COMPITO B

B_1

1) CONV. ASSOLUTA:

$$\sum |a_n| = \sum \frac{n^n}{n! \cdot 3^n}$$

$$|a_n| \sim \frac{\cancel{n^n} e^n}{\cancel{n^n} \sqrt{2\pi n} \cdot 3^n} = \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{3}\right)^n$$

criterio della radice:

$$\sqrt[n]{\frac{1}{\sqrt{2\pi n}} \left(\frac{e}{3}\right)^n} = \frac{1}{\sqrt{\sqrt[n]{2\pi n}}} \cdot \frac{e}{3} \xrightarrow[n \to \infty]{} \frac{e}{3} < 1$$

convergenza $\Rightarrow$ la serie converge assolutamente $\Rightarrow$ converge semplicemente.

In alternativa: criterio di Leibniz

i) $$a_n = \frac{n^n}{n! \cdot 3^n} \sim \frac{\cancel{n^n} e^n}{\cancel{n^n} \sqrt{2\pi n}\, 3^n} = \frac{\left(\frac{e}{3}\right)^n}{\sqrt{2\pi n}} \to 0$$

ii) $a_{n+1} \le a_n \iff \dfrac{(n+1)^{n+1}}{(n+1)!\cdot \cancel{3^{n+1}}} \le \dfrac{n^n}{n!\cdot \cancel{3^n}}$ (B2)

$$\iff \frac{(n+1)^{n+\cancel{1}}}{\cancel{(n+1)}n^n} \le 3 \iff \left(\frac{n+1}{n}\right)^n \le 3$$

$$\iff \left(1+\frac{1}{n}\right)^n \le 3 \qquad \forall n \in \mathbb{N}$$

$\Rightarrow$ convergente della serie

2) $D = \{x = 1\}$

f sempre strettamente positiva

$$f(0) = \operatorname{arctg}\left(\frac{1}{|-1|}\right) = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

$$\lim_{x \to 1^\pm} f(x) = \operatorname{arctg}\left(\frac{1}{|0^\pm|}\right) = \operatorname{arctg}(+\infty) = \frac{\pi}{2}.$$

SINGOLARITÀ ELIMINABILE.

NO AS. VERT.

$$\lim_{x \to \pm\infty} f(x) = \operatorname{arctg}\left(\frac{1}{|\pm\infty|}\right) = 0$$

B3

AS. ORIZZ. per $x \to \pm\infty$.

$$f(x) = \begin{cases} \operatorname{arctg}\left(\dfrac{1}{x-1}\right) & \text{se } x > 1 \\ \operatorname{arctg}\left(\dfrac{1}{1-x}\right) = -\operatorname{arctg}\left(\dfrac{1}{x-1}\right) & \text{se } x < 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{1}{1+\left(\dfrac{1}{x-1}\right)^2} \cdot \left[\dfrac{-1}{(x-1)^2}\right] = \dfrac{-1}{(x-1)^2+1} < 0 & \text{se } x > 1 \\ \dfrac{1}{(x-1)^2+1} > 0 & \text{se } x < 1 \end{cases}$$

f cresce in $(-\infty, 1)$; decresce in $(1, +\infty)$

$\nexists$ MAX-MIN. ASS. o REL.

