$X_x^\alpha(t) = xe^{-\int_0^t \alpha(s)\ ds} = x - axt + o(t) = x + o(1), \quad a := \alpha(0^+),$$

and hence

$$u(X_x^\alpha(t)) = u(x) - axu'(x)t + o(t)$$

if $t \searrow 0$. Furthermore, recall that

$$e^{-2t} = 1 - 2t + o(t)$$

if $t \to 0$.

Using these relations, we deduce from (6) for every admissible *constant* control $\alpha = a$ that

$$\begin{aligned} u(x) &\le \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t} \\ &= \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(x) - axu'(x)t - 2u(x)t + o(t), \end{aligned}$$

whence

$$2u(x) + axu'(x) \le \frac{1}{t}\int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + o(1).$$

Letting $t \to 0$ we obtain that

$$2u(x) - |x| + \big\{axu'(x) - |a|\big\} \le 0.$$

Maximizing with respect to a, we conclude that

$$2u(x) - |x| + \max_{|a|\le 1}\big\{axu'(x) - |a|\big\} \le 0.$$

In order to show the inverse inequality, fix $t > 0$ and $\varepsilon > 0$ arbitrarily. Using (6) there exists an admissible control α such that

$$u(x) > \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t} - \varepsilon t.$$

Assuming for simplicity that this control is sufficiently regular, using the above estimates of $X_x^\alpha(t)$, $u(X_x^\alpha(t))$ and e^{-2t}, it follows that

$$u(x) > \int_0^t |x| + |a| + o(1)\ ds + u(x) - axu'(x)t - 2u(x)t + o(t) - \varepsilon t,$$

so that

$$2u(x) - |x| + \{axu'(x) - |a|\} \geq o(1) - \varepsilon.$$

Maximizing with respect to a this yields

$$2u(x) - |x| + \max_{|a|\leq 1} \{axu'(x) - |a|\} \geq o(1) - \varepsilon.$$

Now letting $t \to 0$ and then letting $\varepsilon \to 0$ we conclude that

$$2u(x) - |x| + \max_{|a|\leq 1} \{axu'(x) - |a|\} \geq 0. \quad \square$$

4. A brief remaind on Hopf-Lax formula

We assume H smooth, convex, coercive, $u_0 \in$Lip$(\mathbb{R}^N)$, $u_0 \in B(\mathbb{R}^N)$ (B reads bounded).

$$H^*(x) = \max_y \{xy - H(y)\}$$

As a reference to this part we may refer to L.C. Evans's book [5], in which the problem is split in three parts H smooth, convex, coercive $H(p)/|p| \to +\infty$ as $|p| \to +\infty$. $H^\star$ dual convex.

$$H^\star(x) = \max_y \{xy - H(y)\}$$

- **Variational Approach.**

$$\tilde{u}(x,t) = \inf \left\{ \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y) : \zeta(0) = y, \zeta(t) = x \right\}$$

- **PDEs.**
  Consider the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} v_t + H(Dv) = 0 & \text{in} \quad \mathbb{R}^{\mathrm{N}} \times (0, +\infty) \\ v(0,x) = u_0 \end{cases} \tag{7}$$

- **Hopf-Lax formula.**

$$u(x,t) = \min_{y \in R^N} \left\{ tH^*\left(\frac{x-y}{t}\right) + u_0(y) \right\} \qquad (Hopf - Lax\ formula)$$

4.1. **Reminds on viscosity solutions.** The notion of viscosity solution was introduced by M. G. Crandall and P. L. Lions [3]. Let us recall their notion using test function, as introduced in M.G. Crandall, L.C. Evans, and P.-L. Lions [4]. We use, to simplify, the vanishing viscosity method. The key point is that the notion gives a meaning to the solution of the equation also if the solution has very weak property of regularity (for example u is just a continuous function or even less regular).

