A SHORT INTRODUCTION TO FRACTIONAL CALCULUS

RAUL DE MAIO

0.1. **Introduction.** In this notes, we will give a brief introduction to fractional calculus. In the last years, this subject has been studied in two different ways, though close. The first approach is probabilistic and we think it is the first step a mathematician has to do to build and investigate some model. There are several works which follows this approach; in particular we suggest the papers by F. Mainardi and papers and books by M.M. Meerschaert (see [1, 3]).
Their work is an overview about fractional differential models and we recommend it as a first step to understand the differences with "classical" models.
The second approach is analytical and it is the one chosen in the following. There are some reasons for this choice; fractional derivatives are interesting objects but not so easy to handle, then we decided to write this informal notes as an exercise to get more familiar with them. For this reasons, we will often stress the differences with the classical derivative and we will point out details that we find interesting from my point of view and research. On the other hand, these pages are a result for one of the courses I've attended during my Ph.D.
Moreover, we suggest the book "Fractional Differential Equations"(1999) by I. Podlubny, which is the main reference we've followed.

1. Fractional derivatives

Let's introduce the main objects of these notes. The plural in the title of this section is due to the presence of several definitions of fractional derivative. In the following, we start with the Grunwald-Letnikov's one to move then to the Riemann-Liouville's and the Caputo's ones, which will be discussed in details in the following.
In classical analysis, it is well known that applying the definition of first order derivative we obtain the formula for the derivative of order $p \in \mathbb{N}$ as

(1.1) {?} $$f^{(p)}(t) = \lim_{h\to 0} \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \left(\begin{array}{c} p \\ r \end{array} \right) f(t-rh),$$

where $n \geq p$.
We can also consider negatives values of p. Indeed, for $p \in \mathbb{N}$, we define

$$\left(\begin{array}{c} -p \\ r \end{array} \right) = (-1)^r \left(\begin{array}{c} p \\ r \end{array} \right);$$

then, replacing in the previous equation, we get

(1.2) {?} $$f^{(-p)}(t) = \lim_{h\to 0} \frac{1}{h^p} \sum_{r=0}^{n} \left(\begin{array}{c} p \\ r \end{array} \right) f(t-rh).$$

Let $a \in \mathbb{R}$ be a point and define $h = \frac{t-a}{n}$. Then, substituting h and taking the limit for $n \to +\infty$, we define the limit value

(1.3) {?} $${}_aD_t^p f(t) := \lim_{n\to +\infty} \left(\frac{n}{t-a} \right)^p \sum_{r=0}^{n} \left(\begin{array}{c} p \\ r \end{array} \right) f(t - r\frac{t-a}{n}),$$

which can be easily extended to any value of $p \in \mathbb{R}$.
This formula gives the definition of Grunwald-Letnikov derivative, but it is not the easieast to use formula to work with fractional derivative.
A first semplification comes from the Riemann-Liouville fractional derivative. Assume that the function f is $m+1-$differentiable in a closed interval $[a,t] \subset \mathbb{R}$, where $m \in \mathbb{N}$ such that $p \in$

$(m, m+1]$. Then, under these assumptions, we can write the Taylor series, with Lagrange term, for f and get

(1.4) {?}
$$\begin{aligned} D^p_{a,t} f(t) &= \sum_{k=0}^{m} \frac{f^{(k)}(t-a)^{k-p}}{\Gamma(1+k-p)} + \frac{1}{\Gamma(1+m-p)} \int_a^t (t-s)^{m-p} f^{(m+1)}(s)ds \\ &= \frac{1}{m+1-p} \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-s)^{m-p} f(s)ds =_a D^{p,RL}_t f(t). \end{aligned}$$

The last expression is it the most known definition of fractional derivative, denoted with $D^{p,RL}$ usually called **Riemann-Liouville derivative**.
Observe that the two definitions coincide only if the function f is $m+1$-differentiable.
From the same formula, we can define the **Caputo derivative** ${}_aD^{p,C}_t = D^{p,C}$

(1.5) {?}
$$D^{p,C} f(t) = \frac{1}{\Gamma(1+m-p)} \int_a^t (t-s)^{m-p} f^{(m+1)}(s)ds.$$