$\inf f = 0$; $\sup f = \dfrac{\pi}{2}$

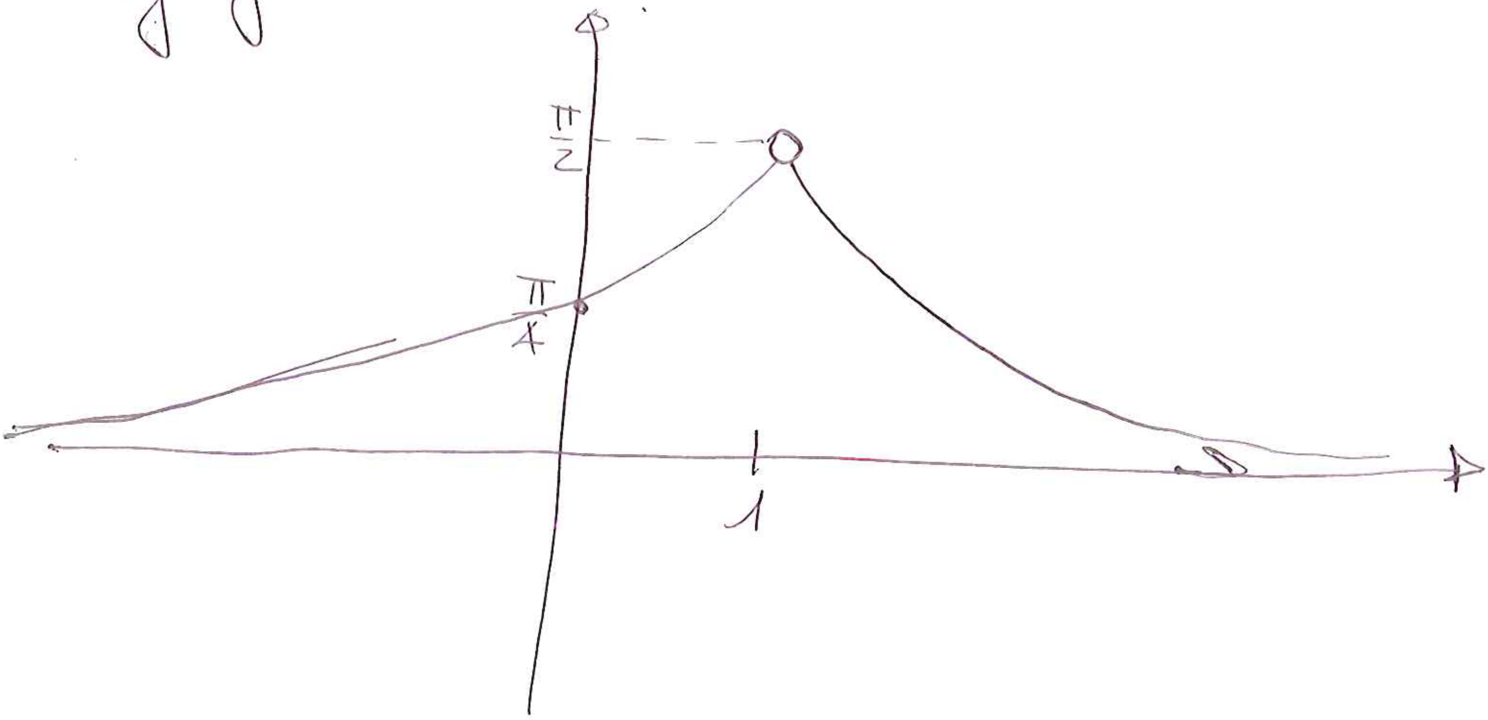
$$f(D) = \left(0, \frac{\pi}{2}\right).$$

B 3 bis

$$f''(x) = \begin{cases} \dfrac{1}{\left[(x-1)^2+1\right]^2}\, 2(x-1) > 0 & \forall x > 1 \\[2ex] \dfrac{-1}{\left[(x-1)^2+1\right]^2}\, 2(x-1) > 0 & \forall x < 1 \end{cases}$$

$\Rightarrow$ f CONVESSA in $(-\infty, 1)$ e in $(1, +\infty)$.

Grafico:



3) OMOGENEA: (B4)

$\alpha^2 - 4\alpha + 3 = 0 \iff \alpha_1 = 1 \; ; \; \alpha_2 = 3$

$\Rightarrow y_0(x) = C_1 e^x + C_2 e^{3x}$

NON OMOGENEA:

Principio di sovrapposizione:

$f_1(x) = e^x$

Poichè $\alpha = 1$ è radice del polinomio caratteristico $\Rightarrow y_1(x) = Ax e^x$

$\Rightarrow y_1'(x) = A(x+1)e^x \; ; \; y_1''(x) = A(x+2)e^x$

$\Rightarrow A[(x+2) - 4(x+1) + 3x]e^x = e^x$

$\Rightarrow A(2-4) = 1 \Rightarrow A = -\frac{1}{2}$

$\Rightarrow y_1(x) = -\frac{1}{2} x e^x$

$$f_2(x) = e^{2x} \Rightarrow y_2(x) = A e^{2x}$$ B5

$$\Rightarrow y_2' = 2A e^{2x} \; ; \; y_2'' = 4A e^{2x}$$

$$\Rightarrow A(4 - 8 + 3) e^{2x} = e^{2x}$$

$$-A = 1 \Rightarrow A = -1$$

$$\Rightarrow y_2(x) = -e^{2x}$$

INT. GENERALE:

$$y(x) = C_1 e^{x} + C_2 e^{3x} - \frac{1}{2} x e^{x} - e^{2x}$$

$$y(0) = C_1 + C_2 - 1 = 0$$

$$y'(x) = C_1 e^{x} + 3C_2 e^{3x} - \frac{1}{2}(x+1) e^{x} - 2e^{2x}$$

$$y'(0) = C_1 + 3C_2 - \frac{1}{2} - 2 = 1$$

$$\Rightarrow \begin{cases} C_1 + C_2 = 1 \\ C_1 + 3C_2 = \frac{7}{2} \end{cases} \Rightarrow \begin{cases} C_2 = \frac{5}{4} \\ C_1 = -\frac{1}{4} \end{cases}$$

$\Rightarrow$ SOLUZIONE del PROBLEMA

$$y(x) = -\frac{1}{4} e^{x} + \frac{5}{4} e^{3x} - \frac{1}{2} x e^{x} - e^{2x}$$

B6

4) a) $\int \left[\log\left(1+\frac{1}{2x}\right) - \frac{1}{2x}\right] dx$

$$= \left[x\log\left(1+\frac{1}{2x}\right) - \int \frac{x}{\left(1+\frac{1}{2x}\right)}\left(-\frac{1}{2x^2}\right)dx\right]$$

$$- \frac{1}{2}\log|x|$$

$$= \left[x\log\left(1+\frac{1}{2x}\right) + \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{2x+1}\,dx\right] - \frac{1}{2}\log(|x|)$$

$$= x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log(|2x+1|) - \frac{1}{2}\log(|x|) + c$$

$$= x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(\left|\frac{2x+1}{x}\right|\right) + c$$

b) $f(x) = \left[\cancel{\frac{1}{2x}} - \frac{1}{8x^2} - \cancel{\frac{1}{2x}} + o\left(\frac{1}{x^2}\right)\right]$

$\sim -\frac{1}{8x^2}$ che è integrabile

in $[1, +\infty) \Rightarrow f$ integrabile

c) $\int_1^{+\infty} f(x)\,dx$ (B_y)

$= \left[x \log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(\frac{2x+1}{x}\right)\right]_1^{+\infty}$

$= \lim_{x\to+\infty} \left[\log\left[\left(1+\frac{1}{2x}\right)^x\right] + \frac{1}{2}\log\left(\frac{2x+1}{x}\right)\right]$

$- \log\left(\frac{3}{2}\right) - \frac{1}{2}\log 3 =$

$= \log\left(e^{\frac{1}{2}}\right) + \frac{1}{2}\log(2) - \log\left(\frac{3}{2}\right) - \frac{1}{2}\log 3$

$= \frac{1}{2} - \frac{1}{2}\log\left(\frac{3}{2}\right) - \log\left(\frac{3}{2}\right)$

$= \frac{1}{2} - \frac{3}{2}\log\left(\frac{3}{2}\right).$

5) $\bar{z}(z\cdot\bar{z}) = z^3$

$z(\bar{z})^2 = z\cdot z^2$

$z\left[(\bar{z})^2 - z^2\right] = 0$

$z=0$ oppure $\bar{z}^2 = z^2$ $\quad$ B_8

$\cancel{x^2} \cancel{-y^2} - 2ixy = \cancel{x^2} \cancel{-y^2} + 2ixy$

$\Rightarrow xy = 0$

$\Rightarrow$ tutti i punti sugli assi sono soluzioni.

In alternativa, con la rappresentazione esponenziale:

$$\rho e^{-i\vartheta} \rho^2 = (\rho e^{i\vartheta})^3$$

$$\rho^3 e^{-i\vartheta} = \rho^3 e^{i3\vartheta} \Rightarrow \boxed{\rho = 0}$$

oppure

$e^{-i\vartheta} = e^{i3\vartheta}$ da cui $-i\vartheta = 3\vartheta + 2k\pi$

cioè $4\vartheta = 2h\pi \Rightarrow \vartheta = h\frac{\pi}{2}$

$$\left.\begin{array}{ll} \vartheta = 0: & \text{asse } x > 0 \\ \vartheta = \frac{\pi}{2}: & \text{asse } y > 0 \\ \vartheta = \pi: & \text{asse } x < 0 \\ \vartheta = \frac{3}{2}\pi: & \text{asse } y < 0 \end{array}\right\} \Rightarrow \text{TUTTI I PUNTI DEGLI ASSI.}$$