NOTES ON LINEAR CONTROL THEORY

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Contents

1. Finite-dimensional control theory

1.1. **Observability.**

We consider the system

$$x' = Ax, \quad x(0) = x_0, \quad y = Bx, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ are given matrices, and $x_0 \in \mathbb{R}^n$.

Here the elements of $\mathbb{R}^n$ and $\mathbb{R}^m$ are considered as column vectors, x is called the *state* of the system at time t, x_0 is the *initial state*, and y the *observation*.

Remark. We identify A and B with linear operators $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $B \in L(\mathbb{R}^n, \mathbb{R}^m)$ in the usual way.

We recall that for any given $x_0 \in \mathbb{R}^n$ the system has a unique solution, given by the formula

$$x(t) = e^{tA}x_0, \quad t \in \mathbb{R},$$

whence

$$y(t) = Be^{tA}x_0, \quad t \in \mathbb{R}.$$

Both functions $x : \mathbb{R} \to \mathbb{R}^n$ and $y : \mathbb{R} \to \mathbb{R}^m$ are analytical.

Fix a positive number T.

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Definition. The system (1.1) is *observable in time* T, if different initial data lead to different observations in $[0,T]$, i.e., if the linear map

$$x_0 \mapsto y|_{[0,T]} \tag{1.2}$$

is injective.

By linearity this is equivalent to the relation

$$x_0 = 0 \quad \Longleftrightarrow \quad y \equiv 0 \quad \text{on} \quad [0,T].$$

We are looking for a characterization of the triplets (A,B,T) for which the system (1.1) is observable. Here and in the sequel we denote by C^*, x^* the adjoint of the matrix C and of the vector x. Hence x^* is a row vector, and the scalar product of $\mathbb{R}$ may be expressed in the form

$$(x,y) = x^* y.$$

The following operator will prove to be very helpful for the solution of our problem:

Definition. The *observability Gramian* $W \in \mathbb{R}^{n\times n}$ associated with (A,B,T) is defined by the formula

$$W := \int_0^T e^{tA^*} B^* B e^{tA}\ dt.$$

Remark. The bilinear form, associated with W may be expressed without using the adjoints:

$$(Wx_0, \tilde{x}_0) := \int_0^T (Be^{tA}x_0, Be^{tA}\tilde{x}_0)\ dt, \quad x_0, \tilde{x}_0 \in \mathbb{R}^n, \tag{1.3}$$

where $(\cdot,\cdot)$ denotes the usual scalar product of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

In what follows we denote by $N(W)$ and $R(W)$ the kernel and range of $W \in \mathbb{R}^{n\times n}$, i.e.,

$$N(W) := \{x \in \mathbb{R}^n \ : \ Wx = 0\} \quad \text{and} \quad R(W) := \{Wx \in \mathbb{R}^n \ : \ x \in \mathbb{R}^n\}.$$

Lemma 1.1. *The observability Gramian has the following basic properties:*

(a) $W^* = W$;
(b) $(Wx_0, x_0) \ge 0$ *for all* $x_0 \in \mathbb{R}^n$;
(c) $(Wx_0, x_0) = 0 \quad \Longleftrightarrow \quad Wx_0 = 0$;
(d) $R(W) \perp N(W)$ *and* $R(W) + N(W) = \mathbb{R}^n$.

Proof. Properties (a), (b) and the implication $\Longleftarrow$ of (c) follow from the definitions.

The implication $\Longrightarrow$ of (c) follows from the generalized Cauchy–Schwarz inequality

$$|(Wx_0, \tilde{x}_0)|^2 \leq (Wx_0, x_0) \cdot (W\tilde{x}_0, \tilde{x}_0).$$

The latter can be proved in the same way as in the case of a real scalar product: the positive semidefiniteness is sufficient instead of the positive definiteness.

Since $R \perp R^\perp$ and $R + R^\perp = \mathbb{R}^n$ for every linear subspace R of $\mathbb{R}^n$, (d) will follow if we show that $R(W)^\perp = N(W)$. The proof is straightforward:

$$\begin{aligned}
x_0 \in R(W)^\perp \quad &\Longleftrightarrow \quad (x_0, W\tilde{x}_0) = 0 \quad \text{for all} \quad \tilde{x}_0 \in \mathbb{R}^n \\
&\Longleftrightarrow \quad (Wx_0, \tilde{x}_0) = 0 \quad \text{for all} \quad \tilde{x}_0 \in \mathbb{R}^n \\
&\Longleftrightarrow \quad Wx_0 = 0 \\
&\Longleftrightarrow \quad x_0 \in N(W).
\end{aligned}$$

We have used (a) and the fact that only the null vector is orthogonal to all $x_0 \in \mathbb{R}^n$. □

Lemma 1.2. *The linear maps* (1.2) *and* W *have the same kernel. Consequently, the system* (1.1) *is* observable *if and only if* $N(W) = \{0\}$.