The equation to consider is

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon) - \epsilon\Delta u^\epsilon = 0 \qquad \text{in} \quad (0,+\infty)\times\mathbb{R}^{\mathrm{N}} \\ u(0,x) = u_0 \end{cases}$$

Fix $\epsilon > 0$ and we consider a subsequence u^{ϵ_j}, such that

$$u^{\epsilon_j} \to u,$$

as $j \to +\infty$. Next, we consider $\phi \in C^2$ such that $u-\phi$ has a strict maximum at (x_0, t_0). We also assume that u is C^2.
Since ϵ_j is small there exists $(x_{\epsilon_j}, t_{\epsilon_j})$ such that $u^{\epsilon_j} - \phi$ has a max in $(x_{\epsilon_j}, t_{\epsilon_j})$ with

$$(x_{\epsilon_j}, t_{\epsilon_j}) \to (x_0, t_0).$$

Moreover,

$$Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = D\phi(x_{\epsilon_j}, t_{\epsilon_j})$$
$$u_t(x_{\epsilon_j}, t_{\epsilon_j}) = \phi_t(x_{\epsilon_j}, t_{\epsilon_j})$$
$$-\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \geq -\Delta\phi(x_{\epsilon_j}, t_{\epsilon_j})$$

$$\phi_t(x_{\epsilon_j}, t_{\epsilon_j}) + H(D\phi_t(x_{\epsilon_j}, t_{\epsilon_j}) = u_t(x_{\epsilon_j}, t_{\epsilon_j}) + H(Du_t(x_{\epsilon_j}, t_{\epsilon_j}) =$$
$$\epsilon_j \Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \leq \epsilon_j \Delta\phi(x_{\epsilon_j}, t_{\epsilon_j})$$

As $j \to +\infty$,

$$\phi_t(x_0, t_0) + H(D\phi_t(x_0, t_0) \leq 0$$

Then, we are now ready to recall the definition using test function.
We say that u is a (viscosity) subsolution of

$$u_t(x,t) + H(Du(x,t)) = 0$$

if for every $\phi \in C^1$ such that $u - \phi$ has a max in (x_0, t_0)

$$\phi_t(x_0, t_0) + H(Du(x_0, t_0)) \leq 0$$

We say that u is a viscosity supersolution of

$$u_t(x,t) + H(Du(x,t)) = 0$$

if for every $\phi \in C^1$ such that $u - \phi$ has a min in x

$$\phi_t(x_0, t_0) + H(D\phi(x_0, t_0)) \geq 0$$

A viscosity solution of $u_t(x,t) + H(Du(x,t)) = 0$is a viscosity subsolution and a viscosity supersolution (of $u_t(x,t) + H(Du(x,t)) = 0$)

4.2. **Equivalence of the three problems.** The three problems are equivalent, that is the $u = \tilde{u} = v$ (v being the unique viscosity solution of (7)). We give the reference of the complete proof to [5], however here the rewrite the equivalence between u and $\tilde{u}$ to give an idea how to argue in this topic. We define the trajectory

$$\zeta(s) = y + \frac{s}{t}(x-y), \quad 0 \leq s \leq t, \quad \dot{\zeta}(s) = \frac{x-y}{t}$$

By definition, for this trajectory

$$\inf\left\{\int_0^t H^*(\zeta(s))ds + u_0(y) : \zeta(0) = y, \zeta(t) = x\right\} \leq \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y) =$$

$$\int_0^t H^*\left(\frac{x-y}{t}\right)ds + u_0(y),$$

which immediately shows

$$\tilde{u}(x,t) \leq u(x,t) \tag{8}$$

Jensen's inequality gives (H^*convex)

$$H^*\left(\frac{1}{t}\int_0^t \dot{\zeta}(s)ds\right) \leq \frac{1}{t}\int_0^t H^*(\dot{\zeta}(s))ds$$

Since

$$\int_0^t \dot{\zeta}(s)ds = \zeta(t) - \zeta(0) = x - y$$

$$tH^*\left(\frac{x-y}{t}\right) \leq \int_0^t H^*(\dot{\zeta}(s))ds$$

$$tH^*\left(\frac{x-y}{t}\right) + u_0(y) \leq \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y)$$

