Fractional Pearson diffusions

Nikolai N. Leonenko
School of Mathematics, Cardiff University, UK

Abstract

This is a joint work with Mark M. Meerschaert and Alla Sikorskii (Michigan State University, USA)

Fractional diffusion equations are an important and useful tool in many areas of science and engineering, see, e.g., [8,9] and the references therein. In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space.

Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients. Pearson diffusions have stationary distributions of Pearson type. They includes Ornstein-Uhlenbeck, Cox-Ingersoll-Ross, and several others processes, see, e.g., [1,2,3,4]. Their stationary distributions solve the Pearson equation, developed by Pearson in 1914 to unify some important classes of distributions (e.g., normal, gamma, beta, reciprocal gamma, Student, Fisher-Snedencor). Their eigenfunction expansions involve the traditional classes of orthogonal polynomials (e.g., Hermite, Laguerre, Jacobi), as well as some less known the finite systems of classical orthogonal polynomials (e.g., Bessel, Routh-Romanovski and Fisher-Snedecor), which are orthogonal with respect to heavy-tailed distributions.

We develop fractional Pearson diffusions $X_\alpha(t) = X_1(S_t^{(\alpha)}), t \geq 0, 0 < \alpha < 1$, where $X_1(t)$ is a Pearson diffusion process and $S_t^{(\alpha)}$ is the standard inverse α-stable subordinator independent of X_1., see [5,6]. Their transition densities are shown to solve a time-fractional analogue to the diffusion equation with polynomial coefficients. Because this process is not Markovian, the stochastic solution provides additional information about the movement of particles that diffuse under this model, which is important in applications of a anomalous diffusions in physics, geophysics, chemistry, and finance. Then the correlation function of the corresponding fractional Pearson diffusion is given by

$$\mathrm{Corr}[X_\alpha(t), X_\alpha(s)] = E_\alpha(-\theta t^\alpha) + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz$$

for $t \geq s > 0$, where $E_\alpha(z)$ is the Mittag-Leffler function. It follows that fractional Pearson diffusions exhibit long-range dependence, see [6,7].

References

[1] Avram, F, Leonenko, N.N and Šuvak, N. (2013), On spectral analysis of heavy-tailed Kolmogorov-Pearson diffusion, Markov Processes and Related Fields. Volume 19, N 2 , 249-298.

[2] Avram, F., Leonenko, N. N. and Šuvak, N. (2013) Spectral representation of transition density of Fisher-Snedecor diffusion, Stochastics, 85, no. 2, 346–369.

[3] D'Ovidio, M. (2013) From Sturm-Liouville problems to fractional and anomalous diffusions. Stochastic Process. Appl., 122, no. 10, 3513–3544.

[4] D'Ovidio, M and Orsingher, E. (2011) Bessel processes and hyperbolic Brownian motions stopped at different random times. Stochastic Process. Appl. 121, no. 3, 441–465.

[5] Leonenko, N.N., Meerschaert, M.M and Sikorskii, A. (2013) Fractional Pearson diffusion, Journal of Mathematical Analysis and Applications, vol. 403, 532-546.

[6] Leonenko, N.N., Meerschaert, M.M and Sikorskii, A. (2013) Correlation structure of fractional Pearson diffusion, Computers and Mathematics with Applications, 66, 737-745.

[7] Leonenko, N.N., Meerschaert, M.M and Sikorskii, A. (2013) Covariance structure of time changed Lévy processes, Preprint.

[8] Meerschaert, M.M. and Sikorskii, A. (2012) Stochastic Models for Fractional Calculus. de Gruyter Studies in Mathematics, 43. Walter de Gruyter & Co., Berlin.

[9] Orsingher, E. and Beghin, L. (2009) Fractional diffusion equations and processes with randomly varying time. Ann. Probab. 37, no. 1, 206–249.