Remark. By elementary linear algebra we have

$$N(W) = \{0\} \quad \Longleftrightarrow \quad R(W) = \mathbb{R}^n \quad \Longleftrightarrow \quad W \quad \text{is invertible.}$$

Proof. We infer from (1.3) that

$$y \equiv 0 \quad \text{in} \quad \Longleftrightarrow \quad (Wx_0, x_0) = 0.$$

We conclude by applying the preceding lemma:

$$(Wx_0, x_0) = 0 \quad \Longleftrightarrow \quad Wx_0 = 0.$$ □

Next we express the kernel of W directly in terms of A and B:

Lemma 1.3. *The following equality holds:*

$$N(W) = \bigcap_{k=0}^{\infty} N(BA^k).$$

Remark. Observe that the kernel of W does not depend on the particular value of T. Hence the observability of the system (1.1) is independent of the choice of T.

Proof. By the preceding lemma we have to show that $y(t) \equiv 0$ in $[0,T]$ if and only if $BA^k x_0 = 0$ for all $k = 0, 1, \dots$.

Since $y(t) = Be^{tA}x_0$ is an analytic function, we have $y(t) \equiv 0$ in $[0,T]$ if and only if $y^{(k)}(0) = 0$ for all $k = 0, 1, \dots$.

We conclude by observing that $y^{(k)}(0) = BA^k x_0$ for all $k = 0, 1, \dots$ by repeated differentiation. □

Everything that we have done until now remains valid if A and B are continuous linear operators in infinite-dimensional Hilbert spaces. The following lemma, however, uses the fact that we are working in finite dimension here:

Lemma 1.4. *The following equality holds:*

$$N(W) = \bigcap_{k=0}^{n-1} N(BA^k). \tag{1.4}$$

Proof. It suffices to show that if $BA^k x_0 = 0$ for $k = 0, \dots, n-1$, then we have also $BA^m x_0 = 0$ for all $m \geq n$.

By the Cayley–Hamilton theorem A^n is a linear combination of $I, A, \dots, A^{n-1}$. It follows by induction that, more generally, A^m is also a linear combination of $I, A, \dots, A^{n-1}$ for all $m \geq n$:

$$A^m = \sum_{k=0}^{n-1} c_k^m A^k.$$

Consequently we have

$$BA^m x_0 = \sum_{k=0}^{n-1} c_k^m BA^k x_0 = \sum_{k=0}^{n-1} 0 = 0. \qquad \square$$

Using the matrix notation we obtain finally the following very useful result:

Theorem 1.5. *(Kalman) The system* (1.1) *is* observable *if and only if the $n \times nm$ matrix*

$$\begin{pmatrix} B \\ BA \\ \vdots \\ BA^{n-1} \end{pmatrix}$$

has (the maximal) rank n.

In particular, the observability does not depend on T.

Proof. By Lemmas 1.2 and 1.4 the system (1.1) is *not* observable if and only there exists a nonzero vector $x_0\mathbb{R}^n$ such that $BA^kx_0 = 0$ for $k = 0, \dots, n-1$. The latter condition means exactly that the column vectors of the above matrix are linearly dependent. □

1.2. **Controllability.** Now we consider the system

$$z' = Cz + Du, \quad z(0) = z_0, \tag{1.5}$$

where $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ are given matrices, or equivalently

$$C \in L(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad D \in L(\mathbb{R}^m, \mathbb{R}^n)$$

are given linear operators, and $z_0 \in \mathbb{R}^n$.

Fix $T > 0$ again. We recall that for any given $z_0 \in \mathbb{R}^n$ and a continuous function $u : [0, T] \to \mathbb{R}^m$, the system has a unique continuously differentiable solution $x : [0, T] \to \mathbb{R}^n$, given by the *variation of constants* formula

$$z(t) = e^{tC}z_0 + \int_0^t e^{(t-s)C}Du(s)\ ds, \quad t \in [0, T]. \tag{1.6}$$

Definition.

- The state z_0 is *null controllable* if there exists a continuous function $u : [0, T] \to \mathbb{R}^m$ such that the solution of (1.5) satisfies $z(T) = 0$. We say also that the *control* u drives the system from state z_0 to rest in time T.
- The system (1.5) is *null controllable* if every state z_0 is null controllable.
- The system (1.5) is *controllable* if for any two states $z_0, z_T \in \mathbb{R}^n$ there exists a continuous function $u : [0, T] \to \mathbb{R}^m$ such that the solution of (1.5) satisfies $z(T) = z_T$. We say also that the *control* u drives the system from state z_0 to z_T in time T.

Our first result shoes that it will suffice to study the null controllability of our system.

Proposition 1.6. *The system* (1.5) *is controllable if and only if it is null controllable.*

Proof. The implication $\Longrightarrow$ is obvious. Conversely, assume that the system is null controllable, and fix $z_0, z_T \in \mathbb{R}^n$ arbitrarily. Let us choose a control u driving the system from the initial state $z_0 - e^{-CT}z_T$ to rest. Then using (1.6) we have

$$e^{TC}(z_0 - e^{-CT}z_T) + \int_0^T e^{(T-s)C}Du(s)\ ds = 0,$$

which may be rewritten as

$$e^{TC}z_0 + \int_0^T e^{(T-s)C}Du(s)\ ds = z_T.$$

The latter means that u drives the system (1.5) from z_0 to z_T. $\square$

We are going to characterize the set R of null controllable states. Since our system is linear, R is a linear subspace of $\mathbb{R}^n$, and the system (1.5) is controllable if and only if $R = \mathbb{R}^n$.