Then, fixed $m \in \mathbb{N}$, $a \in \mathbb{R}$, for $t \geq a$, we have for $p \in (m-1, m]$ the following derivatives:

(1.6) RLdev
$$D^{p,RL} f(t) = \frac{1}{\Gamma(1+m-p)} \left(\frac{d}{dt}\right)^{m} \int_a^t (t-s)^{m-1-p} f(s)ds,$$

the Caputo derivative

(1.7) ?Cdev?
$$D^{p,C} f(t) = \frac{1}{\Gamma(m-p)} \int_a^t (t-s)^{m-1-p} f^{(m)}(s)ds;$$

connected together by the formula

(1.8) change_formula
$$D^{p,RL} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-p}}{\Gamma(1+k-p)} + D^{p,C} f(t).$$

The last formula is fundamental. It links the two derivatives and highlights the first differences between the derivatives. Indeed, it is easy to see that the two differential operators have different domain, since the Caputo derivative requires the $m-$differentiability for the function, while the Riemann-Lioville's requires the integrability of the functions.
Another important difference between the two derivatives is given by the fractional derivative of a constant. Let $f = C > 0$, then the Caputo derivative is clearly 0 as in the classical case.
On the other hand, formula (1.8) shows that

$$D^{p,RL} C = C \frac{(t-a)^{-p}}{\Gamma(1-p)}.$$

we remark that we am assuming that $t \geq a$; however it is possible to generalise the previous definitions also for $t \leq a$. In the following, we always assume the first case and for this reason we omit a and t in our notation.

1.1. **The Riemann-Liouville integral.** In the previous section, we started from the classical notion of derivative of natural order to define the fractional one. We have seen that there is not a unique way. In this section, we stress this lack of "uniqueness" and highlight the fractional derivative as integro-differential operators.

Define the **Riemann-Liouville integral** as

(1.9) ?Ialpha?
$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} dt,$$

where Γ is the Gamma function, a a fixed point and α a positive constant.
Clearly, if $\alpha = 1$, I^1 is the Lebesgue integral.

Fixed a bounded interval $(a,b)\subset\mathbb{R}$, I^α is a linear operator over $L^1(a,b)$. Moreover, thanks to Fubini's theorem, this operator is also continuous on L^1 and it holds

$$\|I^\alpha f\|_1 \le \frac{|b-a|^\alpha}{\alpha|\Gamma(\alpha)|}\|f\|_1.$$

Futhermore, we have that, for $\alpha\to 0$, $I^\alpha f\to f$ in L^1,i.e.

$$\lim_{\alpha\to 0^+}\|I^\alpha f - f\|_1 = 0.$$

Once this integral is defined, it is easy to observe that, if $\alpha\in(m,m+1]$ for $m\in\mathbb{N}$, then

$$D^{\alpha,RL}f(t)=\frac{d^{m+1}}{dt^{m+1}}\left(I^{m+1-\alpha}f\right)(t),$$

and

$$D^{\alpha,C}f(t)=I^{m+1-\alpha}\left(\frac{d^{m+1}f}{dt^{m+1}}\right)(t).$$

1.2. **Behaviour near the lower terminal.** A first question would be the behaviour of the Riemann Liouville derivative for $t\to a^+$.
Assume that f is analytic, at least, in the interval $[a,a+\epsilon]$ for some small $\epsilon>0$.
Then, we can represent it by the Taylor series

$$f(t)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{1.10 {?}}$$

By linearity we get

$$D_t^p f(t)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{\Gamma(k-p+1)}(t-a)^{k-p}; \tag{1.11 {?}}$$

it follows that if f is analytic then, for $t\to a^+$, $D_t^p f(t)$ has the same behaviour of

$$\frac{f(a)}{\Gamma(1-p)}(t-a)^{-p} \tag{1.12 {?}}$$

and

$$\lim_{t\to a^+} D_t^p f(t)=\begin{cases}0, & p<0,\\ f(a), & p=0,\\ +\infty, & p>0.\end{cases} \tag{1.13 {?}}$$

