SVOLGIMENTO PROVA SCRITTA di ANALISI 1 del 5/2/2018

A1

1) OMOGENEA:

$$\alpha^2 - 3\alpha + 2 = 0$$

$$\Rightarrow \alpha_1 = 2 \ ; \ \alpha_2 = 1$$

$$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 e^{x}$$

NON OMOGENEA:

Principio di sovrapposizione

$f_1(x) = e^x$ $\qquad$ $\alpha = 1$ è radice del polinomio caratteristico $\Rightarrow y_1(x) = Axe^x$

$$\Rightarrow y_1'(x) = A(x+1)e^x \ ; \ y_1''(x) = A(x+2)e^x$$

$$\Rightarrow A\left[(x+2) - 3(x+1) + 2x\right]e^x = e^x$$

$$\Rightarrow A[2-3] = 1 \Rightarrow A = -1 \Rightarrow y_1(x) = -xe^x$$

$$f_2(x) = e^{3x} \Rightarrow y_2(x) = Ae^{3x}$$

$$\Rightarrow y_2'(x) = 3Ae^{3x} \ ; \ y_2''(x) = 9Ae^{3x}$$

$$\Rightarrow A(9-9+2)e^{3x} = e^{3x} \quad \Rightarrow A = \frac{1}{2} \quad A_2$$

$$\Rightarrow y_2(x) = \frac{1}{2}e^{3x}$$

$$\Rightarrow y(x) = C_1 e^{2x} + C_2 e^{x} - xe^{x} + \frac{1}{2}e^{3x}$$

(INT. GEN.)

$$y(0) = C_1 + C_2 + \frac{1}{2} = 0$$

$$y'(x) = 2C_1 e^{2x} + C_2 e^{x} - (x+1)e^{x} + \frac{3}{2}e^{3x}$$

$$y'(0) = 2C_1 + C_2 - 1 + \frac{3}{2} = 1$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{2} \\ 2C_1 + C_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -\frac{3}{2} \end{cases}$$

SOLUZIONE DEL PROBLEMA

$$y(x) = e^{2x} - \frac{3}{2}e^{x} - xe^{x} + \frac{1}{2}e^{3x}$$

2) a) $\displaystyle\int \left[\log\left(1+\frac{2}{x}\right) - \frac{2}{x}\right] dx$ A3

$$= \left[x \log\left(1+\frac{2}{x}\right) - \int \frac{x}{1+\frac{2}{x}} \left(-\frac{2}{x^2}\right) dx \right]$$

$$-2\log|x|$$

$$= x \log\left(1+\frac{2}{x}\right) + \int \frac{2}{x+2}\, dx - 2\log|x|$$

$$= x \log\left(1+\frac{2}{x}\right) + 2\log(|x+2|) - 2\log(|x|) + c$$

$$= x \log\left(1+\frac{2}{x}\right) + 2\log\left(\left|\frac{x+2}{x}\right|\right) + c$$

b) $$f(x) = \left(\cancel{\frac{2}{x}} - \frac{2}{2x^2} - \cancel{\frac{2}{x}} + o\left(\frac{1}{x^2}\right)\right)$$

$$\sim -\frac{1}{x^2} \leftarrow \text{integrabile in } [1, +\infty)$$

$\Rightarrow f(x)$ integrabile

$$\int_1^{+\infty} f(x)\,dx = \left[x\log\left(1+\frac{2}{x}\right) + 2\log\left(\left|\frac{x+2}{x}\right|\right)\right]_1^{+\infty}$$

$$= \lim_{x\to+\infty} \left[\log\left[\left(1+\frac{2}{x}\right)^x\right] + 2\log\left(\frac{x+2}{x}\right)\right] \quad A_4$$

$$-\log 3 - 2\log 3$$

$$= \log\left(e^2\right) + 2\log 1 - 3\log 3 = 2 - 3\log 3$$

3) $z(z\cdot\bar{z}) - (\bar{z})^3 = 0$

$$\bar{z}\left(z^2 - (\bar{z})^2\right) = 0$$

$\bar{z} = 0$ oppure $z^2 = (\bar{z})^2$

$\Rightarrow z = 0$

$\cancel{x^2} - \cancel{y^2} + 2ixy = \cancel{x^2} - \cancel{y^2} - 2ixy$

$\Rightarrow xy = 0$

$\Rightarrow x = 0$ oppure

$y = 0$

In alternativa, si possono usare le coordinate esponenziali:

$$\rho e^{i\vartheta} \rho^2 = (\rho e^{-i\vartheta})^3$$

$$\cancel{\rho^3} e^{i\vartheta} = \cancel{\rho^3} e^{-i3\vartheta} \qquad \Rightarrow \quad \rho = 0$$

oppure

$$e^{i\vartheta} = e^{-i3\vartheta}$$

da cui

$$\vartheta = -3\vartheta + 2k\pi$$

ovvero

$$4\vartheta = 2k\pi$$

$$\Rightarrow \vartheta = \frac{k\pi}{2}$$

$\vartheta = 0$: asse $x > 0$

$\vartheta = \frac{\pi}{2}$: asse $y > 0$

$\vartheta = \pi$: asse $x < 0$

$\vartheta = \frac{3}{2}\pi$: asse $y < 0$

Quindi le soluzioni sono tutti i punti appartenenti agli assi

(A_5)

4) CONVERGENZA ASSOLUTA:

$$\sum \left| (-1)^n \frac{n!\,2^n}{n^n} \right| = \sum \frac{n!\,2^n}{n^n}$$

$$|a_n| \sim \frac{\cancel{n^n}\sqrt{2\pi n}}{e^n \cancel{n^n}} 2^n = \left(\frac{2}{e}\right)^n \sqrt{2\pi n}$$

Criterio radice:

$$\sqrt[n]{\left(\frac{2}{e}\right)^n \sqrt{2\pi n}} = \left(\frac{2}{e}\right) \sqrt{\sqrt[n]{2\pi n}}$$

$$\xrightarrow[n \to \infty]{} \left(\frac{2}{e}\right) < 1$$

$\Rightarrow$ convergente $\Rightarrow$ la serie converge assolutamente $\Rightarrow$ converge semplicemente

In alternativa: criterio di Leibniz

(A6)

i) $$a_n = \frac{n!\,2^n}{n^n} \sim \frac{\cancel{n^n}\sqrt{2\pi n}\,2^n}{\cancel{n^n}e^n} =$$

$$= \left(\frac{2}{e}\right)^n \sqrt{2\pi n} = \frac{\sqrt{2\pi n}}{\left(\frac{e}{2}\right)^n} \xrightarrow[n\to\infty]{} 0$$

ii) $$a_{n+1} \le a_n \iff \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \le \frac{n!\,2^n}{n^n}$$

$$\iff \frac{\cancel{(n+1)}\,2\,n^n}{(n+1)^{n+\cancel{1}}} \le 1 \iff \frac{2}{\left(\frac{n+1}{n}\right)^n} \le 1$$

$$\iff 2 \le \left(1+\frac{1}{n}\right)^n \quad \text{vero } \forall n \ge N.$$

5) $D = \{x \neq -2\}$ A7

f sempre positiva

$$f(0) = \text{arctg}(1) = \frac{\pi}{4}$$

$$\cancel{f(x) = \begin{cases} \text{arctg}\left(\frac{2}{x+2}\right) & \text{se } x > -2 \\ \text{arctg}\left(\frac{2}{-x-2}\right) & \text{se } x < -2 \end{cases}}$$

$$\lim_{x \to -2^{\pm}} f(x) = \text{arctg}\left(\frac{2}{|0^{\pm}|}\right) = \text{arctg}(+\infty) = \frac{\pi}{2}$$

SINGOLARITÀ ELIMINABILE.

NO AS. VERT.

$$\lim_{x \to \pm\infty} f(x) = \text{arctg}(0) = 0$$

AS. ORIZZ. a $\pm\infty$: $y = 0$.

$$f(x) = \begin{cases} \operatorname{arctg}\left(\dfrac{2}{x+2}\right) & \text{se } x > -2 \\ \operatorname{arctg}\left(\dfrac{2}{-x-2}\right) = -\operatorname{arctg}\left(\dfrac{2}{x+2}\right) & \text{se } x < -2 \end{cases}$$

A8

$$f'(x) = \begin{cases} \dfrac{1}{1+\left(\dfrac{2}{x+2}\right)^2} \cdot \left[\dfrac{-2}{(x+2)^2}\right] = \dfrac{-2}{(x+2)^2+4} < 0 & \text{se } x > -2 \\ \dfrac{2}{(x+2)^2+4} > 0 & \text{se } x < -2 \end{cases}$$

f cresce in $(-\infty, -2)$; decresce in $(-2, +\infty)$

$\nexists$ MAX. o MIN. REL. e ASS.