We set $A := -C^*$, $B := -D^*$, and we consider the system of the preceding subsection:

$$x' = Ax, \quad x(0) = x_0, \quad y = Bx. \tag{1.7}$$

It will be convenient to rewrite the system (1.5) in the form

$$z' = -A^*z - B^*u, \quad z(0) = z_0. \tag{1.8}$$

Introducing the controllability Gramian W as in the preceding subsection, we prove the following

Theorem 1.7. *We have $R = R(W)$. Consequently, the controllability of the system* (1.8) *is equivalent to the observability of the system* (1.7).

Proof of the inclusion $R \subset R(W)$. The solution of (1.8) is given by the formula

$$z(t) = e^{-tA^*}z_0 - \int_0^t e^{-(t-s)A^*}B^*u(s)\ ds, \quad t \in (0, T].$$

If u drives the system to rest, then

$$0 = z(T) = e^{-TA^*}\left(z_0 - \int_0^T e^{sA^*}B^*u(s)\ ds\right),$$

and therefore

$$z_0 = \int_0^T e^{sA^*}B^*u(s)\ ds.$$

It follows that $z_0 \perp N(W)$ and thus $z_0 \in R(W)$. Indeed, if $x_0 \in N(W)$, then

$$\begin{aligned}(z_0, x_0) &= \int_0^T \left(e^{sA^*}B^*u(s), x_0\right)\ ds\\ &= \int_0^T \left(u(s), Be^{sA}x_0\right)\ ds\\ &= 0,\end{aligned}$$

because $x_0 \in N(W)$ implies $Be^{sA}x_0 \equiv 0$ in $[0, T]$. $\square$

Proof of the inclusion $R(W) \subset R$. Given $z_0 \in R(W)$ arbitrarily, we have to find a control u driving the system to rest from z_0.

Let us seek a control of the form $u := Bx$, where x is the solution of (1.7) with $x_0 \in \mathbb{R}^n$ to be precised later. Using the matrix notation for commodity, we have

$$\begin{aligned} z(T) &= e^{-TA^*}\left(z_0 - \int_0^T e^{sA^*}B^*u(s)\ ds\right) \\ &= e^{-TA^*}\left(z_0 - \int_0^T e^{sA^*}B^*Be^{sA}x_0\ ds\right) \\ &== e^{-TA^*}(z_0 - Wx_0). \end{aligned}$$

Hence, choosing $x_0 \in \mathbb{R}^n$ satisfying $Wx_0 = z_0$ (this is possible by our assumption $z_0 \in R(W)$) we conclude that $z(T) = 0$. ☐

Using the matrix notation and combining Theorems 1.5 and 1.7 we obtain another fundamental result:

Theorem 1.8. *(Kalman) The following equality holds:*

$$R(W) = \sum_{k=0}^{n-1} R(C^kD). \tag{1.9}$$

Consequently, the system (1.5) *is* controllable *if and only if the $mn \times n$ matrix*

$$(D, CD, \ldots, C^{n-1}D)$$

has (maximal) rank n. In particular, the controllability does not depend on T.

Remark. The theorem also shows that the controllable states are the linear combinations of the column vectors of the above matrix.

We need a lemma:

Lemma 1.9. *We have*

$$R(T^*)^\perp = N(T) \tag{1.10}$$

for every linear operator $T \in L(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. Indeed, we have

$$\begin{aligned} x \in R(T^*)^\perp &\iff (x, T^*y) = 0 \quad \text{for all} \quad y \in \mathbb{R}^m \\ &\iff (Tx, y) = 0 \quad \text{for all} \quad y \in \mathbb{R}^m \\ &\iff Tx = 0 \\ &\iff x \in N(T). \end{aligned}$$

☐

Proof. Using (1.4) and applying the preceding lemma we have the following equivalences:

$$\begin{aligned}
x_0 \in R(W)^{\perp} \quad &\Longleftrightarrow \quad x_0 \in N(W) \\
&\Longleftrightarrow \quad x_0 \in \bigcap_{k=0}^{n-1} N(BA^k) \\
&\Longleftrightarrow \quad x_0 \in \bigcap_{k=0}^{n-1} R((A^*)^k B^*)^{\perp} \\
&\Longleftrightarrow \quad x_0 \in \left(\sum_{k=0}^{n-1} R((A^*)^k B^*) \right)^{\perp}.
\end{aligned}$$

Using the equalities $A := -C^*$ and $B := -D^*$ we conclude that

$$R(W) = \sum_{k=0}^{n-1} R((A^*)^k B^*) = \sum_{k=0}^{n-1} R(C^k D). \qquad \square$$

Remark. Let us discuss briefly the connection of the preceding theorem with optimal control theory.