Assume now that $f(t)$ has an integrable singularity at $t=a$, then we can write it as $f(t)=(t-a)^q g(t)$, where $g(a)\neq 0$ and $q\in(-1,0)$. Suppose that g can be represented by a Taylor series as before, hence

$$f(t)=\sum_{k=0}^{\infty}\frac{g^{(k)}(a)}{k!}(t-a)^{k+q}; \tag{1.14 {?}}$$

applying the Riemann-Liouville fractional derivative, we obtain

$$D_t^p f(t)=\sum_{k=0}^{\infty}\frac{g^{(k)}(a)}{k!}\frac{\Gamma(q+k+1)}{\Gamma(q+k+1-p)}(t-a)^{q+k-p} \tag{1.15 {?}}$$

from which follows that, for $t\to a^+$, behaves as

$$g(a)\frac{\Gamma(q+1)}{\Gamma(q-p+1)}(t-a)^{q-p}, \tag{1.16 {?}}$$

and

$$\lim_{t\to a^+} D_t^p f(t)=\begin{cases}0, & p<q\\ g(a)\frac{\Gamma(q+1)}{\Gamma(q-p+1)}, & p=q,\\ +\infty, & p>q.\end{cases} \tag{1.17 {?}}$$

1.3. **Behaviour far from the Lower Terminal.** we want know to answer to a question similar to the previous one and understand what happens far from the lower terminal a.
Let f be analytic, then, applying the Grunwald-Letnikov definition, we get

$$D_t^p f(t) = \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) \frac{(t-a)^{k-p}}{\Gamma(k-p+1)} f^{(k)}(t).$$

Applying the definition of binomial coefficient an the reflection formula for the Γ function, we can write

$$D_t^p f(t) = \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (t-a)^{k-p}}{(p-k)k!} f^{(k)}(t). \tag{1.18}$$

{?}

Assume that t is far from a, then

$$(t-a)^{k-p} = t^{k-p} \left(1 - \frac{(k-p)a}{t} + O\left(\frac{a^2}{t^2}\right) \right).$$

Therefore, $D_t^p f(t)$ behaves as

$$\frac{\Gamma(p+1)\sin(p\pi)}{\pi} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k t^{k-p}}{(p-k)k!} f^{(k)}(t) + \frac{a}{t^{p+1}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k f^{(k)}(t)}{k!} \right\},$$

i.e., for large t it behaves as

$$D_{0,t}^p f(t) + \frac{a\Gamma(p+1)\sin(p\pi)f(0)}{\pi t^{p+1}}.$$

The last expression is interesting because the impact of the initial instant a vanishes as $t \to \infty$.

2. Fractional differential equations

Once the fractional derivative is defined, it is natural to ask if it possible to solve a differential equation which involves a fractional derivative.
In this section, I'll expose the theory of fractional differential equations for the Riemann-Liouville derivative. This choice is fundamental as we will see about the initial conditions, because the their depends on the fractional derivative definition.
It is also possible to provide the same theory for the Caputo derivative thanks to formula (1.8).
Let $\alpha_j \in (0,1]$ for $j = 1, \ldots, n$ and define $\sigma_k = \sum_{j=1}^{k} \alpha_j$, for $k = 1, \ldots, n$.
For $T < \infty$, we want to study the linear problem

$$\begin{cases} D_t^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t) D_t^{\sigma_{n-j}} y(t) + p_n(t) y(t) = f(t), & t \in (0,T) \\ \left[D^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, & k = 1, \ldots, n. \end{cases} \tag{2.1}$$

fCauchy

where

$$D_t^{\sigma_k} = D_t^{\alpha_k} \ldots D_t^{\alpha_1},$$
$$D_t^{\sigma_k - 1} = D^{\alpha_k - 1} D_t^{\alpha_{k-1}} \ldots D_t^{\alpha_1},$$

and $f \in L^1(0,T)$.The definition of the operator D^{σ_k} and parameters α_j is necessary because the fractional derivative are not commutative.
We have the following theorems and we will omit the respective proofs:

⟨thm1⟩ **Theorem 2.1.** *If $f \in L^1(0,T)$, then the equation*

$$D_t^{\sigma_n} y(t) = f(t), \tag{2.2}$$

{?}

has the unique solution $y \in L^1(0,T)$, which satisfies the initial conditions in (2.1)*, given by*

$$y(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-s)^{\sigma_n - 1} f(s) ds + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1}. \tag{2.3}$$

