SVOLGIMENTO PROVA SCRITTA DI ANALISI MAT. 1 del 13/1/2015

A1

COMPITO A

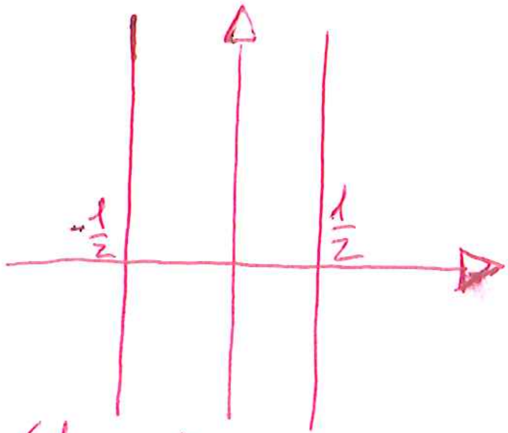
1) $z = x + iy$; $e^{8\pi i} = 1$

$$\Rightarrow \quad (2iy)^2 + 4(x^2+y^2) = 1$$

$$\cancel{-4y^2} + 4x^2 + \cancel{4y^2} = 1$$

$$x = \pm \frac{1}{2}. \quad \Longrightarrow \quad z = \left(\frac{1}{2}, y\right) \quad z = \left(-\frac{1}{2}, y\right)$$

INFINITE SOLUZIONI



2) $$\sum_{n=2}^{\infty} \left(1 - \frac{1}{(\log n)^2}\right)^{n^2}$$

serie a termini positivi

Criterio radice:

$$\sqrt[n]{a_n} = \left(1 - \frac{1}{(\log n)^2}\right)^{n} = \left[\left(1 - \frac{1}{(\log n)^2}\right)^{(\log n)^2}\right]^{\frac{n}{(\log n)^2}}$$

$$\longrightarrow \left(\frac{1}{e}\right)^{\infty} = 0 < 1$$

$\Rightarrow$ convergente.

3) La funzione integranda è pari:

$$f(-x) = (-x)^2 \cos(-2x) = x^2 \cos(2x) = f(x)$$

$$\Rightarrow 2 \int_0^{\pi} x^2 \cos(2x)\, dx$$

Per parti: $$= 2\left[\cancel{\frac{1}{2}\sin(2x) \cdot x^2 \Big|_0^{\pi}} - \int_0^{\pi} \frac{1}{\cancel{2}} \cancel{2} x \sin(2x)\, dx\right]$$

$$= -\cancel{2}\left[-\frac{1}{\cancel{2}}\cos(2x)\,x\Big|_0^{\pi} + \frac{1}{\cancel{2}}\int_0^{\pi}\cos(2x)\,dx\right]$$

A2

$$= \pi - \left[\frac{1}{2}\sin(2x)\right]_0^{\pi} = \pi$$

L'integrale NON rappresenta un'area perché la funzione integranda non è costantemente di segno non negativo.

$$g(x) = x^2\sin(2x) \text{ è dispari} : g(-x) = -g(x)$$

$$\Rightarrow \int_{-\pi}^{\pi} g(x)\,dx = 0.$$

4)
$$f(x) \underset{x\to+\infty}{\sim} \frac{\cancel{1}+\left[\cancel{\frac{1}{x}}-\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right]-\left[\cancel{1}+\cancel{\frac{1}{x}}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right]}{\left[\cancel{\frac{1}{x}}-\frac{1}{6x^3}+o\left(\frac{1}{x^3}\right)\right]-\left[\cancel{\frac{1}{x}}+\frac{1}{3x^3}+o\left(\frac{1}{x^3}\right)\right]}$$

$$= \frac{-\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)}{-\frac{1}{2x^3}+o\left(\frac{1}{x^3}\right)} \sim \frac{2x^3}{x^2} = 2x \xrightarrow[x\to+\infty]{} +\infty.$$

5) $D = \{x\in\mathbb{R} \mid x \neq \pm 1\} = (-\infty,-1)\cup(-1,1)\cup(1,+\infty).$

Segno: $f(x) > 0 \iff \dfrac{1}{e^{|x^2-1|}} > 1$

$\Longleftrightarrow \dfrac{1}{|x^2-1|} > 0 \qquad \forall x \neq \pm 1.$ (A3)

f sempre strettamente positiva $\forall x \in D$.