In order to drive a given initial state $z_0 \in \mathbb{R}^n$ to rest, let us try to drive it as close as possible to zero. This amounts to find a control $u \in X := C([0,T];\mathbb{R}^n)$ minimizing $z(T)$, where z is the corresponding solution of (1.5). Since

$$z(T) = e^{TC} z_0 + \int_0^T e^{(T-s)C} Du(s)\ ds,$$

this is equivalent to minimize the differentiable convex function $F : X \to \mathbb{R}$ defined by the formula

$$F(u) := \left\| e^{TC} z_0 - Mu \right\|^2 \quad \text{with} \quad Mu := -\int_0^T e^{(T-s)C} Du(s)\ ds.$$

(Observe that $M \in L(X, \mathbb{R}^n)$.)

Since F is differentiable and convex, it has a global minimum in u if and only if $F'(u) = 0$. Since we have

$$F'(u)h = -2(e^{TC} z_0 - Mu, Mh)$$

by a simple computation, $F'(u) = 0$ if and only if $e^{TC} z_0 - Mu$ is orthogonal to the range $R(M)$ of M, i.e., if and only if Mu is the orthogonal projection of $e^{TC} z_0$ onto the subspace $R(M)$. Hence the minimum of F is equal to the square of the distance of $e^{TC} z_0$ from $R(M)$.

It follows that z_0 is controllable if and only if $e^{TC}z_0 \in R(M)$. This relation may be shown to be equivalent to $z_0 \in R(W)$.

Exercise. (Space control problems) We investigate the controllability of the system

$$z' = Cz + Du, \quad z(0) = z_0 \tag{1.11}$$

with either

$$C = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.12}$$

or

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.13}$$

where ω is a given positive constant.

(a) Study the controllability of (1.11) and (1.12).
(b) Study the controllability of (1.11) and (1.13).
(c) Study the controllability of (1.11) and (1.13) when one of the two columns of B is removed..

1.3. **Stabilizability.** We consider again the system

$$z' = Cz + Du, \quad z(0) = z_0, \tag{1.14}$$

with

$$C \in \mathbb{R}^{n\times n}, \quad D \in \mathbb{R}^{n\times m} \quad \text{and} \quad z_0 \in \mathbb{R}^n.$$

Definition. The system (1.14) is *stabilizable* if there exists $F \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that setting $u := Fz$, the solutions of the new system

$$z' = (C + DF)z, \quad z(0) = z_0 \tag{1.15}$$

satisfy

$$z(t) \to 0 \quad \text{as} \quad t \to \infty \tag{1.16}$$

for all $z_0 \in \mathbb{R}^n$.

Here F is called a *feedback*, and $u = Fz$ un *feedback control*.

Theorem 1.10. *Every controllable system is stabilizable. Moreover, for each fixed $\omega > 0$ there exists $F = F_\omega$ and a constant $M > 0$ such that the solutions of the system 1.15 satisfy the estimates*

$$\|z(t)\| \le M\,\|z_0\|\,e^{-\omega t} \tag{1.17}$$

for all $z_0 \in \mathbb{R}^n$ and $t \ge 0$.

Remark. A stabilizable systems is not necessarily controllable: consider for example the system with $C = -I$ and $D = 0$.

Proof. Setting $A := -C^*$ and $B := -D^*$ again, we rewrite (1.14) in the form

$$z' = -A^*z - B^*u, \quad z(0) = z_0, \tag{1.18}$$

and we consider also the dual system:

$$x' = Ax, \quad x(0) = x_0, \quad y = Bx. \tag{1.19}$$

Fix $T > 0$ and $\omega > 0$ arbitrarily, and set

$$W := e^{-2\omega s} \int_0^T e^{tA^*} B^* B e^{tA} \, dt$$

or equivalently

$$(Wx_0, \tilde{x}_0) := \int_0^T e^{-2\omega s} (Be^{As}x_0, Be^{As}\tilde{x}_0) \, ds, \quad x_0, \tilde{x}_0 \in \mathbb{R}^n.$$

Since the system (1.15) is observable by the controllability of (1.18) and by Theorem 1.7, W is a selfadjoint operator, and the associated quadratic form is positive definite. Indeed, we may repeat the proof given in Subsection 1.1 for $\omega = 0$. It follows that W is invertible, and the formula

$$(x_0, \tilde{x}_0)_W := (W^{-1}x_0, \tilde{x}_0)$$

defines a new scalar product in $\mathbb{R}^n$, equivalent to the original one. There exists thus two positive constants α, β such that

$$\alpha \|x\| \le \|x\|_W \le \beta \|x\| \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Set $F := BW^{-1}$, i.e., consider the following system:

$$z' = -(A^* + B^*BW^{-1})z, \quad z(0) = z_0.$$

We are going to show that the solutions of this system satisfy the inequality

$$\frac{d}{dt} \|z\|_W^2 \le -2\omega \|z\|_W^2 \quad \text{in} \quad \mathbb{R}_+. \tag{1.20}$$