{?}

We propose some examples. Assume $\sigma_n = \alpha \in (0,1]$ and $f(t) \equiv 0$. Then, the solution would be

$$y(t) = \frac{b_n}{\Gamma(\sigma_n)} t^{\alpha-1}.$$

Apply formula (1.6), to verify that

$${}_0D_t^{\alpha} y(t) = \frac{d}{dt} \pi csc(\alpha\pi) = 0.$$

Under the same hypothesis, if $f(t) \equiv C \in \mathbb{R}$, then

$$y(t) = -t^{\alpha} \frac{C}{\Gamma(\alpha+1)} + \frac{b_n}{\Gamma(\alpha)} t^{\alpha-1}$$

From the previous theorem, it follows some results

Theorem 2.2. *If $f \in L^1(0,T)$, and p_j, with $(j = 1,\dots,n)$ are continuous functions in the closed interval $[0,T]$, the initial-value problem (2.1) has a unique solution in $L^1(0,T)$.*

Theorem 2.3. *If f and p_j $(j = 1,\dots,n)$ are continuous functions in the closed interval $[0,T]$, then the initial value problem (2.1) has a unique solution $y(t)$ which is continuous in $[0,T]$.*

2.1. **Fractional Differential Equation of a general form.** Let us consider now the initial value problem given by

(2.4) genfCauchy
$$\begin{cases} D_t^{\sigma_n} y(t) = f(t,y), & t \in (0,T) \\ \left[D^{\sigma_k - 1} y(t)\right]_{t=0} = b_k, & k = 1,,n, \end{cases}$$

where where

$$D_t^{\sigma_k} = D_t^{\alpha_k} \dots D_t^{\alpha_1},$$
$$D_t^{\sigma_k - 1} = D^{\alpha_k - 1} D_t^{\alpha_{k-1}} \dots D_t^{\alpha_1},$$

for $a_j \in (0,1]$ for $j = 1,\dots,n$ and $\sigma_k = \sum_{j=1}^{k} \alpha_j$.
Suppose that $f(t,y)$ is defined on a domain G of a plane (t,y) and define a regione $R(h,K) \subset G$, whose points satisfy the following inequalities:

$$0 < t < h$$
$$\left| t^{1-\sigma_1} y - \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - \sigma_1} \right| \leq K,$$

where h, K are constant.
Observe that, for $n = 1$ and $\sigma_n = 1$ we obtain the classical case and $R(h,K)$ is a cylinder. In the fractional case, this set is not easy to visualize. However, it is possible to give a qualitative description.
Assume that $y(t) \in \mathbb{R}$ for all $t \in (0,h)$. Then, the previous inequality can be rewritten as

$$-Kt^{\sigma_1 - 1} + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} \leq y \leq Kt^{\sigma_1 - 1} + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1},$$

equivalent to

$$\frac{1}{t}\left(-Kt^{\sigma_1} + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i} \right) \leq y \leq \frac{1}{t}\left(Kt^{\sigma_1} + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i} \right).$$

Then, the region $R(h,K)$ is not bounded, since for small $t > 0$ the singular term $\frac{1}{t}$ wins over the fractional polynomial.

We can finally give the following theorem

Theorem 2.4. *Let $f(t,y)$ be a real-valued continuous function, defined in the domain G, satisfying in G the Lipschitz condition with respect to y such that f is bounded by $M > 0$ in G. Let also*

$$K \geq \frac{Mh^{\sigma_n - \sigma_1 + 1}}{\Gamma(1+\sigma_n)}.$$

Then there exists in the region $R(h,K)$ a unique and continuous solution $y(t)$ of problem (2.4).

?⟨21junc⟩?

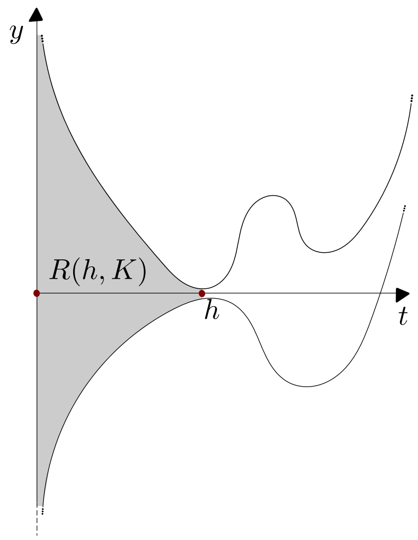

FIGURE 1. Example of $R(h,K)$

2.2. **Understanding Fractional ODE.** we want to conclude this notes with some exercises and observation to help the reader to understand and think about fractional derivative in differential equations.