Passing to the inf

$$u(x,t) \leq \tilde{u}(x,t), \tag{9}$$

hence

$$u(x,t) = \tilde{u}(x,t) \tag{10}$$

To give just an idea how to pass the equation, it is relevant to show a property of u, showing a semigroup property.

$$u(x,t) = \min_{y \in R^N}\left\{(t-s)H^*\left(\frac{x-y}{t-s}\right) + u(y,s)\right\} \tag{11}$$

Select $\hat{x}$ such that

$$u(x,t) = tH^*\left(\frac{x-\hat{x}}{t}\right) + u_0(\hat{x})$$

$$y = \frac{s}{t}x + (1 - \frac{s}{t})\hat{x}$$

$$\frac{x-y}{t-s}=\frac{x-\hat{x}}{t}=\frac{y-\hat{x}}{s}$$

$$(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)=(t-s)H^*\left(\frac{x-\hat{x}}{t}\right)+u(y,s)\le$$

$$(t-s)H^*\left(\frac{x-\hat{x}}{t}\right)+sH^*\left(\frac{y-\hat{x}}{s}\right)+u_0(\hat{x})=$$

$$tH^*\left(\frac{x-\hat{x}}{t}\right)+u_0(\hat{x})=u(x,t)$$

Passing to the min

$$\min_{y\in R^N}\left\{(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)\right\}\le u(x,t) \tag{12}$$

Next, choose $z\in R^N$

$$u(y,s)=sH^*\left(\frac{y-z}{s}\right)+u_0(z)$$

$$\frac{x-z}{t}=(1-\frac{s}{t})\frac{x-y}{t-s}+\frac{s}{t}\frac{y-z}{s}$$

By the convexity of H^*

$$H^*\left(\frac{x-z}{t}\right)\le(1-\frac{s}{t})H^*\left(\frac{x-y}{t-s}\right)+\frac{s}{t}H^*\left(\frac{y-z}{s}\right)$$

Then

$$u(x,t)\le$$

$$tH^*\left(\frac{x-z}{t}\right)+u_0(z)\le(t-s)H^*\left(\frac{x-y}{t-s}\right)+sH^*\left(\frac{y-z}{s}\right)+u_0(z)=$$

$$(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)$$

The result follows since y can be chosen in arbitrary way.
Now the check how it is possibile to connect the problem to the PDEs, assuming regularity for the function u and using the semigroup formula.
Fix $q\in\mathbb{R}^N$ $h>0$

$$u(x+hq,t+h)=\min_{y\in R^N}\left\{(t-s)H^*\left(\frac{x+hq-y}{h}\right)+u(y,t)\right\}\le$$

$$hH^*(q)+u(x,t)$$

From which we deduce that

$$\frac{u(x+hq,t+h)-u(x,t)}{h}\le H^*(q)$$

$h\to0^+$

$$qDu+u_t-H^*(q)\le0,$$

the inequality being true also for the max yields

$$u_t + H(Du) \leq 0.$$

We will not show the other inequality, refering to [5]

5. Eikonal equations

We now consider the problem of minimal exit time from an open set. Consider a system satisfying the *state equation*

$$\begin{cases} \dot{X}(s) = \alpha(s) & \text{in the open interval } \Omega = (-1,1) \\ X(0) = x \end{cases} \tag{13}$$

where the *controls* α are bounded : $|\alpha(s)| \leq 1 \quad \text{for all} \quad s$. Such a control is called *admissibile.*

Problem: find α such that the system attains the boundary of Ω in the smallest possible time $T(x)$. A direct computation shows that $T(x) = 1 - |x|$ for all $x \in [-1,1]$, and for each fixed $x \in [-1,1] - \{0\}$ un optimal control is the constant function

$$\alpha(s) = \text{sign of } x, \quad 0 \leq s \leq T(x).$$

Moreover if $x \neq 0$ the control is unique, and depends on the time only via the system:

$$\alpha(s) = \text{sign of } X(s).$$

(so called called *feedback controls*), while for $x = 0$ there are two optimal controls: the constant functions $\alpha = 1$ and $\alpha = -1$.