It will follow that

$$\frac{d}{dt}\Big(e^{2\omega t} \|z(t)\|_W^2\Big) \le 0 \quad \text{in} \quad (0, +\infty]$$

and hence, integrating between 0 and t, we get

$$e^{2\omega t} \|z(t)\|_W^2 \le \|z_0\|_W^2 \quad \text{for all} \quad t \ge 0.$$

Equivalently,

$$\|z(t)\|_W \le \|z_0\|_W \, e^{-\omega t} \quad \text{for all} \quad t \ge 0,$$

and hence, setting $M := \beta/\alpha$,

$$\|z(t)\| \le M \|z_0\| e^{-\omega t} \quad \text{for all} \quad t \ge 0,$$

For the proof of (1.20) we set $x := W^{-1}z$ for brevity, so that $z = Wx$ and

$$Wx' = -(A^*W + B^*B)x.$$

Since

$$\|z\|_W^2 = (W^{-1}z, z) = (x, Wx),$$

we have the following equalities:

$$\begin{aligned}
\frac{d}{dt}\|z\|_W^2 &= (x', Wx) + (x, Wx') \\
&= (Wx', x) + (x, Wx') \\
&= -((A^*W + B^*B)x, x) - (x, (A^*W + B^*B)x) \\
&= -((A^*W + WA + 2B^*B)x, x).
\end{aligned}$$

It remains to show that

$$((A^*W + WA + 2B^*B)x, x) \ge 2\omega(Wx, x) \tag{1.21}$$

for all $x \in \mathbb{R}^n$. Indeed, applying this with $x := W^{-1}z$ we will deduce (1.20) from the preceding inequality.

For the proof of (1.21) we evaluate the integral

$$\int_0^T \frac{d}{ds}\left(e^{-2\omega s} e^{sA^*} B^*Be^{sA}\right)\, ds$$

in two different ways.

First, using the Newton–Leibniz formula, it is equal to

$$e^{-2\omega s} e^{TA^*} B^*Be^{TA} - B^*B.$$

Secondly, differentiating the product by the formula of Leibniz, the integral is equal to

$$-2\omega W + A^*W + WA.$$

Equating the two expressions we get

$$A^*W + WA + 2B^*B - 2\omega W = e^{-2\omega s} e^{TA^*} B^*Be^{TA} + B^*B,$$

and we conclude by observing that

$$e^{-2\omega s} x^* e^{TA^*} B^*Be^{TA}x + x^*B^*Bx = e^{-2\omega s}\left\|Be^{TA}x\right\|^2 + \|Bx\|^2 \ge 0$$

for all $x \in \mathbb{R}^n$. $\square$

2. Observation and control of vibrating strings

In this section we investigate the one-dimensional wave equation by using the explicit form of the solutions given by d'Alembert's formula. We will investigate systems of the form

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad (0,\ell)\times(0,T), \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,\ell], \\ u_t(x,0) = u_1(x) \quad \text{for} \quad x \in [0,\ell], \\ u(0,t) = v_0(t) \quad \text{for} \quad t \in [0,T], \\ u(\ell,t) = v_\ell(t) \quad \text{for} \quad t \in [0,T], \end{cases} \tag{2.1}$$

where c,ℓ,T are given positive constants, and $u_0 \in H^1(0,\ell)$, $u_1 \in L^2(0,\ell)$, $v_0, v_\ell \in H^1(0,T)$ are given functions, satisfying the compatibility conditions

$$v_0(0) = u_0(0) \quad \text{and} \quad v_\ell(0) = u_0(\ell). \tag{2.2}$$

They modelize the small transversal vibrations of a string of length ℓ.

We recall the following result:

Proposition 2.1. *The system* (2.1) *has a unique solution satisfying* $u \in C([0,T], H^1(0,\ell))$ *and* $u_t \in C([0,T], L^2(0,\ell))$.

Furthermore, there exist two functions $f \in H^1(0,\ell+cT)$ *and* $g \in H^1(-cT,\ell)$ *such that*

$$u(x,t) = f(x+t) + g(x-t)$$

for all $x \in [0,\ell]$ *and* $t \in [0,T]$.

2.1. Observability.

In this subsection we consider the a vibrating string with fixed endpoints:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad (0,\ell)\times(0,T), \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,\ell], \\ u_t(x,0) = u_1(x) \quad \text{for} \quad x \in [0,\ell], \\ u(0,t) = u(\ell,t) = 0 \quad \text{for} \quad t \in [0,T]. \end{cases} \tag{2.3}$$

The compatibility condition is equivalent to consider initial data $u_0 \in H_0^1(0,\ell)$ and $u_1 \in L^2(0,\ell)$, and the solution satisfies

$$u \in C([0,T]; H_0^1(0,\ell)) \quad \text{and} \quad u_t \in C([0,T]; L^2(0,\ell)).$$

We investigate the question whether by measuring the force exerted by the string at the endpoints it is possible to recover the unknown initial data. More precisely, is it possible to express the initial data by the functions $u_x(0,t)$ and $u_x(\ell,t)$ for $t \in [0,T]$?

