COMPITO B

(1B)

1) $B(y) = \frac{1+y^3}{2y^2} \in C^\infty((-\infty,0)\cup(0,+\infty))$

dato il Problema di Cauchy, cerco soluzioni con valori in $(0,+\infty)$

$A(x) = \cos x \in C^\infty(\mathbb{R})$

$\Rightarrow \exists I(0)$ t.c. $\exists!$ sol. $y(x) \in C^1(I(0))$

$B(y) = 0 \iff y = -1$ (sol. singolare)

La sol. singolare NON soddisfa il Problema di Cauchy.

Metodo di separazione variabili:

$$\int \frac{3y^2}{1+y^3}\,dy = \int \cos x\,dx$$

Ma $y \in (0,+\infty)$
$\Rightarrow \log(1+y^3)$

$$\Rightarrow \log(|1+y^3|) = \sin x + C$$

$$y(0) = 1 \Rightarrow \log 2 = C$$

$$\Rightarrow \log(1+y^3) = \sin x + \log 2$$

$$1+y^3 = e^{\sin x + \log 2} = 2e^{\sin x}$$

$$\Rightarrow \quad y^3 = 2e^{\sin x} - 1$$

$$\Rightarrow \quad y = \sqrt[3]{2e^{\sin x} - 1}$$

2) $$\lim_{x \to 0^-} \frac{e^{\alpha x^2} - 1}{x^2} = \alpha$$

$$\lim_{x \to 0^+} \frac{\sin x}{x} = \underline{1}$$

$\Rightarrow$ se $\alpha = \beta = 1 \qquad f \in C^0$ in $x = 0$.

$$f'(x) = \begin{cases} \dfrac{2xe^{x^2} \cdot x^2 - (e^{x^2} - 1)2x}{x^4} & \text{se } x < 0 \\ \\ \dfrac{\cos x \cdot x - \sin x}{x^2} & \text{se } x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{2}{x^3}\left[(x^2 - 1)e^{x^2} + 1\right] & \text{se } x < 0 \\ \dfrac{\cos x \cdot x - \sin x}{x^2} & \text{se } x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{2}{x^3}\left[(x^2-1)\left[1+x^2+\dfrac{x^4}{2}+o(x^4)\right]+1\right] & \text{se } x<0 \\ \dfrac{x\left[1-\dfrac{x^2}{2}+o(x^2)\right]-\left(x-\dfrac{x^3}{6}+o(x^3)\right)}{x^2} & \text{se } x>0 \end{cases}$$

(3B)

$$= \begin{cases} \dfrac{2}{x^3}\left[x^4 \cancel{-1} - \dfrac{x^4}{2}+o(x^4) \cancel{+1}\right] & \text{se } x<0 \\ \dfrac{\cancel{x}-\dfrac{x^3}{2}\cancel{-x}+\dfrac{x^3}{6}+o(x^3)}{x^2} & \text{se } x>0 \end{cases}$$

$$= \begin{cases} x & \text{se } x<0 \\ -\dfrac{1}{3}x & \text{se } x>0 \end{cases}$$

$\lim_{x\to 0^+} f'(x) = f'_-(0) = 0 = \lim_{x\to 0^-} f'(x) = f'_+(0)$

$\Rightarrow f$ è derivabile in $x=0$.

Poiché per $x\neq 0$ $f\in C^\infty$, allora

~~$f\in C^1(\mathbb{R})$~~ f è ~~de~~ continua e

derivabile in $\mathbb{R}$ solo per $\alpha=\beta=1$.

3) f definita per $|e^{3x}-1|>0$ 4B

$\Rightarrow e^{3x}-1\neq 0 \Rightarrow$ ~~x≠ log~~ $x\neq 0$

$D=\mathbb{R}-\{0\}=(-\infty,0)\cup(0,+\infty)$.

$$\lim_{x\to 0^{\pm}} f(x)=\log(0^+)=-\infty$$

$x=0$ ASINTOTO VERTICALE DX e SX

$$f(x)=\begin{cases}\log(e^{3x}-1) & \text{se } e^{3x}-1>0 \Rightarrow x>0\\ \log(1-e^{3x}) & \text{se } 1-e^{3x}>0 \Rightarrow x<0\end{cases}$$

Per $x\to+\infty$

$$f(x)=\log\left[e^{3x}\left(1-e^{-3x}\right)\right]=3x+\underbrace{\log\left(1-e^{-3x}\right)}_{\to 0}$$

Poiché $\lim\limits_{x\to+\infty}\left[f(x)-3x\right]=0$

$\Rightarrow$ $y=3x$ è ASINTOTO OBLIQUO per $x\to+\infty$

$$\lim_{x\to-\infty} f(x) = \log 1 = 0$$

$y=0$ ASINTOTO ORIZZONTALE per $x \to -\infty$.

ANCHE SE NON RICHIESTO, completiamo il grafico.

$$f(x)=0 \iff |e^{3x}-1| = 1 \iff e^{3x}-1 = \pm 1$$

$$\Rightarrow e^{3x} = 2 \Rightarrow x = \frac{\log 2}{3}.$$

$$f(x)>0 \iff |e^{3x}-1| > 1 \iff$$

$$e^{3x}-1 > 1 \quad \text{oppure} \quad \underbrace{e^{3x}-1 < -1}_{\text{MAI}}$$

$$\Rightarrow e^{3x} > 2 \Rightarrow \underbrace{x > \frac{\log 2}{3}}$$

$$f'(x) = \frac{1}{e^{3x}-1} 3e^{3x} = 3 + \frac{3}{e^{3x}-1}$$

$$f'(x) > 0 \iff e^{3x}-1 > 0 \iff x > 0$$

f decresce in $(-\infty, 0)$ e cresce in $(0, +\infty)$.

