Fractional Laplacians - Navier vs Dirichlet

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Let Ω be a bounded domain with smooth boundary. We compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the "Navier" and the "Dirichlet" ones. We denote their quadratic forms by $Q^N_{s,\Omega}$ and $Q^D_{s,\Omega}$ respectively.

Theorem 1. *Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \mathrm{Dom}\,(Q^D_{s,\Omega})$, $u \not\equiv 0$, the following relations hold:*

$$Q^N_{s,\Omega}[u] > Q^D_{s,\Omega}[u], \textit{ if } 2k < s < 2k+1,\ k \in \mathbb{N}_0;$$

$$Q^N_{s,\Omega}[u] < Q^D_{s,\Omega}[u], \textit{ if } 2k-1 < s < 2k,\ k \in \mathbb{N}_0.$$

Moreover, for $u \in \mathrm{Dom}\,(Q^D_{s,\Omega})$, the following facts hold (here $F(\Omega)$ stands for the class of smooth and bounded domains containing Ω).

$$Q^D_{s,\Omega}[u] = \inf_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u] \quad , \textit{ if } \quad 2k < s < 2k+1,\ k \in \mathbb{N}_0;$$

$$Q^D_{s,\Omega}[u] = \sup_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u] \quad , \textit{ if } \quad 2k-1 < s < 2k,\ k \in \mathbb{N}_0.$$

We also give a quantitative version of the last statement.

The following theorem gives pointwise comparison of fractional Laplacians.

Theorem 2. *Let $0 < |s| < 1$, and let $f \in \mathrm{Dom}\,(Q^D_{s,\Omega})$, $f \geq 0$, $f \not\equiv 0$. Then the following relations hold:*

$$(-\Delta_\Omega)^s_N f > (-\Delta_\Omega)^s_D f, \textit{ if } 0 < s < 1;$$

$$(-\Delta_\Omega)^s_N f < (-\Delta_\Omega)^s_D f, \textit{ if } -1 < s < 0.$$

Here all inequalities are understood in the sense of distributions.

This talk is based on joint papers with Roberta Musina, see [1-3].

References

1. R. Musina, A. I. Nazarov, *On fractional Laplacians*, Comm. in PDEs, **39** (2014), N9, 1780-1790.

2. R. Musina, A. I. Nazarov, *On fractional Laplacians-2*, Annales de l'Institut Henri Poincare. Analyse Nonlineaire. DOI 10.1016/j.anihpc.2015.08.001. 7p.

3. R. Musina, A. I. Nazarov, *On fractional Laplacians-3*, 2015. Preprint arxiv.org/abs/1503.00271. To appear in ESAIM: COCV.