By introducing a new unknown function by the formula

$$w(x,t) := u\left(\frac{x}{\ell}, \frac{ct}{\ell}\right)$$

we may assume without loss of generality that $\ell = c = 1$. Choosing also $T = 1$ we consider henceforth the following simplified problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad (0,1)\times(0,1), \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,1], \\ u_t(x,0) = u_1(x) \quad \text{for} \quad x \in [0,1], \\ u(0,t) = u(1,t) = 0 \quad \text{for} \quad t \in [0,1]. \end{cases} \tag{2.4}$$

By d'Alembert's formula the solution is given by the expression

$$u(x,t) = f(x+t) + g(x-t)$$

with suitable functions

$$f \in H^1(0,2) \quad \text{and} \quad g \in H^1(-1,1).$$

The initial conditions are equivalent to

$$f(x) + g(x) = u_0(x)$$

and

$$f'(x) - g'(x) = u_1(x)$$

for $x \in [0,1]$. The second equation is equivalent to

$$f(x) - g(x) = U_1(x) \quad \text{for} \quad x \in [0,1]$$

with some suitable primitive U_1 of u_1. It follows that

$$2f(x) = (u_0 + U_1)(x) \quad \text{for} \quad x \in [0,1] \tag{2.5}$$

and

$$2g(x) = (u_0 - U_1)(x) \quad \text{for} \quad x \in [0,1]. \tag{2.6}$$

Next we observe that the boundary conditions are equivalent to

$$f(t) + g(-t) = 0$$

and

$$f(1+t) + g(1-t) = 0$$

for $t \in [0,1]$. Using (2.5) and (2.6) it follows that

$$2f(y) = -2g(2-y) = -(u_0 - U_1)(2-y) \quad \text{for} \quad y \in [1,2] \tag{2.7}$$

and

$$2g(y) = -2f(-y) = -(u_0 + U_1)(-y) \quad \text{for} \quad y \in [-1,0]. \tag{2.8}$$

Notice that the formulas (2.5) and (2.7) (resp. (2.6) and (2.8)) coincide for $x = y = 1$ (resp. for $x = y = 0$) because $u_0 \in H_0^1(0,1)$ implies that $u_0(0) = u_0(1) = 1$.

Next we deduce from d'Alembert's formula that

$$u_x(0,t) = f'(t) + g'(-t)$$

and

$$u_x(1,t) = f'(1+t) + g'(1-t)$$

for $t \in [0,1]$. Using the equations (2.5)–(2.8) they may be rewritten in the form

$$u_x(0,t) = (u_0' + u_1)(t)$$

and

$$u_x(1,t) = (u_0' - u_1)(1-t)$$

for $t \in [0,1]$. Writing the latter as

$$u_x(1,1-t) = (u_0' - u_1)(t) \quad \text{for} \quad t \in [0,1]$$

and solving the resulting linear system we find that

$$2u_0'(t) = u_x(0,t) + u_x(1,1-t)$$

and

$$2u_1(t) = u_x(0,t) - u_x(1,1-t)$$

for $t \in [0,1]$. Since

$$u_0(x) = \int_0^x u_0'(t)\ dt,$$

we conclude that

$$u_0(x) = \int_0^x u_x(0,t) + u_x(1,1-t)\ dt$$

and

$$u_1(x) = u_x(0,x) - u_x(1,1-x)$$

for $x \in [0,1]$.

Returning to the original system (2.3) we obtain finally the following:

Proposition 2.2. *If $T \geq \ell/c$, then the initial data of the solutions of* (2.3) *may be expressed by the observations*

$$v_0(t) := u_x(0,t) \quad \textit{and} \quad v_\ell(t) := u_x(\ell,t)$$

via the formulas

$$u_0(x) = \frac{1}{2}\int_0^x v_0\left(\frac{x}{c}\right) + v_\ell\left(\frac{\ell - x}{c}\right)\ dx$$

and

$$u_1(x) = \frac{c}{2}\left(v_0\left(\frac{x}{c}\right) - v_\ell\left(\frac{\ell - x}{c}\right)\right)$$

for all $x \in [0, \ell]$.

