RETTA MINIMI QUADRATI

$$\min\left[\sum_{i=1}^{N}\left(y_i-(A+Bx_i)\right)^2\right] \qquad \sigma_x^2 \ll \sigma_y^2$$

$$\Downarrow$$

La funzione $(A+Bx_i)$ deve essere tale da minimizzare la somma dei quadrati delle distanze tra i dati MISURATI e quelli della forma funzionale $(A+Bx_i)$

Minimizzare tale funzione vuol dire DERIVARE rispetto ad A e B e porre $=\emptyset$

$$\frac{\partial}{\partial A}\left[\sum_{i=1}^{N}\left(y_i-(A+Bx_i)\right)^2\right]=2(-1)\left[\sum_{i=1}^{N}y_i-NA-B\sum_{i=1}^{N}x_i\right]$$

$$NA+B\sum_{i=1}^{N}x_i-\sum_{i=1}^{N}y_i=\emptyset$$

PRIMA EQ. NORMALE

$$\frac{\partial}{\partial B}\left[\sum_{i=1}^{N}\left(y_i-(A+Bx_i)\right)^2\right]=2\left[\sum_{i=1}^{N}\left(y_ix_i-Ax_i-Bx_i^2\right)\right]$$

$$2\sum_{i=1}^{N}x_iy_i-2A\sum_{i=1}^{N}x_i-2B\sum_{i=1}^{N}x_i^2=\emptyset$$

$$\sum_{i=1}^{N}x_i\,A+\sum_{i=1}^{N}x_i^2\,B=\sum_{i=1}^{N}x_iy_i$$

SECONDA EQ. NORMALE

$$\begin{cases} NA + \sum_{i=1}^{N} x_i B = \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i A + \sum_{i=1}^{N} x_i^2 B = \sum_{i=1}^{N} x_i y_i \end{cases}$$

Soluzione mediante metodo di Cramer

$$A = \frac{\begin{vmatrix} \sum y_i & \sum x_i \\ \sum x_i y_i & \sum x_i^2 \end{vmatrix}}{\Delta} = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{\Delta}$$

dove $\Delta = N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i \right)^2$

$$B = \frac{\begin{vmatrix} N & \sum y_i \\ \sum x_i & \sum x_i y_i \end{vmatrix}}{\Delta} = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

Trovati A e B possiamo valutare le incertezze su questi due valori σ_A e σ_B propagando l'incertezza σ_y su A e B. Per fare ciò utilizzeremo la relazione:

$$\sigma_A^2 = \sum_{j=1}^{N} \left(\frac{\partial A}{\partial y_j} \right)^2 \sigma_y^2 \quad e \quad \sigma_B^2 = \sum_{j=1}^{N} \left(\frac{\partial B}{\partial y_j} \right)^2 \sigma_y^2$$

prima però valutiamo le grandezze $\frac{\partial A}{\partial y_j}$ e $\frac{\partial B}{\partial y_j}$.

$$\frac{\partial A}{\partial y_j} = \frac{\sum_{i=1}^{N} x_i^2 - x_j \sum_{i=1}^{N} x_i}{\Delta}$$

$$\frac{\partial B}{\partial y_j} = \frac{N x_j - \sum_{i=1}^{N} x_i}{\Delta}$$

Calcoliamo σ_A^2:

$$\sigma_A^2 = \sum_{j=1}^{N} \left(\frac{\sum_{i=1}^{N} x_i^2 - x_j \sum_{i=1}^{N} x_i}{\Delta} \right)^2 \sigma_y^2$$

$$= \frac{\sigma_y^2}{\Delta^2} \sum_{j=1}^{N} \left(\left(\sum_{i=1}^{N} x_i^2\right)^2 - 2 x_j \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i^2 + x_j^2 \left(\sum_{i=1}^{N} x_i\right)^2 \right)$$

$$= \frac{\sigma_y^2}{\Delta^2} \left[N \left(\sum_{i=1}^{N} x_i^2\right)^2 - 2 \underbrace{\sum_{j=1}^{N} x_j \sum_{i=1}^{N} x_i}_{\left(\sum_{i=1}^{N} x_i\right)^2} \sum_{i=1}^{N} x_i^2 + \sum_{j=1}^{N} x_j^2 \left(\sum_{i=1}^{N} x_i\right)^2 \right]$$

$$= \frac{\sigma_y^2}{\Delta^2} \left[N \left(\sum_{i=1}^{N} x_i^2\right)^2 - \underbrace{\sum_{i=1}^{N} x_i^2}_{\text{raccogliendo questo termine}} \left(\sum_{i=1}^{N} x_i\right)^2 \right]$$

$$= \frac{\sigma_y^2}{\Delta^2} \left(\sum_{i=1}^{N} x_i^2\right) \underbrace{\left[N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 \right]}_{\Delta}$$

$$\Longrightarrow \sigma_A^2 = \frac{\sigma_y^2}{\Delta^2} \sum_{i=1}^{N} x_i^2 \cdot \cancel{\Delta} = \frac{\sigma_y^2}{\Delta} \sum_{i=1}^{N} x_i^2$$

Calcoliamo σ_B^2:

$$\sigma_B^2 = \sum_{j=1}^{N} \left(\frac{N x_j - \sum_{i=1}^{N} x_i}{\Delta} \right)^2 \sigma_y^2$$

$$= \frac{\sigma_y^2}{\Delta^2} \sum_{j=1}^{N} \left(N^2 x_j^2 - 2 N x_j \sum_{i=1}^{N} x_i + \left(\sum_{i=1}^{N} x_i \right)^2 \right)$$

$$= \frac{\sigma_y^2}{\Delta^2} \left[N^2 \sum_{j=1}^{N} x_j^2 - 2N \underbrace{\sum_{j=1}^{N} x_j \sum_{i=1}^{N} x_i}_{\left(\sum_{i=1}^{N} x_i\right)^2} + N \left(\sum_{i=1}^{N} x_i \right)^2 \right]$$

$$= \frac{\sigma_y^2}{\Delta^2} N \underbrace{\left[N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i \right)^2 \right]}_{\Delta}$$

$$\Rightarrow \sigma_B^2 = \frac{\sigma_y^2}{\Delta^{\cancel{2}}} \cdot N \cdot \cancel{\Delta} = \frac{N}{\Delta} \sigma_y^2$$

In conclusione la forma funzionale che approssima meglio il vostro set di punti sperimentali sarà:

$$\boxed{y = (A \pm \sigma_A) + (B \pm \sigma_B) \cdot x}$$