~~lim~~ f PARI

$$\lim_{x\to\pm\infty} f(x) = e^0 - 1 = 0$$

$y=0$ AS. ORIZZ. per $x \to \pm\infty$

$$\lim_{x\to\pm 1^\pm} f(x) = e^{\frac{1}{0^+}} - 1 = +\infty$$

$x = \pm 1$ AS. VERTICALI

$$f(x) = \begin{cases} e^{\frac{1}{x^2-1}} - 1 & \text{se } x<-1;\ x>1 \\ e^{\frac{1}{1-x^2}} - 1 & \text{se } -1<x<1. \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{-2x}{(x^2-1)^2}\, e^{\frac{1}{x^2-1}} & \text{se } x<-1;\ x>1 \\ \dfrac{2x}{(1-x^2)^2}\, e^{\frac{1}{1-x^2}} & \text{se } -1<x<1 \end{cases}$$

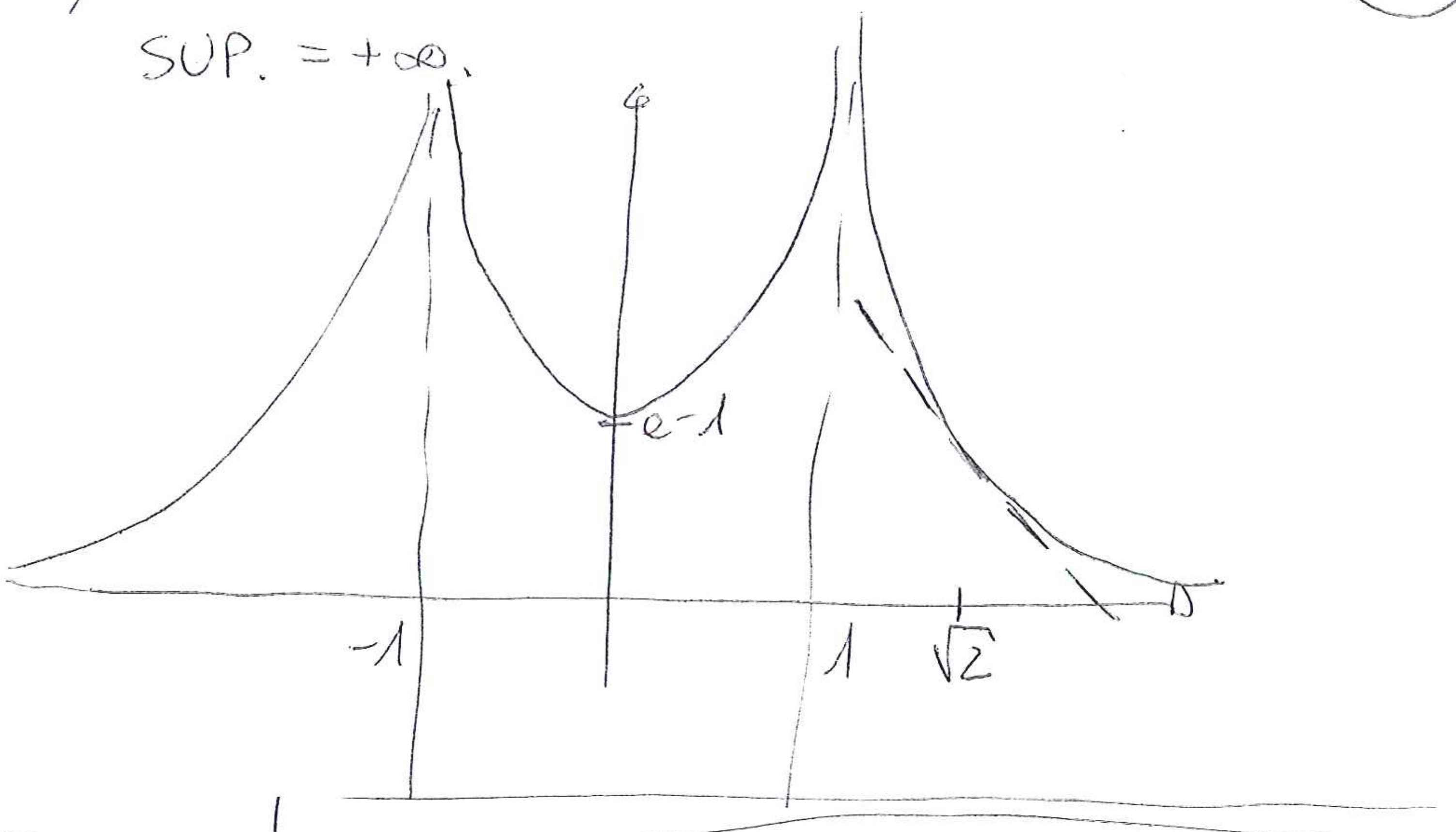
$$\frac{2x}{(x^2-1)^2} = \frac{2x}{(1-x^2)^2} > 0 \Longleftrightarrow x > 0$$