$$\inf f = 0$$

$$\sup f = \frac{\pi}{2}$$

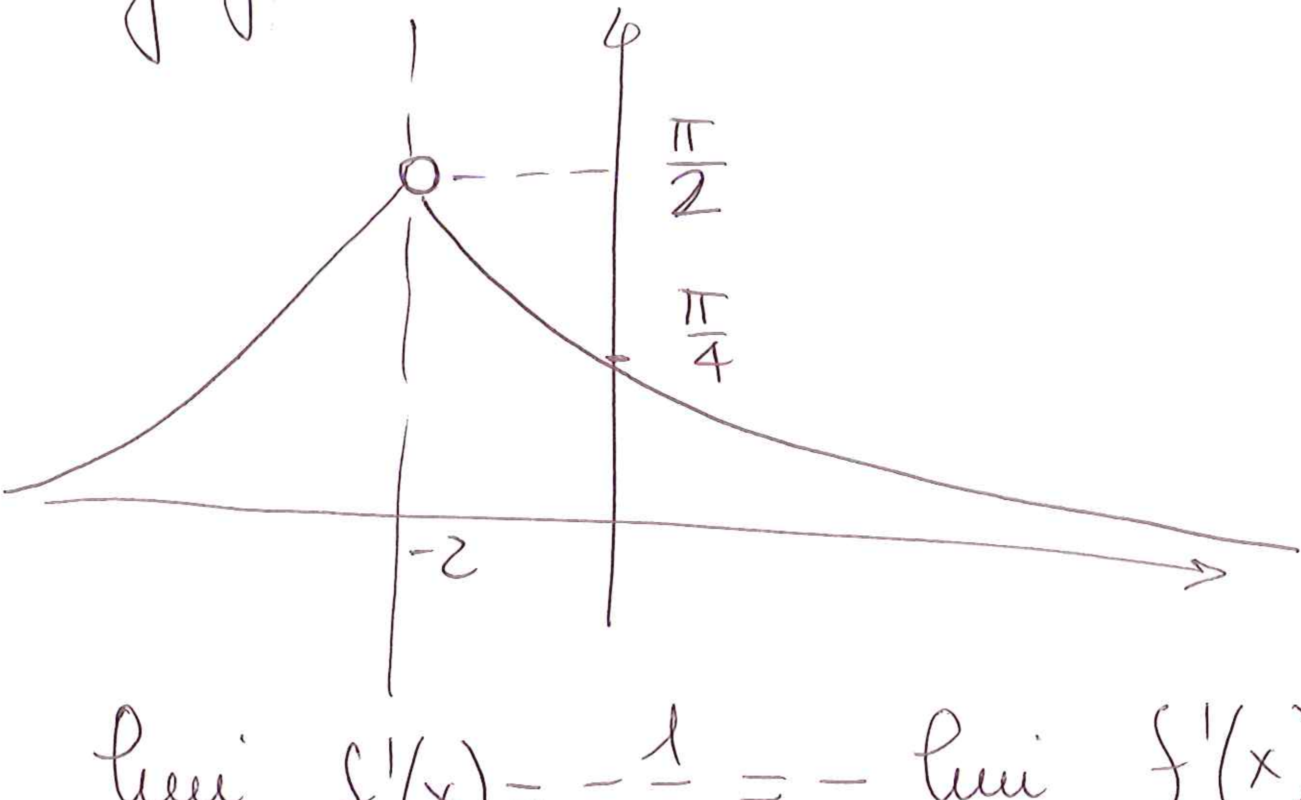
$$f(D) = \left(0, \frac{\pi}{2}\right).$$

Ag

$$f''(x) = \begin{cases} \dfrac{2}{\left[(x+2)^2+4\right]^2} \cdot 2(x+2) > 0 & \text{se } x > -2 \\[2ex] \dfrac{-4(x+2)}{\left[(x+2)^2+4\right]^2} > 0 & \text{se } x < -2 \end{cases}$$

f convesse in $(-\infty, -2)$ e in $(-2, +\infty)$

Grafico:



$$\lim_{x \to -2^+} f'(x) = -\frac{1}{2} = -\lim_{x \to -2^-} f'(x)$$