In the last years, fractional calculus has been used to build differential models for thermodynamics, viscoelasticity, earthquakes, diffusion in heterogenous and porous media and other interesting application.
It is interesting that sometimes fractional ODEs fit experimental data better than classical ODEs. The fractional "behaviour comes into play when we calibrate also the derivation order through data.
In this first example, we want to understand what happens to the easy classical ODE

$$\dot{y} = \lambda y,$$

when we change the order of derivation to ϵ, closed to 1:

$$y^{(\epsilon)} = \lambda y, \tag{2.5}$$ {?}

where $\lambda \in \mathbb{R}$.
From Theorem (2.1) it seems that our solution takes the form

$$y(t) = \frac{1}{\Gamma(\epsilon)} \int_0^t \lambda (t-s)^{\epsilon-1} y(s) ds + \frac{C}{\Gamma(\epsilon)} t^{\epsilon-1}.$$

Then, substituting in the previous formula, we would get

$$\begin{aligned} y(t) &= \frac{C}{\Gamma(\epsilon)} t^{\epsilon-1} + \frac{C}{\Gamma(\epsilon)^2} \int_0^t \lambda (t-s)^{\epsilon-1} s^{\epsilon-1} ds + \frac{1}{\Gamma(\epsilon)^2} \int_0^t \lambda^2 (t-s)^{\epsilon-1} \int_0^s (s-r)^{\epsilon-1} dr \\ &= \frac{C}{\Gamma(\epsilon)} t^{\epsilon-1} + \frac{C}{\Gamma(\epsilon)^2} t^{2\epsilon-1} \lambda \int_0^1 (1-z)^{\epsilon-1} z^{\epsilon-1} dz + \frac{1}{\Gamma(\epsilon)^2} \int_0^t \lambda^2 (t-s)^{\epsilon-1} \int_0^s (s-r)^{\epsilon-1} dr \\ &= \frac{C}{\Gamma(\epsilon)} t^{\epsilon-1} + \frac{C}{\Gamma(\epsilon)^2} t^{2\epsilon-1} \lambda \frac{\Gamma(\epsilon)^2}{\Gamma(2\epsilon)} + \frac{1}{\Gamma(\epsilon)^2} \int_0^t \lambda^2 (t-s)^{\epsilon-1} \int_0^s (s-r)^{\epsilon-1} dr. \end{aligned}$$

Then, applying iteratively, we get the following expression for y:

(2.6) {?}
$$y(t) = \frac{C}{\Gamma(\epsilon)} t^{\epsilon-1} + \int_0^t \sum_{j=1}^{+\infty} \frac{1}{\Gamma(j\epsilon)} (t-s)^{j\epsilon-1} \lambda^j \frac{C}{\Gamma(\epsilon)} s^{\epsilon-1} ds.$$

With some computation, we find out then that
$$y(t) = Ct^{\epsilon-1}\mathbb{E}_{\epsilon,\epsilon}(\lambda t^{\epsilon}),$$
where $\mathbb{E}_{\alpha,\beta}(z)$ is the Mittag Leffler function defined by
$$\mathbb{E}_{\alpha,\beta}(z) := \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$
Observe that for $\epsilon \in (0,1)$, the function $y(t)$ has a singularity at $t=0$. Indeed, by the illustrated theory, the solution to a fODE belongs to $L^1(0,T)$ and it is not necessarily continuous.
For $\epsilon = 1$, we formally have $\overline{y}(t) = y(t)_{\epsilon=0} = C\mathbb{E}_{1,1}(\lambda t)$.
It is possible to verify that $y = y(\cdot,\epsilon) \to \overline{y}$ for $\epsilon \to 1$ respect to the L^1 convergence.