2.2. **Controllability.** In this section we investigate the boundary controllability of the system (2.1). Given some initial and final states $u_0, z_0 \in H_0^1(0,\ell)$, $u_1, z_1 \in L^2(0,\ell)$, we ask whether there exist suitable *control functions* $v_0, v_\ell \in H^1(0,T)$ satisfying (2.2) and such that the solution of (2.1) satisfies the final conditions

$$u(x,T) = z_0(x) \quad \text{and} \quad u_t(x,T) = z_1(x) \quad \text{for} \quad x \in [0,\ell]? \tag{2.9}$$

Similarly to the preceding section we assume without loss of generality that $\ell = c = 1$. Choosing $T = 1$ again we consider the system

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad (0,1)\times(0,1), \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,1], \\ u_t(x,0) = u_1(x) \quad \text{for} \quad x \in [0,1], \\ u(0,t) = v_0(t) \quad \text{for} \quad t \in [0,1], \\ u(1,t) = v_1(t) \quad \text{for} \quad t \in [0,1], \end{cases} \tag{2.10}$$

and the compatibility and final conditions become

$$u_0(0) = v_0(0) \quad \text{and} \quad u_0(1) = v_1(0) \tag{2.11}$$

and

$$u(x,1) = z_0(x) \quad \text{and} \quad u_t(x,1) = z_1(x) \quad \text{for} \quad x \in [0,1]. \tag{2.12}$$

Seeking again the solutions in the form

$$u(x,t) = f(x+t) + g(x-t)$$

with suitable functions

$$f \in H^1(0,2) \quad \text{and} \quad g \in H^1(-1,1),$$

we may repeat the earlier computations leading to (2.5) and (2.6):

$$2f(x) = (u_0 + U_1)(x) \quad \text{for} \quad x \in [0,1] \tag{2.13}$$

and

$$2g(x) = (u_0 - U_1)(x) \quad \text{for} \quad x \in [0,1]. \tag{2.14}$$

Furthermore, repeating the computations leading to (2.7) and (2.8), but now using the nonhomogeneous boundary conditions

$$f(t) + g(-t) = v_0(t)$$

and

$$f(1+t) + g(1-t) = v_1(t)$$

for $t \in [0,1]$, we obtain instead of (2.7) and (2.8) the relations

$$2f(y) = 2v_1(y-1) - (u_0 - U_1)(2-y) \quad \text{for} \quad y \in [1,2] \tag{2.15}$$

and

$$2g(y) = 2v_0(-y) - (u_0 + U_1)(-y) \quad \text{for} \quad y \in [-1,0]. \tag{2.16}$$

The formulas (2.13) and (2.15) (resp. (2.14) and (2.16)) coincide for $x = y = 1$ (resp. for $x = y = 0$) because of the compatibility conditions (2.11).

Using these formulas we may express the solutions explicitly through the initial and boundary data. For example, if $t \ge \max\{x, 1-x\}$, then using (2.15)–(2.16) we obtain that

$$\begin{aligned} 2u(x,t) &= 2f(x+t) + 2g(x-t) \\ &= 2v_1(x+t-1) - (u_0 - U_1)(2-x-t) \\ &\qquad + 2v_0(t-x) - (u_0 + U_1)(t-x), \end{aligned}$$

whence differentiating we get

$$\begin{aligned} 2u_t(x,t) = 2v_1'(x+t-1) &+ (u_0' - u_1)(2-x-t) \\ &+ 2v_0'(t-x) - (u_0' + u_1)(t-x). \end{aligned}$$

Choosing $t = 1$ we conclude that

$$u(x,1) = v_1(x) + v_0(1-x) - u_0(1-x)$$

and

$$u_t(x,1) = v_1'(x) + v_0'(1-x) - u_1(1-x)$$

for all $x \in [0,1]$. Hence the final conditions (2.12) are equivalent to the following equations for all $x \in [0,1]$:

$$\begin{aligned} v_1(x) + v_0(1-x) &= u_0(1-x) + z_0(x), \\ v_1'(x) + v_0'(1-x) &= u_1(1-x) + z_1(x). \end{aligned}$$

Integrating the second equation and fixing some primitives U_1, Z_1 of u_1, z_1, we obtain the equivalent algebraic equation

$$v_1(x) - v_0(1-x) = -U_1(1-x) + Z_1(x) + c$$

with some constant c. Resolving the linear system we obtain the formulas

$$2v_1(x) = u_0(1-x) + z_0(x) - U_1(1-x) + Z_1(x) + c$$

and

$$2v_0(1-x) = u_0(1-x) + z_0(x) + U_1(1-x) - Z_1(x) - c$$

for all $x \in [0,1]$.

However, we have to choose the constant c so as to satisfy the compatibility conditions (2.11). They lead to the conditions

$$u_0(1) = z_0(0) - U_1(1) + Z_1(0) + c$$

and

$$u_0(0) = z_0(1) + U_1(0) - Z_1(1) - c.$$

Eliminating c we see that we may satisfy both conditions simultaneously if and only if

$$u_0(0) + u_0(1) + \int_0^1 u_1(s)\ ds = z_0(0) + z_0(1) - \int_0^1 z_1(s)\ ds. \tag{2.17}$$

Turning back to the original system (2.1), we obtain thus the following:

Proposition 2.3. *Let $T = \ell/c$. If $u_0, z_0 \in H_0^1(0,\ell)$ and $u_1, z_1 \in L^2(0,\ell)$ satisfy the condition*

$$u_0(0) + u_0(\ell) + \frac{1}{c}\int_0^\ell u_1(s)\ ds = z_0(0) + z_0(\ell) - \frac{1}{c}\int_0^\ell z_1(s)\ ds, \tag{2.18}$$

then there is a unique choice of control functions $v_0, v_1 \in H^1(0,T)$ satisfying (2.11) *and such that the solution of the system satisfies the final conditions* (2.12).