Proposition 5.1.
(a) We have $T(x) = 1 - |x|$ for all $x \in [-1,1]$.

(b) For each fixed $x \in [-1,1]$, $x \neq 0$ an optimal control is the constant function

$$\alpha(s) = \text{sign of } x, \quad 0 \leq s \leq T(x).$$

Proof. If $0 \leq t < 1 - |x|$, then for every admissible control α we have

$$|X_x^\alpha(t)| = \left| x + \int_0^t \alpha(s)\ ds \right| \leq |x| + |t| < 1,$$

whence

$$T(x) \geq 1 - |x|\,.$$

Moreover, for $x \neq 0$ we have equality in the above estimate if and only if $t = 1 - |x|$ and $\alpha(s) = \text{sign of } x$ for all $0 \leq s \leq t$. ☐

- The proof shows that for $x \neq 0$ the control is unique, and it depends on the time only via the system

$$\alpha(s) = \text{sign of } X(s).$$

Controls of this type, called *feedback controls*, have much interest in the applications because they allow us to modify the state of the system on the basis of the sole knowledge of its actual state.

In the proof of the second inequality we assumed that the controls are regular. This can be avoided by an indirect argument, contained in several references cited at the end of these notes. However, we prefered to give a direct and more transparent proof.

In case $x = 0$ there are two optimal controls: the constant functions $\alpha = 1$ and $\alpha = -1$.

Proposition 5.2. The function $T : [-1, 1] \to \mathbb{R}$ satisfies the following conditions:

(a) $T(-1) = T(1) = 0$;

(b) T is Lipschitzian;

(c) $|T'(x)| - 1 = 0$ in every point $x \in (-1, 1)$ where T is differentiable and $T(x) > 0$.

Proof.

(a) Obvious from the definition.

For the proof of (b) and (c), observe that the principle of dynamic programming yields

$$T(x) = \inf_{\alpha}[T(X_x^\alpha(t)) + t] \quad \text{for every} \quad 0 \le t \le T(x). \tag{14}$$

(b) We prove that

$$|T(x) - T(y)| \le |x - y| \tag{15}$$

for every $x, y \in [-1, 1]$. Assume by symmetry that $T(x) \ge T(y)$.

The case $T(x) \le |x - y|$ is obvious:

$$T(x) - T(y) \le T(x) \le |x - y| .$$

If $T(x) > |x - y| =: t$, then take an admissible control α such that

$$\alpha(s) = \text{sign of } (y - x) \quad \text{for} \quad 0 \le s \le t := |x - y| .$$

Then $X_x^\alpha(t) = y$, so that, applying (14) we obtain $T(x) \le t + T(y)$, i.e., (15).

(c) For every sufficiently regular admissible control we have

$$X_x^\alpha(t) = x + \int_0^t \alpha(s)\ ds = x + at + o(t) = x + o(1), \quad a := \alpha(0^+),$$

and hence

$$T(X_x^\alpha(t)) = T(x) + T'(x)at + o(t)$$

if $t \searrow 0$. Choosing a constant control $\alpha = a$ and using these relations, we deduce from (14) the estimate

$$T(x) \leq T(X_x^\alpha(t)) + t = T(x) + T'(x)at + o(t) + t,$$

whence

$$-aT'(x) - 1 \leq o(1).$$

Letting $t \searrow 0$ and then maximizing with respect to a, we conclude that

$$\left|T'(x)\right| - 1 \leq 0.$$

In order to show the inverse inequality, fix $0 < t < T(x)$ and $\varepsilon > 0$ arbitrarily. Using (14), there exists an admissible control α such that

$$T(x) > t + T(X_x^\alpha(t)) - \varepsilon t.$$

Assuming that this control is regular, using the above estimate of $T(X_x^\alpha(t))$ it follows that

$$T(x) > t + T(x) + T'(x)at + o(t) - \varepsilon t,$$

whence

$$-aT'(x) - 1 > o(1) - \varepsilon.$$

Maximizing with respect to a, this yields the inequality

$$\left|T'(x)\right| - 1 > o(1) - \varepsilon.$$

Finally, letting $t \to 0$ and then $\varepsilon \to 0$, we conclude that

$$\left|T'(x)\right| - 1 \geq 0. \quad \square$$

More general, eikonal equations are

$$Du(x) = n(x),$$

for suitable functions n.