We want to conclude this notes with an exercise to understand what happens when a fractional derivative appears in an ordinary differential equation, i.e. which is the role of a small perturbation given by a fractional derivative.
Consider the *fODE*

(2.7) disturb
$$\dot{y} + \epsilon y^{(\alpha)} + y = 0,$$
where $\epsilon << 1$ and $\alpha \in (0,1)$.
The initial condition are given by $y(0) = A$ and $y^{(\alpha-1)}(0) = B$, with $A, B \in \mathbb{R}$.
Since this problem would seem similar to the Duffing's oscillator, we try at first to look for a solution to this problem with perturbative development: assume that
$$y = \sum_{i\geq 0} \epsilon^i y_i(t),$$

and substitute it in (2.7); we get

(2.8) {?}
$$y_0' + \epsilon y_1' + \epsilon^2 y_2' + \ldots + \epsilon y_0^{(\alpha)} + \epsilon^2 y_1^{(\alpha)} + \ldots + y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots = 0.$$
Dividing by the order of ϵ, we get

(2.9) {?}
$$\begin{cases} \dot{y}_0 + y_0 = 0, & \\ \dot{y}_n + y_n = -y_{n-1}^{(\alpha)}, & n \in \mathbb{N}, \\ y(0) = A, & y^{(\alpha-1)}(0) = B. \end{cases}$$
The solution to the first equation is clearly,
$$y_0(t) = C_0 e^{-t}.$$
On the other hand, for $n \in \mathbb{N}$, we get
$$y_n(t) = e^{-t}\left[C_n + \int_0^t e^s y_{n-1}(s) ds\right],$$

which means
$$\begin{aligned} y(t) &= \sum_{i\geq 0} \epsilon^i y_i(t) \\ &= C_0 e^{-t} + e^{-t} \sum_{i\geq 1} \epsilon^i \left[\int_0^t e^s y_{i-1}^{(\alpha)}(s) ds + C_i\right] \\ y(t) &= \sum_{i\geq 0} \epsilon^i C_i e^{-t} + \int_0^t e^{s-t} \left(\sum_{i\geq 1} \epsilon^i y_{i-1}^{(\alpha)}(s)\right) ds. \end{aligned}$$

From the previous expression we can observe that

$$A = y(0) = C_0 + \sum_{i\geq 1} \epsilon^i C_i,$$

then it follows that $C_0 = A$ and $C_i = 0$ for all $i \geq 1$. Then, we have

$$y(t) = Ae^{-t} + \sum_{i\geq 1} \epsilon^i \int_0^t e^{s-t} y_{i-1}^{(\alpha)}(s) ds. \tag{2.10}$$ {?}

We would need now to verify that $y^{(\alpha-1)}(0) = B$, but it is not easy to deal with this calculus. For this reason, we will propose another method which is based on the Laplace transform.
Indeed, if we apply the Laplace transform on (2.7), we have

$$sY - A + \epsilon s^{\alpha} Y - \epsilon B + Y = 0, \tag{2.11}$$ {?}

where $Y(s)$ is the Laplace transform of y. It follows then that

$$Y(s) = (A + \epsilon B) \frac{1}{s + \epsilon s^{\alpha} + 1},$$

using the Taylor series for $\epsilon = 0$ it is equal to

$$Y(s) = (A + \epsilon B) \left(\frac{1}{1+s} \sum_{n\geq 1} \frac{(-\epsilon s^{\alpha})^n}{(1+s)^{n+1}} \right).$$

Applying the inverse Laplace transform, we finally have the solution written as a series:

$$y(t) = (A + \epsilon B) \left(e^{-t} + \sum_{n\geq 1} (-\epsilon)^n \left(\frac{t^{-\alpha n - 1}}{\Gamma(-\alpha n)} \right) \star \left(\frac{e^{-t} t^n}{\Gamma(n+1)} \right) \right), \tag{2.12}$$ {?}

where $\star$ is the convolution operator.

References

ms [1] Meerschaert, M.; Sikorskii, A., Stochastich Models for Fractional Calculus, De Gruyter Studies in Mathematics, 43, December 2011.

?pob? [2] Podlubny, I., Fractional Differential Equations, Elsevier, Series: Mathematics in Science and Engineering, Vol. 198, October 1998

mainardi [3] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London (2010),

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, "Sapienza" Università di Roma, Via Scarpa 16, 00161 Rome, Italy

E-mail address: raul.demaio@sbai.uniroma1.it