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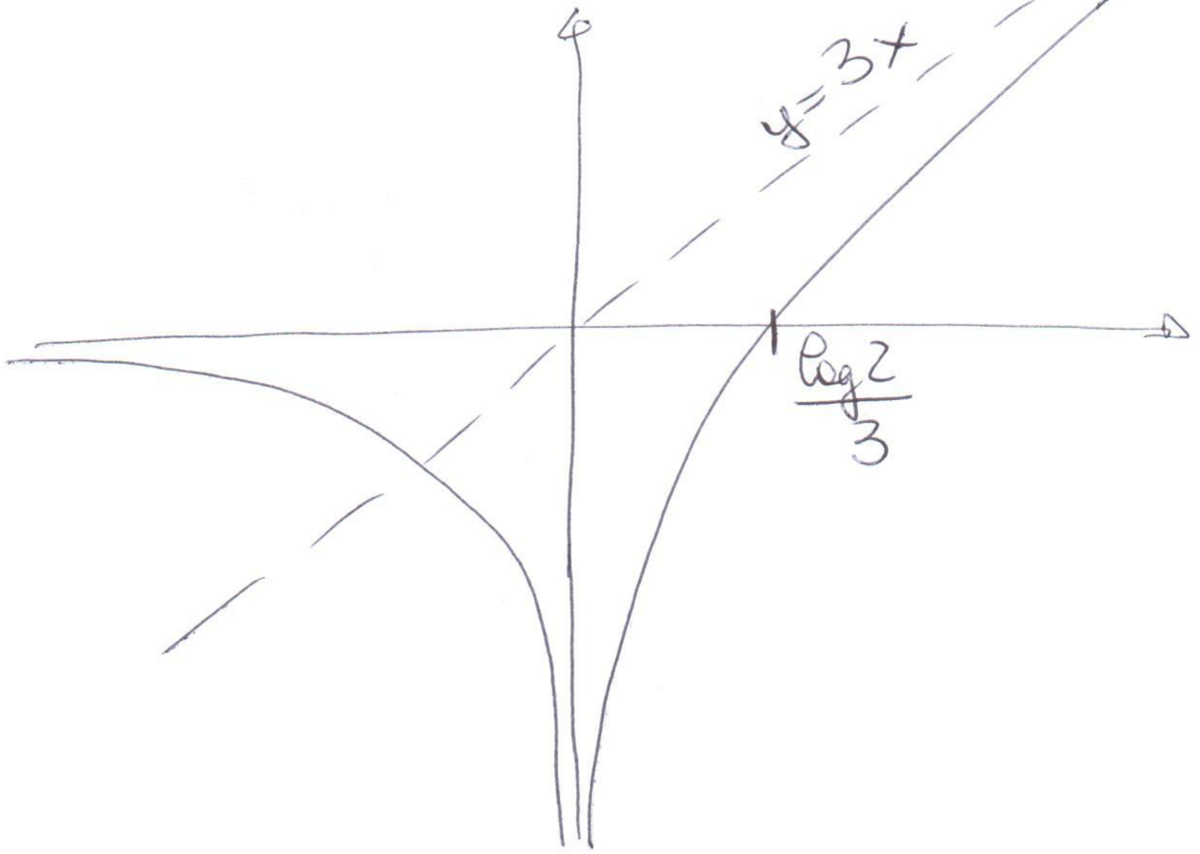
$$f''(x) = \frac{-3}{(e^{3x}-1)^2} \cdot 3e^{3x} < 0 \quad \forall x \in D$$

$\Rightarrow f$ concava in $(-\infty, 0)$ e in $(0, +\infty)$.



4) $\bar{z} \neq -2i \Rightarrow z \neq 2i$

$$\bar{z} + 2i = \frac{1}{i^{320}} = (-i)^{320} = [(-i)^4]^{80} = 1^{80} = 1$$

$$\Rightarrow x - iy + 2i = 1 \Rightarrow \begin{cases} x = 1 \\ y = 2. \end{cases}$$

$$\Rightarrow z = 1 + 2i$$

5) $a_n = \frac{n-1}{2n^2+1} \geq 0 \quad \forall n \in \mathbb{N}$ 7B

Serie a segno alterno.

Criterio di Leibniz.

i) $a_n \xrightarrow[n \to +\infty]{} 0$

ii) $a_n \geq a_{n+1} \iff \frac{n-1}{2n^2+1} \geq \frac{n}{2(n+1)^2+1}$

$\iff \left[2(n+1)^2+1\right](n-1) \geq n(2n^2+1)$

$(2n^2+4n+3)(n-1) \geq 2n^3+n$

$\cancel{2n^3}+2n^2-n-3 \geq \cancel{2n^3}+n$

$2n^2-2n-3 \geq 0$

Studiamo $2x^2-2x-3=0$

$x_{1,2} = \frac{1 \pm \sqrt{1+6}}{2} = \frac{1 \pm \sqrt{7}}{2}$

$\Rightarrow a_n$ decresce $\forall n > \frac{1+\sqrt{7}}{2}$ cioè $\forall n \geq 2$

$\Rightarrow$ la serie converge.