6. Uniqueness

6.1. **Harmonic functions and the maximum principle.** The following resulo show the strict minimum points of the constrained problem

(16) $$\text{Min}\, f(x) : \Delta f(x) = 0,\, x \in \overline{\Omega}$$

are located in the boundary of Ω.

Let Ω an open, bounded set. Then its boundary $\partial\Omega$ is a compact set. If f is a continuous function in $\overline{\Omega}$, the following real numbers are well defined

$$m_{\partial\Omega} = \text{Min}\,\{f(x); x \in \partial\Omega\}, \qquad M_{\partial\Omega} = \text{Max}\,\{f(x); x \in \partial\Omega\},$$

$$m_{\overline{\Omega}} = \text{Min}\,\{f(x); x \in \partial\Omega\}, \qquad M_{\overline{\Omega}} = \text{Max}\,\{f(x); x \in \partial\Omega\},$$

Then

Theorem 6.1. If $f \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ verifies

$$\Delta f = \sum_{i=1^N} \frac{\partial^2 f}{\partial x_i} = 0,$$

then

$$m_{\partial\Omega} \leq m_{\overline{\Omega}} \qquad M_{\partial\Omega} \geq M_{\overline{\Omega}}$$

Proof. For any $\epsilon > 0$ we consider

$$g_\epsilon(x) = f(x) - \epsilon\, |x|^2$$

If follows that $g \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, and

$$\Delta g_\epsilon(x) = \Delta f(x) - \epsilon \Delta(|x|^2) = -2N\epsilon < 0.$$

Then the absolute minimum of g_ϵ in $\overline{\Omega}$ has to be assumed on the boundary of Ω and

$$g_\epsilon \geq \min \{f(x) - \epsilon\, |x|^2 \, ; \, x \in \partial\Omega\}.$$

Since Ω is a bounded set there exists $R > 0$ such that $|x| \leq R$, $\forall x \in \overline{\Omega}$. Then

$$f(x) - \epsilon\, |x|^2 = g_\epsilon(x) \geq m_{\partial\Omega} - \epsilon R^2.$$

As $\epsilon \to 0$,

$$f(x) \geq m_{\partial\Omega}.$$

In an analogous way we argue for the maximum taking

$$h_\epsilon(x) = f(x + \epsilon\, |x|^2)$$

□

6.2. **Viscosity Solutions.**

Remark. Let us explain the idea of the proof. Assume that the continuous function $u - v$ admits a global minimum in some point b and a global maximum in some point c. If u and v are also differentiable in these two points, then

$$(u - v)'(b) = (u - v)'(c) = 0,$$

so that

$$u'(b) = v'(b) \quad \text{and} \quad u'(c) = v'(c).$$

Therefore we deduce from the equation (**??**) that

$$u(b) = v(b) \quad \text{and} \quad u(c) = v(c),$$

i.e.,

$$(u - v)(b) = (u - v)(c) = 0.$$

Since

$$(u - v)(b) \leq (u - v)(x) \leq (u - v)(c)$$

for every x, by definition of b and c we conclude that $u = v$.

There are two technical difficulties here:

- it is not sure that $u-v$ has maximal and minimal values because $\mathbb{R}$ is not compact;
- even if there exist such points, it is not sure that u and v are differentiable in b and c.

We overcome these difficulties by using a penalization method.

Proof. One part (a) is established, parts (b) readily follow from propositions **??** and **??**. Fix $\delta > 0$ arbitrarily. We prove the inequality $u \le v$ os part (a) in three steps.