If the condition (2.20) *is not satisfied, then there are no suitable control functions.*

Summarizing, we have the following

Exercise.

(a) Show that in case $T < \ell/c$ the condition (2.20) is not sufficient for the existence of suitable control functions.
(b) Is the condition (2.20) necessary for $T > \ell/c$?

2.3. **Stabilizability.** Let us try to construct stabilizing boundary feedbacks for the system (2.1), i.e,, to define the controls v_0, v_ℓ by some given rule through the actual state of the system. We are looking for some mechanism which dissipate the energy

$$E(t) := \int_0^\ell u_x(x,t)^2 + u_t(x,t)^2 \, dx$$

as $t \to \infty$.

A formal computation shows that

$$\begin{aligned} E'(t) &= 2\int_0^\ell u_x u_{xt} + u_t u_{tt} \, dx \\ &= 2\int_0^\ell u_x u_{xt} + u_t u_{xx} \, dx \\ &= 2\int_0^\ell (u_x u_t)_x \, dx \\ &= 2u_x(\ell,t)u_t(\ell,t) - 2u_x(0,t)u_t(0,t). \end{aligned}$$

Hence, if we take the boundary conditions

$$u_x(0,t) = u_t(0,t) \quad \text{for} \quad u_x(\ell,t) = -u_t(\ell,t)$$

then we will get

$$E'(t) = -2u_x(\ell,t)^2 - 2u_x(0,t)^2 \le 0$$

for all t, and we may hope that $E(t) \to 0$ as $t \to \infty$.

We are thus led to investigate the following system:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in [0,\ell], \\ u_t(x,0) = u_1(x) \quad \text{for} \quad x \in [0,\ell], \\ u_x(0,t) = u_t(0,t) \quad \text{for} \quad t \in [0,\infty), \\ u_x(\ell,t) = -u_t(\ell,t) \quad \text{for} \quad t \in [0,\infty). \end{cases} \tag{2.19}$$

We admit that for any given initial data $u_0 \in H_0^1(0,\ell)$ and $u_1 \in L^2(0,\ell)$, this problem has a unique solution satisfying

$$u \in C([0,\infty); H^1(0,\ell)) \quad \text{and} \quad u_t \in C([0,\infty); L^2(0,\ell)).$$

It turns out that not only we have $E(t) \to 0$, but we have even *extinction* in finite time:

Proposition 2.4. *If $u_0 \in H_0^1(0,\ell)$ and $u_1 \in L^2(0,\ell)$ satisfy the condition*

$$u_0(0) + u_0(\ell) + \frac{1}{c}\int_0^\ell u_1(s) \, ds = 0, \tag{2.20}$$

then the solution of (2.19) *satisfies the equality*

$$u(x,t) = u_t(x,t) = 0$$

for all $x \in [0,\ell]$ *and* $t \geq T := \ell/c$.

Proof. We may assume as usual that $\ell = c = T = 1$. It is sufficient to verify that if we choose $z_0 = z_1 \equiv 0$, then the solution of the system (2.10) with the corresponding control functions

$$v_1(x) \equiv \frac{1}{2}(u_0 - U_1)(1-x)$$

and

$$v_0(1-x) \equiv \frac{1}{2}(u_0 + U_1)(1-x),$$

where U_1 is the primitive of u_1 for which $u_0(0) = U_1(0)$ and $u_0(1) = -U_1(1)$, satisfies the boundary conditions in (2.19). Indeed, then extending it by zero for $t > T$ we obtain a function satisfying (2.19), and hence *the* solution of (2.19) by the uniqueness of the solution.

During the proof of Proposition 2.3 we have computed the solution of (2.10) in the upper triangle defined by the inequalities $t \geq \max\{x, 1-x\}$ by using the formulas (2.15) and (2.16) for f and g.

In order to compute the solution of (2.10) in the left triangle defined by the inequalities $x \leq t \leq 1-x$, we have to apply (2.13) and (2.16). We obtain

$$2u(x,t) = (u_0 + U_1)(x+t) + (2v_0 - u_0 - U_1)(t-x) = (u_0 + U_1)(x+t).$$

Since the solution in this region depends only on $x+t$, we conclude that $u_x = u_t$ in this region. This yields the required boundary condition in $x = 0$.

Similarly, to compute the solution of (2.10) in the right triangle defined by the inequalities $1-x \leq t \leq x$, we have to apply (2.14) and (2.15). We obtain

$$\begin{aligned} 2u(x,t) &= 2v_1(x+t-1) - (u_0 - U_1)(2-x-t) + (u_0 - U_1)(x-t) \\ &= (u_0 - U_1)(x-t). \end{aligned}$$

Since the solution in this region depends only on $x-t$, we conclude that $u_x = -u_t$ in this region. This yields the required boundary condition in $x = 1$. $\square$