f cresce in $(-\infty,-1)$; decresce in $(-1,0)$; cresce in $(0,1)$; decresce in $(1,+\infty)$.

$x=0$ punto di MIN. REL. $f(0) = e-1$.

$\nexists$ MIN. ASS. ma INF $= 0$

SUP. $= +\infty$.



$$\sqrt{2} > 1 \implies f'(\sqrt{2}) = \left.\frac{-2x}{(x^2-1)^2} e^{\frac{1}{x^2-1}}\right|_{x=\sqrt{2}}$$

$$= \frac{-2\sqrt{2}}{(2-1)^2} \cdot e^{\frac{1}{2-1}} = -2\sqrt{2}\,e$$

inoltre

$f(\sqrt{2}) = e-1$

$\implies$ la retta tangente ha equazione

$$y = e-1 - 2\sqrt{2}\,e\,(x-\sqrt{2}) =$$
$$= 5e-1 - 2\sqrt{2}\,e x$$

FACOLTATIVO:

A_5

$$f''(x)=\begin{cases}\dfrac{2e^{\frac{1}{x^2-1}}}{(x^2-1)^4}(3x^4-1) & x<-1;\ x>1\\ \dfrac{-2e^{\frac{1}{1-x^2}}}{(x^2-1)^4}(3x^4-4x^2-1) & -1<x<1\end{cases}$$

Per $x<-1;\ x>1$

$$f''(x)>0 \iff 3x^4-1>0$$

$$\iff x^4>\frac{1}{3} \iff x<\frac{-1}{\sqrt[4]{3}};\ x>\frac{1}{\sqrt[4]{3}}$$

quindi $f''(x)>0 \quad \forall x\in(-\infty,-1)\cup(1,+\infty)$.

Per $-1<x<1$

$$f''(x)>0 \iff 3x^4-4x^2-1<0$$

$$\iff 0\le x^2<\frac{2+\sqrt{7}}{3}$$

$$\iff -\sqrt{\frac{2+\sqrt{7}}{3}}<x<\sqrt{\frac{2+\sqrt{7}}{3}}$$

quindi $f''(x)>0 \quad \forall x\in(-1,1)$

$\Rightarrow f$ convessa in $(-\infty,-1)$, in $(-1,1)$, in $(1,+\infty)$.