(i) For every fixed $\varepsilon > 0$, consider the continuous function

$$w(x,y) := u(x) - v(y) - \frac{(x-y)^2}{2\varepsilon} - \frac{\delta}{2}(x^2+y^2).$$

Since the functions u and v are Lipschitzian, they increase at most linearly at infinity, so that

$$w(x,y) \to -\infty \quad \text{if} \quad |x|+|y| \to \infty.$$

Consequently, w has a global maximum in some point $(x_\varepsilon, y_\varepsilon)$.

Then the function

$$x \mapsto u(x) - v(y_\varepsilon) - \frac{(x-y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x^2+y_\varepsilon^2)$$

has a maximum in x_ε. Therefore

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon \in D^+u(x_\varepsilon)$$

and hence

$$u(x_\varepsilon) + H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon\Big) \le 0$$

because u is a subsolution. Analogously, the function

$$y \mapsto -u(x_\varepsilon) + v(y) + \frac{(x_\varepsilon - y)^2}{2\varepsilon} + \frac{\delta}{2}(x_\varepsilon^2 + y)$$

has a minimum in y_ε. Consequently,

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon \in D^-v(y_\varepsilon)$$

and therefore

$$v(y_\varepsilon) + H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) \ge 0$$

because u is a supersolution. Combining the two inequalities we obtain that

$$u(x_\varepsilon) - v(y_\varepsilon) \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta y_\varepsilon\Big).$$

For every fixed x, using the relation

$$w(x,x) \le w(x_\varepsilon, y_\varepsilon)$$

we have

$$\begin{aligned} u(x) - v(x) - \delta x^2 &\le u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2) \\ &\le u(x_\varepsilon) - v(y_\varepsilon) \end{aligned}$$

and hence

$$u(x) - v(x) - \delta x^2 \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon\Big). \tag{17}$$

(ii) Next we prove that the three sequences

$$(x_\varepsilon), \quad (y_\varepsilon) \quad \text{and} \quad \Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big)$$

are bounded. The relation

$$w(0,0) \le w(x_\varepsilon, y_\varepsilon)$$

implies the inequality

$$u(0) - v(0) \le u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2).$$

Consequently, denoting by L a Lipschitz constant of both u and v, we have

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} + \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2) \le u(x_\varepsilon) - u(0) + v(0) - v(y_\varepsilon) \le L(|x_\varepsilon| + |y_\varepsilon|).$$

Hence

$$(|x_\varepsilon| + |y_\varepsilon|)^2 \le 2(x_\varepsilon^2 + y_\varepsilon^2) \le \frac{4L}{\delta}(|x_\varepsilon| + |y_\varepsilon|)$$

and therefore

$$|x_\varepsilon| + |y_\varepsilon| \le \frac{4L}{\delta}. \tag{18}$$

Now using the inequality

$$w(x_\varepsilon, x_\varepsilon) + w(y_\varepsilon, y_\varepsilon) \le 2w(x_\varepsilon, y_\varepsilon)$$

we have

$$u(x_\varepsilon) - v(x_\varepsilon) + u(y_\varepsilon) - v(y_\varepsilon) \le 2u(x_\varepsilon) - 2v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon}.$$

Consequently,

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \le u(x_\varepsilon) - u(y_\varepsilon) + v(x_\varepsilon) - v(y_\varepsilon) \le 2L\,|x_\varepsilon - y_\varepsilon|$$

and therefore

$$\Big|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big| \le 4L.$$

(iii) Since the function H is continuous, letting $\delta \to 0$ in (17) and using (18) we obtain for every x the inequality

$$u(x) - v(x) \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big).$$

Observe that the arguments of H are bounded with respect to ε and that $x_\varepsilon - y_\varepsilon \to 0$ if $\varepsilon \to 0$. Since H is uniformly continuous in every compact set, leting $\varepsilon \to 0$ we conclude that

$$u(x) - v(x) \leq 0$$

for every x. □

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